Abstract. — Let $X$ be a modular curve and consider a sequence of Galois orbits of CM points in $X$, whose $p$-conductors tend to infinity. Its equidistribution properties in $X(\mathbb{C})$ and in the reductions of $X$ modulo primes different from $p$ are well understood. We study the equidistribution problem in the Berkovich analytification $X_{\mathbb{Q}_p}$.

We partition the set of CM points of sufficiently high conductor in $X_{\mathbb{Q}_p}$ into finitely many basins $B_V$, indexed by the irreducible components $V$ of the mod-$p$ reduction of the canonical model of $X$. We prove that a sequence $z_n$ of local Galois orbits of CM points with $p$-conductor going to infinity has a limit in $X_{\mathbb{Q}_p}$ if and only if it is eventually supported in a single basin $B_V$. If so, the limit is the unique point of $X_{\mathbb{Q}_p}$ whose mod-$p$ reduction is the generic point of $V$.

The result is proved in the more general setting of Shimura curves over totally real fields. The proof combines Gross’s theory of quasicanonical liftings with a new formula for the intersection numbers of CM curves and vertical components in a Lubin–Tate space.

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1. Introduction

The equidistribution properties of CM points have been studied from various points of view, with remarkable applications ranging from cases of the André–Oort conjecture to the non-triviality of Heegner points. The main results to date are archimedean or ℓ-adic.

To illustrate the situation, let

\[(z_n)_{n \in \mathbb{N}}\]

be a sequence of CM points on a modular curve \(X\), with \(p\)-part of the conductor going to infinity. We view the \(z_n\) as scheme theoretic points of \(X\), or equivalently as Galois orbits of geometric points.

Duke [5], Clozel–Ullmo [3], Zhang [15] and others considered the images of the \(z_n\) in \(X(\mathbb{C})\), and proved the following archimedean equidistribution result: the sequence \(\mu_n\) of averaged delta measures at the orbits \(z_n\) converges to the hyperbolic probability measure on \(X(\mathbb{C})\). Cornut and Vatsal [4] considered the reduction modulo \(\ell\) of the \(z_n\), when \(\ell\) is a prime nonsplit in the fields of complex multiplication (or more generally simultaneous reductions at a finite set of such primes); they proved that within each connected component \(C\) of the mod-\(\ell\) reduction \(\mathcal{X}_F\), the images of the orbits \(z_n\) equidistribute to the counting probability measure on the finite set \(\mathcal{X}_F \cap C\) of supersingular points.

The purpose of this paper is to prove a \(p\)-adic equidistribution result for sequences of CM points. Consider the Berkovich analytic space \(X_p^{an}\) attached to the base-change \(X_Q\). It is a compact Hausdorff topological space, containing the set of closed point of \(X_Q\) and equipped with a reduction map to the special fibre \([\mathcal{X}_F]\) of any model \(\mathcal{X} \rightarrow \mathbb{Z}_p\) of \(X_Q\); the generic point of any irreducible component \(V\) of \([\mathcal{X}_F]\) is the reduction of a unique point \(\zeta_V \in X_p^{an}\). We consider in particular the canonical model defined by Katz–Mazur [8], whose fibre is a union of finitely many irreducible curves intersecting at the supersingular points.

Our main result is most easily described for sequences \((z_n)\) of CM points of \(X_Q\), equivalently local Galois orbits of geometric CM points. We partition the set of such CM points of sufficiently large \(p\)-adic conductor into finitely many basins \(\mathcal{B}_V\) indexed by the irreducible components of the special fibre of \(\mathcal{X}_F\). Then our main theorem is that the sequence of measures \(\mu_n := \delta_{z_n}\) on \(X_p^{an}\) has a limit if and only if the sequence \(z_n\) is eventually supported in a single basin \(\mathcal{B}_V\), in which case the \(\mu_n\) converge to the delta measure at \(\zeta_V\). Equivalently, the sequence \(z_n\) converges to \(\zeta_V\) in the plain topological sense.

The theorem is proved in the more general context of Shimura curves over totally real fields. The rest of this introduction is dedicated to explaining its statement and the idea of proof, as well as the intersection formula which lies at its core.

1.1. CM points on Shimura curves and their integral models. — Let \(F\) be a totally real field. Let \(B\) be a quaternion algebra over \(F\) whose ramification set \(\Sigma\) contains all the infinite places but one which we denote by \(\sigma : F \hookrightarrow \mathbb{R}\). We may attach to \(B\) a tower of Shimura curves \(X_U/F\), where \(U\) runs over compact open subgroups of \(B_{\infty}^x := (B \otimes_{\mathbb{Q}} \mathbb{A}_{\infty}^x)^x\). This curve and its canonical integral model were studied by Carayol [2], to which we refer for more details (see also [14] for a discussion of CM points).

The points of \(X_U\) over \(\mathbb{C}\) \(\hookrightarrow F\) can be identified with

\[(X_U(\mathbb{C})) \cong B^x \backslash \mathbb{H}^x \times B_{\infty}^x / U \cup \{\text{cusps}\}\]

where \(\mathbb{H}^x \cong \mathbb{C} - \mathbb{R}\) is the right quotient of \(B_{\infty}^x\) by its maximal connected compact subgroup, and the finite set \(\{\text{cusps}\}\) is non-empty only if \(B = M_2(\mathbb{Q})\) (it plays no role in this work). The group \(B_{\infty}^x\) acts by right translations (trivially on the components \(B_{\nu}^x\), \(\sigma \neq \sigma' : F \hookrightarrow \mathbb{R}\), and the action at the finite places is defined over \(F\).

From now on we fix an arbitrary level \(U = U_{\tau} U_c \subset B_{\infty}^x\), assumed to be sufficiently small so that the \(B^x\)-action in (1.1.1) has no nontrivial stabilisers. We let \(X := X_U\).
**CM points.** — Let $E$ be a CM quadratic extension of $F$, such that each finite $v \in \Sigma$ is nonsplit in $E$. Then the set of $F_v$-algebra embeddings $\psi: E_v \hookrightarrow B_\Lambda$ is non-empty. For any such $\psi$, the group $\psi(E^\times) \subset B_\Lambda^\times$ acts on the right on $X$. The fixed-point subscheme $X_{\psi(E^\times)}$, called the scheme of points with CM by $(E, \psi)$, is isomorphic (as $F$-subscheme) to $\Spec F \psi^{-1}(U)$ where $E\Sigma$ denotes the abelian extension of $E$ with Galois group $C \subset E^\times \setminus E_A^\times$. The CM (ind)-subscheme of $X_U$ is the union

$$X_{\text{CM}} = \bigcup_{(E, \psi)} X_{\psi(E^\times)}.$$  

**Local integral models and their irreducible components.** — Fix a finite place $v$ of $F$ not in $\Sigma$ and an identification $B_v = M_2(F_v)$. Let $\sigma_v$ be a uniformiser of $F_v$, $\mathcal{O}_{F_v}$ the ring of integers, $F_v$ the residue field. Carayol [2] defined a canonical integral model $\mathcal{X} = \mathcal{X}_U$ of $X$ over $\mathcal{O}_{F_v}$. It carries a sheaf $\mathcal{G}$ of divisible $\mathcal{O}_{F_v}$-modules of rank 2 together with, if $U_v = U_{n,v} \coloneqq \Ker(\GL_2(\mathcal{O}_{F_v}) \to \GL_2(\mathcal{O}_{F_v}/\mathcal{O}_{F_v}^n))$, a Drinfeld level structure$^{(1)}$

$$\alpha: (\sigma_v^{-n})^{\mathcal{O}_{F_v}/\mathcal{O}_{F_v}^n} \to \mathcal{G}(\mathcal{X}).$$

In general, $\mathcal{X} = \mathcal{X}_U \coloneqq \mathcal{X}'_{U, U_{n,v}}/(U_v/U_{n,v})$ for any $U_{n,v} \subset U_v$.

The special fibre $\mathcal{X}_E$ is a union of connected components permuted simply transitively by an action of $F^\times \setminus F_A^\times /\mathcal{O}_{F_v}^\times \det U_v$. The (supersingular) locus $\mathcal{X}_E^{\text{ss}}$ where $\mathcal{G}$ is connected is finite; its complement $\mathcal{X}_E^{\text{ord}} := \mathcal{X}_E - \mathcal{X}_E^{\text{ss}}$ (ordinary locus) has smooth reduction. Each connected component $C \subset \mathcal{X}_E$, is a union of irreducible components, each intersecting all of the others in each of the finitely many supersingular points.

The irreducible components $V$ within a given connected component are canonically parametrised by

$$\mathbf{P}^1(F_v)/U_v,$$

where we view $\mathbf{P}^1(F_v)$ as the space of rank-1 quotients of $\mathcal{O}_{F_v}^2$ or $(F_v/\mathcal{O}_{F_v})^2$. Explicitly, for $\lambda \in \mathbf{P}^1(F_v)/U_v$, the component $V(\lambda)_U = V(\lambda)_U^{(C)} \subset C$, viewed as subscheme with the reduced structure, is the locus where the Drinfeld level structure $\alpha$ factors through the quotient

$$\lambda: (\sigma_v^{-n})^{\mathcal{O}_{F_v}/\mathcal{O}_{F_v}^n} \to (\sigma_v^{-n})^{\mathcal{O}_{F_v}/\mathcal{O}_{F_v}^n}$$

determined by $\lambda$, when $U_v = U_{n,v}$. In general, $V(\lambda)_U^{(C)} \subset \mathcal{X}_F^{(C)}$ is the image of $V(\lambda)_U^{(C)} \subset \mathcal{X}_U^{(C)}$. $\text{End}_{\Pi_v, \text{Mod}}(\mathcal{G}_v)$ is the $\mathcal{O}_{F_v}$-order in $E_v$ with unit group $\mathcal{O}_{F_v}^{(1)}(U_0)$.

**CM points, their reductions, and basins.** — We consider the base-change (ind)-scheme $X_{F_v}^{\text{CM}} \subset \mathcal{X}_F$ and refer to its points as the CM points in $X_{F_v}$. It is well known that if $z \in X_{F_v}^{\text{CM}}$ is a point with CM by $(E, \psi)$, its reduction is a supersingular point if and only if $v$ is nonsplit in $E$. Assume that this is the case. The ring $\mathcal{O}_z := \text{End}_{\mathcal{G}_v, \text{Mod}}(\mathcal{G}_v)$ is the $\mathcal{O}_{F_v}$-order in $E_v$ with unit group $\mathcal{O}_{F_v}^{(1)}(U_0)$. By the well-known classification, there is a unique integer $s \geq 0$, called the $(v)$-conductor of $z$, such that

$$\mathcal{O}_z = \mathcal{O}_{F_v} + \mathcal{O}_{F_v}^{s}.$$  

It follows that there is an $\mathcal{O}_z^s$-linear isomorphism, unique up to $\mathcal{O}_z^s$, 

$$j : \mathcal{G}_v(\mathcal{O}_z) \cong (\mathcal{O}_{F_v} + \mathcal{O}_{F_v}^{s})^s,$$

where

$$(-)^s : = \text{Hom}_{\mathcal{O}_{F_v}}(-, \mathcal{O}_{F_v}^s)$$

is the $\mathcal{O}_{F_v}$-linear Pontryagin duality functor.

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$^{(1)}$This notion is recalled in §3.2 below.
1.2. Partition into basins and the main result. — Let \( z \in X_{F_v}^{\text{CM}} \) be a point with CM by \( E \) and keep the assumption that \( E \) is non-split. Let \( \alpha: (F_v / \mathcal{O}_v)^2 \rightarrow \mathcal{O}_v(\mathcal{F}_v) \) be any isomorphism extending the level structure \( \alpha_z \) on \( \mathcal{G}_z \). Then
\[
\tau^* := j^{-1} \circ \alpha: (F_v / \mathcal{O}_v)^2 \rightarrow (\mathcal{O}_v + \mathcal{O}_v \mathcal{O}_E)^2
\]
is dual to a unique isomorphism
\[
\tau: \mathcal{O}_v + \mathcal{O}_v \mathcal{O}_E \rightarrow \mathcal{O}_v^2.
\]
We denote by the same name its extension to an \( F \)-linear isomorphism \( \tau: E_v \rightarrow F_v^2 \). Then for any \( n \leq s \) such that \( U_{n,v} \subset U_v \), the subspace
\[
L_n(\tau) := \mathcal{O}_v^{-n} \tau(\mathcal{O}_v / \mathcal{O}_v^2) \subset (\mathcal{O}_v^{-n} \mathcal{O}_v / \mathcal{O}_v^2)^2
\]
is an \( (\mathcal{O}_v / \mathcal{O}_v^n) \)-line, whose class \( U_v L_n(\tau) \) is a well-defined invariant of \( z \).

We may compare it with the the orbit \( U_v L_n(\lambda) \) where
\[
L_n(\lambda) := \text{Ker}[(\mathcal{O}_v^{-n} \mathcal{O}_v / \mathcal{O}_v^2) / \mathcal{O}_v^2] \rightarrow \mathcal{O}_v^{-n} \mathcal{O}_v / \mathcal{O}_v^2.
\]

**Definition.** — Let \( n \) be such that \( U_{n,v} \subset U_v \). We say that the CM point \( z \in X_{F_v}^{\text{CM}} \) belongs to the basin \( \mathcal{B}_V \) of the irreducible component \( V = V(\lambda)_U^O \) if either:

- \( E_v / F_v \) is split and the reduction of \( z \) belongs to \( V \cap \mathcal{A}^{\text{ord}} \), or
- \( E_v / F_v \) is non-split, \( z \) has conductor at least \( n \), the reduction of \( z \) belongs to the connected component \( C \), and with notation as above
\[
L_n(\tau) = L_n(\lambda) \pmod{U_v}.
\]

**The Berkovich curve and the main theorem.** — We consider the Berkovich analytic space \( X_v^{\text{an}} \) attached to \( X_{F_v} \). It is a compact Hausdorff topological space, naturally containing the set of closed points of \( X_{F_v} \). It admits a reduction map \( \text{red}: X_v^{\text{an}} \rightarrow |\mathcal{X}_{F_v}| \), such that for each generic point \( \xi_V \in |\mathcal{X}_{F_v}| \) of an irreducible component \( V \subset \mathcal{X}_{F_v} \), there is a unique point
\[
\xi_V \in X_v^{\text{an}}
\]
reducing to \( \xi_V \).

**Theorem A.** — Let \( (z_n) \) be a sequence of points in \( X_{F_v}^{\text{CM}} \), and denote by the same name the image sequence in \( X_v^{\text{an}} \). Assume that the \( \nu \)-conductor of \( z_n \) tends to infinity.

The sequence \( z_n \) has a limit if and only if it is eventually supported in a single basin. If this is the case for the basin \( \mathcal{B}_V \) of the irreducible component \( V \subset \mathcal{X}_{F_v} \), then
\[
\lim_{n \rightarrow \infty} z_n = \xi_V
\]
in \( X_v^{\text{an}} \).

1.3. Idea of proof, intersection formula, and organisation of the paper. — To illustrate the proof of Theorem A, consider first a closed point \( a \in |\mathcal{X}_{F_v}| \) which is nonsingular in \( \mathcal{X}_{F_v}^{\text{ord}} \); this is the case if the \( \nu \)-level of \( X \) is minimal \((U_v = \text{GL}_2(\mathcal{O}_v)) \) or \( a \) is ordinary. Then \( \text{red}^{-1}(a) \subset X_v^{\text{an}} \) is a (twisted) analytic open unit disc, say with coordinate \( u \), and Gross’s theory of quasicanonical liftings shows that CM points of \( \nu \)-conductor \( s \) inside it lie in a circle \( |u| = 1 - \varepsilon_s \), with explicit \( \varepsilon_s \rightarrow 0 \) as \( s \rightarrow \infty \). This immediately implies that any limit point of \( (z_n) \) is one of the points \( \xi_V \). When the \( \nu \)-level of \( X \) is minimal, irreducible and connected components coincide and this argument is enough to conclude. Theorem A for the modular curves \( X_0(1)/\mathbb{Q} \) is also independently obtained by Herrero–Menares–Rivera-Letelier [7].
The above description on the distribution of CM points in nonsingular residue disks also implies the following: for each closed point \( a \in |X_{\mathbb{F}}| \), the intersection multiplicity at \( a \) between \( X_{\mathbb{F}} \) and the Zariski closure \( Z_n \) of points \( z_n \) of the sequence reducing to \( a \) tends to infinity with \( n \). As the intersection multiplicity between different irreducible components is bounded, this means that for large \( n \), exactly one of the irreducible components of \( X_{\mathbb{F}} \) intersects \( Z_n \) with high multiplicity. We will explain that this component is \( V \) if and only if \( z_n \) lies in a small neighbourhood of \( \zeta_V \) in \( X_{\mathbb{F}}^{CM} \). Thus it suffices to show that as \( z \) varies among CM points in \( |X_{\mathbb{F}}| \), the intersection multiplicity between its Zariski closure \( Z \) and any irreducible components \( V' \neq V \) stays bounded. This argument is developed in §2.

In order to conclude, we need to compute the intersection multiplicity at \( a \in |X_{\mathbb{F}}| \) between irreducible components of \( X_{\mathbb{F}} \) and Zariski closures of CM points, the nontrivial case being when \( a \) is supersingular. Thus the rest of the paper is dedicated to proving a new formula for such multiplicity, which is the following.

**Theorem B.** — Let \( z \in X_{\mathbb{F}}^{CM} \) be a point with CM by \((E, \varphi)\) of \( v \)-conductor \( s \), such that \( E_v/F_v \) is nonsplit. Let \( Z \) be the Zariski closure of \( z \) in \( X_{\mathbb{O}_v} \), and let \( V = V(\lambda)^{(C)} \) be an irreducible component through the reduction \( \overline{z} \in X_{\mathbb{F}} \) of \( z \), endowed with the reduced structure.

Let \( \tau \in \text{Hom}_{\mathbb{F}}(E_v, F_v^2) \) be attached to \( z \) as in (1.2.1); choose any \( \mathcal{O}_{E_v} \)-generator \( \delta^{-1} \in \mathcal{D}_{E_v}^{-1} := \text{Hom}_{\mathcal{O}_{E_v}}(\mathcal{O}_{E_v}, \mathcal{O}_{F_v}) \), and view the element \( \delta \tau \in \text{Hom}_{\mathbb{F}}(E_v, F_v^2) \) as a column vector over \( E_v \). Choose a representative for the class of \( \lambda \) among surjective maps in \( \text{Hom}_{\mathcal{O}_{E_v}}(\mathcal{O}_{F_v}^2, \mathcal{O}_{E_v}) \), and view it as a row vector in \( F_v^2 \).

Then

\[
\frac{m_{\mathcal{O}_v}(Z, V)}{[\mathcal{O}_{E_v}^2 : \varphi^{-1}(U_v)]} = [\mathcal{O}_v : F_v] \int_{U_v} |\lambda g \delta \tau|_{E_v}^{-1} d g.
\]

Here \( q = [F_v] \); \( e \) is the ramification degree of \( E_v/F_v \); the absolute value \( |\cdot|_{E_v} = |\cdot|_{E_v} \); and \( d g \) is the Haar measure such that \( \text{vol}(\text{GL}_2(\mathcal{O}_{E_v})) = 1 \).

If we call special cycles in \( \mathcal{X} \) those which are combinations of Zariski closures of CM points and vertical components, then our formula completes the calculations of intersections of special cycles in \( \mathcal{X} \), where the case of intersections of cycles of the same type was treated respectively by Katz–Mazur ([8], for vertical cycles) and, very recently, Qirui Li ([9], for CM cycles).

The intersection problem in \( \mathcal{X} \) is equivalent to one in a Lubin–Tate space \( \mathcal{M}_{U_v} \) of deformations of the unique formal \( \mathcal{O}_k \)-module \( \mathcal{G}_k \) of height 2 over \( k \). We solve it relying on the beautiful method devised by Li [9] (and in turn inspired by work of Weinstein [12]): after passing to infinite level in the Lubin–Tate tower, the intersection problem can be compared to an easier intersection problem in the formal group \( \mathcal{G}_k^2 \).

The cycles in Lubin–Tate towers and formal groups are defined in §3. The computation of intersections and the completion of the proofs is in §4.

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2. Reduction to intersection multiplicities

In this section, we prove Theorem A in the case of minimal level, and reduce it to a statement on intersection multiplicities in general.
2.1. Geometry of Berkovich curves. — Fix for this subsection a discretely valued field $K$, with uniformiser $\sigma$, ring of integers $O_K$, and residue field $k$. Let $X$ be a compact Hausdorff strictly $K$-analytic Berkovich space over $K$, generic fibre of a topologically finitely presented formal scheme $\mathcal{X}$ flat over $\text{Spf } O_K$. There is a reduction map to the special fibre of $\mathcal{X}$, 

\[
\text{red}: X \to |\mathcal{X}_k|.
\]

If $V \subset \mathcal{X}_k$ is an irreducible component and $\zeta_V \in \mathcal{X}_k$ is its generic point, then by [1, Proposition 2.4.4] there is a unique point $\zeta_V \in X$ with

\[
\text{red}(\zeta_V) = \zeta_V.
\]

We call $\zeta_V$ the Shilov point of $V$.

Suppose now that $X$ is a strictly $K$-analytic curve (that is $X$ is as above and $\dim X = 1$), and that $\mathcal{X}$ is regular. If $a \in |\mathcal{X}_k|$ is a closed point, we denote by

\[
D(a, 1 - \varepsilon), \quad \varepsilon > 0,
\]

an arbitrary increasing collection of compact subsets of $\text{red}^{-1}(a)$ such that $\bigcup_{\varepsilon > 0} D(a, 1 - \varepsilon) = \text{red}^{-1}(a)$. (The reader may keep in mind the case when $a$ is a nonsingular point and $K$ is algebraically closed: then $\text{red}^{-1}(a)$ is an open unit disc and one may take the $D(a, 1 - \varepsilon)$ to be discs of radius $1 - \varepsilon$.)

We define a similar collection of compact subsets of $X$ indexed by the open points of $\mathcal{X}_k$. Thus let $V$ be an irreducible component of $\mathcal{X}_k$, and let $\zeta_V$. As $\mathcal{X}$ is regular, hence locally factorial, there is a finite open cover by flat affine formal schemes $\mathcal{X} = \bigcup_{\mathcal{Y} \in I} \mathcal{Y}$ such that in $\mathcal{O}(\mathcal{Y})$ we may factor

\[
\sigma = \prod_{\mathcal{Y}} \phi_{\mathcal{Y} \cap \mathcal{X}}
\]

where $\phi_{\mathcal{Y} \cap \mathcal{X}}$ is a generator of the (height-1 prime, or unit) ideal $p_{\mathcal{Y} \cap \mathcal{X}} \subset \mathcal{O}(\mathcal{Y})$ and $e_{\mathcal{Y} \cap \mathcal{X}} > 0$. Let $Y \subset X$ denote the Berkovich generic fibre of $\mathcal{Y}$, and recall that by definitions, a point $x \in Y$ 'is' a multiplicative seminorm $| \cdot (x) | : \mathcal{O}(\mathcal{Y}) \otimes_{O_K} K \to \mathbb{R}$ extending the absolute value of $K$. We define

\[
D(\zeta_V, Y, 1 - \varepsilon) \colonequals \{ x \in Y : |\phi_{\mathcal{Y} \cap \mathcal{X}}(x)| \leq 1 - \varepsilon \} \subset Y.
\]

Lemma 2.1.1. — For $\varepsilon \in (0, 1)$, there is a unique compact subset

\[
D(\zeta_V, Y, 1 - \varepsilon) \subset X
\]

such that for all sufficiently small $O_K$ flat affine open $\mathcal{Y} \subset \mathcal{X}$ with generic fibre $Y$,

\[
D(\zeta_V, Y, 1 - \varepsilon) \cap Y = D(\zeta_V, \mathcal{Y}, 1 - \varepsilon) = (2.1.2).
\]

If $V'$ is an irreducible component of $\mathcal{X}_k$ and $\varepsilon > 0$ is sufficiently small, then the Shilov $\zeta_{V'}$ belongs to $D(\zeta_V, Y, 1 - \varepsilon)$ if and only if $V = V'$.

Proof. — For the first part, it suffices to observe the trivial facts that if $\mathcal{Y}'$ is a standard open affine subset of $\mathcal{Y}$ then the image in $\mathcal{O}(\mathcal{Y}')$ of a generator $\phi_{\mathcal{Y} \cap \mathcal{Y}'}$ of $p_{\mathcal{Y} \cap \mathcal{Y}'}$ (resp. of the factorisation (2.1.1)) is a generator of $p_{\mathcal{Y} \cap \mathcal{Y}'}$ (resp. a factorisation of the same form).

For the second part, we may replace $X$ by any open affinoid $Y$ as above such that $V \cap \mathcal{Y} \neq \emptyset$. By definition of the map $\text{red}$ in [1, §2.4], for $\phi \in \mathcal{O}(\mathcal{Y})$ we have $|\phi(\zeta_V)| < 1$ if and only if $\phi \in p_V$. This immediately allows to conclude.

Proposition 2.1.2. — Let $X$ be a compact strictly $K$-analytic Berkovich curve, generic fibre of a regular topologically finitely presented formal scheme $\mathcal{X}$ flat over $\text{Spf } O_K$. Let $V$ be an irreducible component of $\mathcal{X}_k$ and let $\zeta_V \in X$ be the Shilov point of $V$. With notation as above, the open subsets

\[
U(\zeta_V; A, \varepsilon) := X - \bigcup_{a \in A} D(a, 1 - \varepsilon) \subset X \quad A \subset |\mathcal{X}_k| - \{ \zeta_V \}, \varepsilon > 0
\]
form a fundamental system of neighbourhoods of $\zeta_v$ in $X$.

Proof. — By construction, the intersection of all the open sets $U(\zeta_v; A, \epsilon)$ contains no element in $\text{red}^{-1}(a)$ for $a \in |\mathcal{X}_k|$ a closed point. By the previous lemma, it contains $\zeta_v = \text{red}^{-1}(\xi_v)$ but no element $\zeta_v = \text{red}^{-1}(\xi_v)$ for $V' \neq V$. \hfill \Box

2.2. Reduction. — In this subsection we prove the main theorem assuming the following proposition, in which we use the following notation in force throughout the paper.

Notation. — If $z$ is a point of the generic fibre of a formal scheme $\mathcal{X}$ (where $'\text{red}'$ denotes an arbitrary decoration), $Z_z$ denotes the Zariski closure of $z$ in $\mathcal{X}$.

Proposition 2.2.1. — Let $X_v^{\text{an}}$ be the Berkovich analytification of the Shimura curve $X_{F_v}$ as in the introduction, and let $\mathcal{X}$ be its regular model over $\text{Spf} \mathcal{O}_{F_v}$ defined by Carayol [2]. Let $V$ be an irreducible component of $\mathcal{X}_{F_v}$.

1. If $z \in X_v^{\text{CM}}$ has CM by a quadratic extension $E$ in which $v$ splits, then

$$z \in \mathcal{B}_V \iff \text{red}(z) \in V - \bigcup_{V' \neq V} V'.$$

In particular, if $Z$ denotes the reduction of $z \in \mathcal{B}_V$ and $V'$ is an irreducible component of $\mathcal{X}_k$ different from $V$, then

$$m_{\mathcal{X}}(Z, V') = 0.$$

2. Let $Z \in \mathcal{X}_{F_v}$ be a supersingular point and let $V'$ be an irreducible component of $\mathcal{X}_{F_v}$ different from $V$. The intersection multiplicity

$$m_{\mathcal{X}}(Z, V')$$

is bounded as $z$ varies among CM points of $X_v^{\text{CM}}$ in $\mathcal{B}_V$ reducing to $Z$.

Part 1 of this proposition is obvious. Part 2 will be proved at the very end of the paper. Note that its statement is trivially true if $U_v = \text{GL}_2(\mathcal{O}_{F_v})$, because then $\mathcal{X}_{F_v}$ is smooth and irreducible component coincide with connected components. In particular, the proof of Theorem A which follows is already complete in the minimal level case.

Proof of Theorem A, assuming Proposition 2.2.1. — Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of CM points in $X_v^{\text{an}}$ with conductor going to $\infty$. Suppose the sequence is eventually supported in the basin $\mathcal{B}_V$. According to Proposition 2.1.2, we need to show that for each $a \in \mathcal{B}_{F_v}$, the set $\mathcal{X}_{F_v}$ of $a$ eventually leaves the compact set $D(a, 1 - \epsilon)$.

Let $\tilde{F}_v$ be the completion of the maximal unramified abelian extension of $F_v$, and denote by $\tilde{X}_v^{\text{an}}$ the analytification of $X_{\tilde{F}_v}$. Its reduction is $\tilde{\mathcal{X}}_k$, where $k = \mathcal{O}_{\tilde{F}_v}$.

Consider first the closed points $a \in \tilde{\mathcal{X}}_k$. As the transition maps in the tower $(X_{F_v})_{U_v}$ are finite and the images of CM points are CM points, it suffices to consider the case of minimal level $U_v \cong \text{GL}_2(\mathcal{O}_{F_v})$. By the ‘Serre–Tate’ result of Carayol recalled in Proposition 3.2.1 below, for each closed point $a \in |\mathcal{X}_k|$ the inverse image $\text{red}^{-1}(a)$ is canonically identified with the analytic generic fibre of the universal deformation space of the formal group $\mathcal{G}_{a}$, and isomorphic to the analytic open unit disk $D$ generic fibre of $\text{Spf} \mathcal{O}_{F_v}[n]$.

The CM points in $\text{red}^{-1}(a)$ are explicitly described by Gross’s theory of quasicanonical liftings (see [6], [11, 13]). Namely, if $z_n \in X_v^{\text{CM}}$ has conductor $s = s(n) \geq 1$, then any point of $X_{\tilde{F}_v}$ over $z_n$ is associated with the point $\text{Spec} \tilde{E}_{v,s}$, where $\tilde{E}_{v,s}$ is a specific totally ramified abelian (resp. dihedral if $v$ ramifies in $E$) extension of $\tilde{F}_v$ of degree

(2.2.1) $$m_{s(n)} := \zeta_{\tilde{F}_v}(1)\zeta_{\tilde{F}_v}(1)^{-1}eq^{s(n)}.$$
The inclusion of the point, spread out to the formal models, is dual to a map $\hat{O}_F[[u]] \to O_{E_n}$ sending $u$ to a uniformiser, hence $z_n$ lies in the circle $|u| = |x z|^{1/m(n)}$. As $m(n) \to \infty$ when $n \to \infty$, any compact disc in red$^{-1}(a)$ contains only finitely many CM points of $X_v^{CM}$. Therefore all points in the limit set of the sequence $(z_n)$ in $X_v^{an}$ reduce to a non-closed point of $\mathcal{X}_{E_n}$.

Let now $V' \neq V$ be an irreducible component of $\mathcal{X}_{E_n}$ and consider the set $D(1 - \varepsilon)$ (cf. Lemma 2.1.1). By the projection formula for intersection multiplicities, (2.2.1) also implies that if $z_n \in X_v^{CM}$ reduces to $\overline{z} \in |\mathcal{X}_{E_n}|$ and has conductor $s(n)$, then
\[
m_{D}(Z_n, \mathcal{X}_{E_n}) = [F_v(\overline{z}) : F_v] \cdot m_{s(n)} \to \infty.
\]
By Proposition 2.2.1, this implies that
\[
m_{D}(Z_n, V')/m_{D}(Z_n, \mathcal{X}_{E_n}) \to 0.
\]
We show that this means that $z_n$ eventually leaves $D(1 - \varepsilon) \subset X_v^{an}$. Let $\mathcal{Y} \subset \mathcal{X}$ be an affine open neighbourhood of $\overline{z}$, with generic fibre $Y \subset X_v^{an}$. The restriction to $\mathcal{O}(\mathcal{Y})$ of the seminorm (attached to) $Z_n$
\[
| \cdot(z_n)|: \mathcal{O}(Y) \to F_v(z_n) \to \mathbb{R}
\]
(where $|\cdot|$ is the unique absolute value extending the one on $F_v$) is precisely
\[
\phi \mapsto |\phi(z_n)| = |\sigma|^m(\mathcal{O}(\mathcal{Y}))/|\mathcal{O}(\mathcal{Y}^{\phi})|.
\]
Up to shrinking $\mathcal{Y}$, we may choose $\phi := \phi_{V'/\mathcal{Y}}$ to be a generator of $p_{V'/\mathcal{Y}} \subset \mathcal{O}(\mathcal{Y})$. Then (2.2.2) implies that
\[
|\phi_{V'/\mathcal{Y}}(z_n)| \to 1
\]
for all $V' \neq V$. Hence $z_n$ is outside $D(1 - \varepsilon)$ for all sufficiently large $n$.

It remains to prove part 2 of Proposition 2.2.1.

3. Special cycles in Lubin-Tate spaces and in formal groups

The goal of the rest of the paper is to establish an intersection multiplicity formula which will imply Proposition 2.2.1. The formal completion of $\mathcal{X}$ at a supersingular geometric point admits a purely local description as a Lubin-Tate space $\mathscr{M}$. After introducing some notation, we define the Lubin-Tate spaces and the local analogues of the special cycles of interest in it $\mathscr{M}$, comparing them to those in $\mathcal{X}$. Then we define companion cycles in formal groups.

3.1. Notation. — The notation introduced here will be used throughout the rest of the paper unless otherwise noted.

Valued fields. — We fix a prime $v$ of $F$ nonsplit in $E$ and work purely in a local setting, dropping the subscript $v$ from the notation; thus $F$ and $E$ will denote the local fields previously denoted by $F_v$ and $E_v$ respectively. We denote by $e = e_{F/E}$ the ramification degree.

We denote by $\sigma$ a fixed uniformiser in $F$, by $\nu$ the valuation on $F^\times$ (so $\nu(\sigma) = 1$), by $q$ the cardinality of the residue field of $F$, and by $|\cdot| := q^{-\nu(\cdot)}$ the standard absolute value on $F$.

If $K$ is a finite extension of $F$, we denote by $\hat{K}$ the completion of the maximal unramified abelian extension of $K$, $\hat{O}_K$ its ring of integers, and $k$ its residue field (which is independent of the choice of $K$). We endow $K$ with the absolute value $|\cdot|_K := |N_{K/F}(\cdot)|$.

Functors on $\mathcal{O}_f$-modules. — We denote by
\[
M[n] := M/\sigma^n M
\]
the truncation of $\mathcal{O}_F$-modules, and by

$$M^* := \text{Hom}_{\mathcal{O}_F}(M, F/\mathcal{O}_F), \quad M^\vee := \text{Hom}_{\mathcal{O}_F}(M, \mathcal{O}_F)$$

the Pontryagin and, respectively, plain dualities on $\mathcal{O}_F$-modules.

We stipulate the following convention on the order of reading of symbols:

$$\mathcal{O}_K^{\vee}(N) := (\mathcal{O}_K[N])^\vee, \quad \mathcal{O}_K[N] := (\mathcal{O}_K^\vee)^{[N]}.$$

We let

$$\mathcal{O}_K^{\vee \leftarrow} := \text{Hom}_{\mathcal{O}_F}(\mathcal{O}_K, \mathcal{O}_F)$$

denote the relative inverse different of $K/F$, an invertible $\mathcal{O}_K$-module.

**Integers.** — For compact open subgroups $C_K \subset K^\times$, we define integers

$$d_C = e_{K/F}[\mathcal{O}_K^\times : C_E], \quad d_C := [\mathcal{O}_F^\times : C_F]$$

where $e_{K/F}$ is the ramification degree. If $E/F$ is a quadratic extension, we also define

$$d_2 = d_{E,0} = 1, \quad d_{F,0} = e,$$

and, for $n \geq 1$,

$$d_n := q^{2n}, \quad d_{E,n} = e_{E,F}(1)^{-1}q^{2n}, \quad d_{E,0} := \zeta_{E,F}(1)^{-1}q^{2n}.$$

**Miscellaneous.** — We denote by $\mathcal{C}_K$ the category of complete Noetherian local $\mathcal{O}_K$-algebras with residue field $k$, and by $\mathcal{C}_k$ its subcategory consisting of objects which are $k$-algebras.

Finally, we will now freely use the language and notation of the Appendix, which the reader is now invited to skim through.

### 3.2. Lubin–Tate spaces

Let $K$ be a finite extension of $F$. If $\mathcal{G}$ is a formal $\mathcal{O}_K$-module of dimension 1 and height $h$ on an $\mathcal{O}_K$-algebra $A$, a Drinfeld structure on $\mathcal{G}$ of level $n$ on $\mathcal{G}$ is an $\mathcal{O}_F$-module map

$$\alpha : \mathcal{O}_K^{h,n} \rightarrow \text{Hom}_{Spf A}(Spf A, \mathcal{G})$$

such that

$$\alpha(\mathcal{O}_K^{h,n}) := \sum_{x \in \mathcal{O}_K^{h,n}} [\alpha(x)] = \mathcal{G}[[\sigma^n]]$$

as Cartier divisors. A Drinfeld structure of infinite level on $\mathcal{G}$ is a map $\mathcal{O}_K^{h,n} \rightarrow \mathcal{G}(A)$ whose restriction to $\mathcal{O}_K^{h,n}$ is a Drinfeld structure of level $n$ for all $n$.

Let $\mathcal{G}_{h,K}$ be a formal $\mathcal{O}_K$-module of height $h$ over $k$, which is unique up to isomorphism. Consider the functor $\mathcal{M}_{h,K,n}$ on $\mathcal{C}_K$ which associates to an object $A$ the set of equivalence classes of triples

$$(3.2.1)\quad [\mathcal{G}, \alpha, \alpha],$$

where $\mathcal{G}$ is a formal $\mathcal{O}_K$-module over $A$ of height $h$, $\alpha : \mathcal{G} \rightarrow \mathcal{G} \times_{Spf A} Spf k$ is a quasi-isogeny of height 0, and $\alpha : (F/\mathcal{O}_F)^{\mathcal{O}_K} \rightarrow \mathcal{G}(A)$ is a Drinfeld structure of level $n$. A theorem of Drinfeld asserts that $\mathcal{M}_{h,K,n}$ is representable by a regular $\mathcal{O}_K$-formal scheme finite flat over $\mathcal{M}_{h,K,0}$, and that $\mathcal{M}_{h,K,0} \cong Spf \mathcal{O}_K[[u_1, \ldots, u_{b-1}]].$

If $U \subset GL_h(\mathcal{O}_K)$ is the open compact subgroup

$$U = U_{h-K} := \text{Ker}(GL_B(\mathcal{O}_K) \rightarrow GL_B(\mathcal{O}_K/\sigma^n)), \quad (2)$$

we define $\mathcal{M}_{h,K,U} := \mathcal{M}_{h,K,n}/U$. If $U \subset GL_B(\mathcal{O}_K)$ is any compact subgroup containing $U_n$, we define $\mathcal{M}_{h,K,U} := \mathcal{M}_{h,K,n}/(U/U_n)$.

---

(2) We will simply write $U_n$ for $U_{n}^{(h,K)}$ when $h, K$ are clear from context.
The spaces (representing the functors) \( \mathcal{M}_{h,K,U} \) are called Lubin–Tate spaces. In what follows, we will denote

\[ \mathcal{G}_E := \mathcal{G}_E^{(1)}, \quad \mathcal{G}_F := \mathcal{G}_F^{(2)} \]

and identify \( \mathcal{G}_F \) with the image of \( \mathcal{G}_F \) under the forgetful functor. The endomorphism algebra of \( \mathcal{G}_E \) (resp. \( \mathcal{G}_F \)) is the ring of integers in \( E \) (resp. in the unique division algebra \( B \) of rank 4 over \( F \)).

We will also denote

\[ \mathcal{M}_U := \mathcal{M}_{2,F,U}, \quad \mathcal{M}_a := \mathcal{M}_U^{(2)}, \quad \mathcal{M}_C := \mathcal{M}_{1,F,C}, \quad \mathcal{N}_u := \mathcal{N}_U^{(2)}. \]

The space \( \mathcal{N}_C \) is isomorphic to \( \text{Spec} \tilde{O}_{E,C} \), the ring of integers in the abelian extension of \( \tilde{E} \) with Galois group \( \Theta_E^*/C \).

Finally, we denote by

\[ \mathcal{M}_\infty := \lim_{\leftarrow n} \mathcal{M}_n, \quad \mathcal{N}_\infty := \lim_{\leftarrow n} \mathcal{N}_n. \]

These limits exist as formal schemes, but it will be enough to consider them as pro-objects of a category \( \mathcal{F} \) of Noetherian formal schemes (cf. the Appendix), and as the functors on \( \mathcal{G}_F \) (resp. \( \mathcal{G}_E \)) of deformations of \( \mathcal{G}_F \) (resp. \( \mathcal{G}_E \)) together with a Drinfeld structure of infinite level.

We fix for the rest of the paper a level \( U \subset \text{GL}_2(\tilde{O}_F) \) and write \( \mathcal{M} := \mathcal{M}_U \). We denote by \( M \) the Berkovich generic fibre of \( M \).

A Serre–Tate theorem. — We restore the subscript \( v \) just for the following proposition.

**Proposition 3.2.1 (Carayol [2, §7.4]).** — Let \( \mathbb{Z} \in \mathcal{X}_{U,k} \) be a closed point.

1. If \( \mathbb{Z} \) is supersingular, the formal completion of \( \mathcal{X}_{\tilde{O}_k} \) at \( \mathbb{Z} \) is canonically (after identifying \( \mathcal{G}_F = \mathcal{G}_E \) \( \tilde{O}_k \)-isomorphic to the Lubin–Tate space \( \mathcal{M}_U \), via the map sending an \( A \)-valued point \( z \) to \( \mathcal{G}_F \) with its level structure \( \mathcal{A}_z \).

2. If \( \mathbb{Z} \) is ordinary, the formal completion of \( \mathcal{X}_{\tilde{O}_k} \) at \( \mathbb{Z} \) is canonically \( \tilde{O}_k \)-isomorphic to the deformation space of the \( O_{F,v} \)-divisible module \( \mathcal{G}_F^{(1)} \oplus F_v/O_{F,v} \) over \( k \), and non-canonically to \( \text{Spf} \tilde{O}_k/\mathcal{M}_{\infty} \); in particular, it is smooth over \( \tilde{O}_k \).

### 3.3. Special curves in Lubin–Tate spaces.

We consider two special classes of curves in \( \mathcal{M} \), the irreducible components of the special fibre, which we call **vertical curves**, and the Zariski closures of CM points, which we call **CM curves**. Then we compare them with the corresponding objects in the Shimura curve \( \mathcal{X} \).

**Vertical curves.** — Let \( A \) be a complete Noetherian \( k \)-algebra and let \( \mathcal{G} \) be an \( \tilde{O}_f \)-module of height 2 over \( A \). An **Igusa structure** of level \( n \) on \( \mathcal{G} \) is an \( \tilde{O}_f \)-module map

\[ \gamma: \mathcal{G}_{(n)}^* \rightarrow \text{Hom}_{\text{Spf}(A)}(\text{Spf}A, \mathcal{G}) \]

such that \( q^n \cdot \gamma(\mathcal{G}_{(n)}) = \mathcal{G}[\sigma^n] \) as Cartier divisors.\(^{(3)}\)

The functor on \( \mathcal{O}_F \) sending an object \( A \) to the set of isomorphism classes of triples \( [\mathcal{G}, \iota, \gamma] \), where \( [\mathcal{G}, \iota] \) is a deformation of \( \mathcal{G}_F \) as in (3.2.1), and \( \gamma \) is an Igusa structure of level \( n \), is representable by a smooth curve

\[ \text{Ig}_n \rightarrow \text{Spec} k \]

called the \( n \)th Igusa curve. It is of degree \( d_{\text{Ig},n} \) over \( \text{Ig}_0 := \mathcal{M}_{U,k} \cong \text{Spec} k[\mu] \). We define \( \text{Ig}_\infty := \lim_{\leftarrow n} \text{Ig}_n \) and

\[ [\text{Ig}_\infty]^n := \lim_{\leftarrow n} [\text{Ig}_n]^n \in K_1(\text{Ig}_\infty)\mathbb{Q}, \quad [\text{Ig}_n]^n := d_{\text{Ig},n}^{-1} \cdot [\text{Ig}_n]. \]

\(^{(3)}\)Our \( \text{Ig}_n \) should be compared with the ‘exotic Igusa curve’ \( \text{Exgl}(p^n, n) \) of \([8, (12.10.5.1)]\); cf. *ibid.* Proposition 13.7.5.
Let
\[ \lambda \in \mathbb{P}^1(F) = F^\times \setminus \text{Hom}_F(F^2, F). \]
We attach to \( \lambda \) the morphism
\[ f_\lambda : \mathfrak{g}_{\infty} \to \mathcal{M}_{U,k} \hookrightarrow \mathcal{M}_U \]
and the irreducible and reduced curve
\begin{equation}
(3.3.1) \quad f_\lambda(\mathfrak{g}_{\infty}) = V(\lambda)_U \subset \mathcal{M}_{U,k}
\end{equation}
defined by the closed condition that any extension to \( \mathfrak{O}_F^\times \) of the level structure \( \alpha \) of a geometric point 
\([\mathfrak{g}, i, \alpha]\) belonging to \( V(\lambda)_U \) should factor through \( \lambda g : \mathfrak{O}_F^\times \to \mathfrak{O}_F^\times \) for some \( g \in U \). The \( V(\lambda)_U \), for \( \lambda \in \mathbb{P}^1(F)/U \), are the reductions of the irreducible components of \( \mathcal{M}_{U,k} \) and they all meet at the closed point.

**Definition 3.3.1.** — The vertical curve attached to \( \lambda \) in \( \mathcal{M}_\infty \), respectively \( \mathcal{M}_U \), is the curve
\[ V(\lambda) := f_\lambda(\mathfrak{g}_{\infty}) \subset \mathcal{M}_\infty, \quad V(\lambda)_U := f_{\lambda,U}(\mathfrak{g}_{\infty}) \subset \mathcal{M}_U. \]

The corresponding normalised vertical cycle in \( K_*(\mathcal{M}_\infty)_Q \), resp. \( K_*(\mathcal{M}_U)_Q \), is
\[ [V(\lambda)]^\circ := f_{\lambda,*}(\mathfrak{g}_{\infty})^\circ = \lim [V(\lambda)]^\circ_U, \quad [V(\lambda)]^\circ_U = d_U^{-1} \cdot [V(\lambda)]_U. \]

**CM curves.** — We consider a local analogue of CM points and of their Zariski closures in formal models. These will be images of maps of Lubin–Tate towers and we start by defining the data parametrising them.

Consider an element
\[ (\varphi, \tau) \in \text{LT}(E, F) := \text{QIsog}(\mathfrak{g}_E, \mathfrak{g}_F) \times_{E^\times} \text{Isom}(E, F^2) \]
where the notation means that \( \varphi \) is a quasi-isogeny of formal \( \mathfrak{O}_F \)-modules, \( \tau \) is an \( F \)-linear isomorphism, and we consider the quotient of the product by the relation \((\varphi \circ t, \tau \circ t) \sim (\varphi, \tau), \ t \in E^\times\). There exists a shortest integer interval \([r,s]\) such that \( \sigma^r \mathfrak{O}_E^2 \subset \tau(\mathfrak{O}_E) \subset \sigma^{-r} \mathfrak{O}_E^2 \); this depends on the choice of representative \( \tau \).

We define two functions, conductor and height, on \( \text{LT}(E, F) \) by
\[ c(\varphi, \tau) := s - r, \quad \text{ht}(\varphi, \tau) := -\text{ht}(\varphi) + \log_2 [\tau(\mathfrak{O}_E) : \mathfrak{O}_E^2], \]
where \( \text{ht}(\varphi) \) is the usual \( \mathfrak{O}_E \)-height of a quasi-isogeny of \( \mathfrak{O}_E \)-modules, and \([\cdot : \cdot]\) denotes generalised index. We denote by \( \text{LT}(E, F)^0 \) the subspace of pairs of height 0.\(^{(4)}\)

From now on, the representatives \((\varphi, \tau)\) of elements of \( \text{LT}(E, F)^0 \) will always be chosen to satisfy \( r = 0.\(^{(5)}\)\)

In this case \( c(\varphi, \tau) = s = \text{ht}(\varphi) \).

To an element \((\varphi, \tau) \in \text{LT}(E, F)^0\) we attach a morphism \( f_{(\varphi, \tau)} \) of Lubin–Tate towers, composition of the natural projection \( \mathcal{N}_\infty \times_{\mathfrak{g}_E} \mathcal{G}_E \to \mathcal{M}_\infty \) and of a (representable) morphism of (representable) functors on \( \mathcal{G}_E \):
\begin{equation}
(3.3.2) \quad f_{(\varphi, \tau)} : \mathcal{N}_\infty \to \mathcal{M}_\infty \times_{\mathfrak{g}_E} \mathcal{G}_E \to \mathcal{M}_\infty
\end{equation}
where the notation is according to the following commutative diagrams:

\(^{(4)}\)Our definitions are equivalent to those in \([9]\)\( \text{specialised to } b = 1 \), where the space \( \text{LT}(E, F)^0 \) is denoted by \( \text{Equ}_b(E/F) \).

\(^{(5)}\)The reader unwilling to make this assumption will simply need to interpret some maps of formal \( \mathfrak{O}_F \)-modules in the diagrams to follow as quasi-isogenies.
In the above diagrams, the columns are exact, the bars denote reduction modulo the maximal ideal of $\hat{O}_E$, and $\alpha_K$ is the subgroup scheme of $\mathcal{G}$ defined by the product of ideal sheaves of the $\alpha_x$, $x \in K$.

Let $\varphi : E \hookrightarrow \text{GL}_2(F)$ be the $F$-algebra morphism such that $\varphi(t)\tau(x) = \tau(tx)$ for all $t, x \in E$. The composition to level $U$ factors as

\[(3.3.4)\quad f_{(\varphi, \tau), U} := p_U \circ f_{(\varphi, \tau)} : \mathcal{N}_\infty \to \mathcal{N}_C \hookrightarrow \mathcal{M}_U\]

where $C := \varphi^{-1}(U)$ and the second map is a closed immersion. In particular, the morphism of functors defined in infinite level does define a morphism in the pro-category of Noetherian formal schemes $\mathcal{F}'$ defined in the Appendix.

If $U = U_0$ and $(\varphi, \tau)$ has conductor $s$, then $C = (O_E + \sigma s O_E)^s$, and the image of $\mathcal{N}_C$ is a quasi-canonical lift of level $s$.

**Definition 3.3.2.** — The CM curve in $\mathcal{M}_\infty$ (resp. $\mathcal{M}_U$) attached to $(\varphi, \tau)$ is

$$Z(\varphi, \tau) := f_{(\varphi, \tau)}(\mathcal{N}_\infty) \subset \mathcal{M}_\infty, \quad Z(\varphi, \tau)_U := f_{(\varphi, \tau), U}(\mathcal{N}_\infty) \subset \mathcal{M}_U.$$ 

where $f_{(\varphi, \tau)}$ (resp. $f_{(\varphi, \tau), U}$) is the morphism defined by (3.3.2) (resp. (3.3.4)).

The local CM point $z(\varphi, \tau)$ in the Berkovich generic fibre $M$ of $\mathcal{M}$ is the generic fibre of $Z(\varphi, \tau)$; equivalently, it is the image under $f_{(\varphi, \tau)}$ of the single point in the generic fibre of $\mathcal{N}_\infty$ for any sufficiently large $n$.

The normalised CM cycle attached to $(\varphi, \tau)$ in $K_s(\mathcal{M}_\infty)_Q$ (resp. $K_s(\mathcal{M}_U)_Q$) is

$$[Z(\varphi, \tau)]^\circ := f_{(\varphi, \tau)}([\mathcal{N}_\infty]^\circ) = \lim_{U} [Z(\varphi, \tau)]^\circ_U, \quad [Z(\varphi, \tau)]^\circ_U = \frac{1}{\partial^{(\varphi^{-1}(U))}} [Z(\varphi, \tau)]_U.$$

**Basins.** — Let $V = V(\lambda)_U \subset \mathcal{M}_U$ be the irreducible component parametrised by the $U$-class of $\lambda$, and let $n$ be such that $U \supset U_n$. Let $z = z(\varphi, \tau) \in M$ be a local CM point of conductor $s \geq n$, and assume $\tau$ is chosen so that $\sigma s^{-1} \tau \subset \tau(\mathcal{O}_E) \supset \mathcal{O}_E^2$. We say that $z$ belongs to the basin $\mathcal{B}_V(\lambda)_U$ of $V$ if the lines

$$L_n(\tau) := \sigma^{s-n} \tau(\mathcal{O}_E) / \sigma^n \mathcal{O}_E^2, \quad L_n(\lambda) := \ker(\mathcal{O}_{F[n]}^2 \to \mathcal{O}_{F[n]}^+)^{\circ_{\mathcal{O}_{F[n]}^2}} \subset \mathcal{O}_{F[n]}$$

are in the same orbit under the action of $U$.

**Comparison of local and global objects.** — We temporarily restore the subscripts $v$ for local objects. We use the notation of the introduction for CM points on our fixed Shimura curve $X = X_U$, and let $\mathcal{M} = \mathcal{M}_{U_0}$.

**Lemma 3.3.3.** — Let $z \in X^{CM}_{F_v}$ be a point with complex multiplication by $(E, \varphi)$, with $E_v/F_v$ non-split. Let $Z$ be the Zariski closure of $z$ in $\mathcal{X}$. Let $\tau : E_v \to F_v^2$ be such that $\psi_e(t)\tau(x) = \tau(tx)$ for all $t, x \in E_v$. Let $\bar{Z} \subset \mathcal{X}_{F_v}$ be the reduction of $z$ and let $C \subset \mathcal{X}_{F_v}$ be the connected component containing $\bar{Z}$. Let $\lambda \in \mathbb{P}^1(F_v)/U_v$, let

$$V = V(\lambda)^{(C)}_U \subset C \subset \mathcal{X}_{F_v}$$

be the corresponding irreducible component.
Let \( \mathcal{Z} \in \mathcal{X}_k \) be any point over \( \mathcal{Z} \). Let \( z' \in M \) be any point mapping to \( z \in X_F \) under the identification of \( M \) with the inverse image of \( \mathcal{Z} \) in \( X_F \), given by Proposition 3.2.1. Let \( Z' \) be the Zariski closure of \( z' \), and let \( V' \subset \mathcal{M}_{k} \) be any irreducible component mapping to \( V \subset \mathcal{M}_{k} \).

Then:

1. There exist \((\varphi, \tau) \in \text{LT}(E, F)^{\prime}\), whose second component is equal (up to rescaling by \( E_\varphi^\ast \)) to the isomorphism \( \tau: F_\varphi \to F_\varphi^2 \) attached to \( z \), such that \( z' = z(\varphi, \tau) \).

2. \( V' = V(\lambda)\big|_{U_k} \subset \mathcal{M}_{U_k} \).

3. The CM point \( z \) belongs to \( \mathcal{B}_V \) if and only if \( z' \) belongs to \( \mathcal{B}_{V'} \).

4. We have the following relation between intersection multiplicities in \( \mathcal{X} \) and \( \mathcal{M}_{U_k} \):

\[
m_p(Z, V) = [F_\varphi(Z):F_\varphi] \cdot m(Z', V').
\]

Proof. — The first three statements are clear from the definitions. For the last one, observe that the right hand side is the sum of the intersection multiplicities of all the \( \text{Gal}(F_\varphi(Z)/F_\varphi) \)-conjugates of the pair \( (Z', V') \) (as a pair of cycles in the completion of \( \mathcal{X}_{\mathcal{O}_k} \) at \( \mathcal{Z} \)), and all those multiplicities are equal. \( \square \)

3.4. Special cycles in formal groups and their Tate modules. — We define special cycles, analogous to those defined above in \( \mathcal{M}_\infty \), inside \( \mathcal{G}_h^2 \), its truncations or Tate modules. The definitions are variants of those of [9]. Later in §4.1, we will reduce the intersection problem in Lubin–Tate spaces to a similar problem in \( \mathcal{G}_h^2 \).

Let \( K = F \) or \( E \) and let \( \mathcal{G}_h \) be a formal \( \mathcal{O}_k \)-module. For \( ? \in \{\emptyset, [N]\} \) and \( \iota \in \{\ast, \vee\} \), let

\[
\iota_{\mathcal{G}_h^2} := \text{Hom}(\mathcal{O}_h^{\iota}, \mathcal{G}_h)
\]

Then we have identifications and maps

\[
\iota_{\mathcal{G}_h^2} \cong \mathcal{G}_h^{-1} \otimes_{\mathcal{O}_h} T \mathcal{G}_h
\]

(3.4.2)

\[
\iota_{\mathcal{G}_h^{[N]}} \cong \mathcal{G}_h^{-1} \otimes_{\mathcal{O}_h} \mathcal{G}_h^{[N]} = \mathcal{G}_h^{[N]} \otimes_{\mathcal{G}_h^{[N]}} \mathcal{G}_h 
\]

where the Tate module \( T \mathcal{G}_h \) is the \( \mathcal{O}_h \)-module scheme

\[
T \mathcal{G} := \lim_{\longrightarrow} \mathcal{G}_h^{[n]}
\]

with transition maps given by multiplication by \( \mathcal{G}_h \). (The isomorphisms denoted by \( \cong \) depend on the choice of \( \mathcal{G}_h \), which allows to compatibly identify \( \mathcal{O}_h^{[N]} \cong \mathcal{O}_h^{[N]} \). This dependence will be negligible for our purposes.)

Vertical cycles. — Let \( \lambda \in \mathbb{P}^1(F) \), which we may represent as \( \lambda = [a : b] \) with \( a, b \in \mathcal{O}_h \) coprime. We attach to \( \lambda \) the morphisms, all denoted by the same name,

\[
f_\lambda^{\prime} : \iota_{\mathcal{G}_h^{[N]}} \to \iota_{\mathcal{G}_h^{[N]}},
\]

\[
x \mapsto x \circ \lambda
\]

where \( ? \in \{[N], \emptyset\} \) and \( \iota \in \{\ast, \vee\} \).

Definition 3.4.1. — Let \( ? \in \{[N], \emptyset\} \), let \( \iota \in \{\ast, \vee\} \), and let \( \bullet = \emptyset \) if \( \iota = \vee \), \( \bullet = \circ \) if \( \iota = \ast \). The vertical cycle attached to \( \lambda \) is

\[
[V(\lambda)]^{\ast} := f_\lambda^{\prime} \ast \iota_{\mathcal{G}_h^{[N]}} \in K_\ast(\mathcal{G}_h^{[N]}),
\]

where in the case of \( \iota = \ast \), the normalisation is made viewing \( \mathcal{G}_h^{[N]} \) as a scheme (of degree \( d_{[N]} \)) over the base \( k \).

(The symbol \( \iota \) is omitted from the name of the cycle as it may be determined from the superscript \( \bullet \).)
CM cycles. — Let \((\varphi, \tau) \in \mathrm{LT}(E, F)^2\). For \(n \geq c(\varphi, \tau)\), consider the morphism
\[(3.4.3) \quad \sigma^m f_{(\varphi, \tau)}^* : \varpi^2 \to \varpi^2 \quad \gamma \mapsto \varphi \circ \gamma \circ \sigma^m \tau^\vee.
\]

(Its name is meant to suggest the idea that \(\sigma^m f_{(\varphi, \tau)}^*\) may be thought of as the composition of \([\sigma^m]\) and an, in general non-existent, morphism \(f_{(\varphi, \tau)}^*\).

**Definition 3.4.2.** — Let \(\ast \in \{[N], \emptyset\}\), let \(t \in \{\emptyset, \vee\}\), and let \(\bullet = \emptyset\) if \(t = \emptyset\), \(\bullet = \ast\) if \(t = \ast\). The CM cycle attached to \((\varphi, \tau)\) is
\[\pi_N, ([Z^\prime(\varphi, \tau)]^*_{\ast}) := d_n(\sigma^m f_{(\varphi, \tau)}^*), ([\varpi^2])^{\ast} \in K_{G}(\varpi^2)_{Q},\]

where in the case of \(t = \ast\), the normalisation is made viewing \(\varpi^2\) as a scheme of degree \(edN\) over the base \(k\).

It is easy to verify that the definition is independent of \(n \geq c(\varphi, \tau)\).

For \(Z^\prime = Z^\prime(\varphi, \tau), V^\prime(\lambda)\), we lighten the notation by \([Z^\prime]^*_{\ast} := [Z^\prime]^*_{\ast}([N])\).

**Lemma 3.4.3.** — With reference to the maps in (3.4.2), for \(Z^\prime = Z^\prime(\varphi, \tau), V^\prime(\lambda)\) we have
\[\pi_N, ([Z^\prime]^*_{\ast}) = [Z^\prime]^*_{\ast} = d_N^{-1}([Z^\prime]^*_{\ast}) = d_N^{-1}([Z]^*_{\ast}).\]

**Proof.** — Clear from the definitions. \(\square\)

**3.5. Approximation.** — We show that the special cycles in \(\mathcal{M}^2 = \varpi^2\) approximate those in \(\mathcal{M}_\infty\). The comparison is done via the following maps:
\[(3.5.1) \quad \varpi^2 \xrightarrow{\sigma^m a^F} \mathcal{M}_\infty, \quad \varpi^2 \xrightarrow{\sigma^m a_E} \mathcal{M}_\infty, \quad \varpi^2 \xrightarrow{\sigma^m a_I} I_{\mathcal{M}_\infty} \]

\[\alpha \longmapsto [\varpi^2, \text{id}, \sigma^m \alpha], \quad \beta \longmapsto [\varpi^2, \text{id}, \sigma^m \beta], \quad \gamma \longmapsto [\varpi^2, \text{id}, \sigma^m \gamma].\]

(The notation is meant to evoke non-existent maps \(a^F, a_E, a_I\)).

**Lemma 3.5.1.** — For all \(m \geq 1\), the maps \(\sigma^m a^F, \sigma^m a_E, \sigma^m a_I\) are well-defined closed immersions of finite origin (Definition A.1.1). More precisely, for any \(N\), those maps are respectively the pullbacks of closed immersions
\[(3.5.2) \quad \sigma^m a^F: \varpi^2 \hookrightarrow \mathcal{M}_{N+m}, \quad \sigma^m a_E: \varpi^2 \hookrightarrow \mathcal{M}_{N+m}, \quad \sigma^m a_I: \varpi^2 \hookrightarrow I_{\mathcal{M}_{N+m}}.\]

For \(N = 0\), the maps (3.5.2) are simply the inclusion of the closed point \(k\).

**Proof.** — The well-definedness follows from [9, Lemma 3.7]. The rest of the statement follows, at least for \(a^F\) and \(a_E\), from Proposition 3.9 ibid. its part (2), in which one should take all the three extensions under consideration to be equal and \(\varphi = \text{id}, \tau = \text{id}\), shows that the maps are of finite origin; its part (1), in whose notation the map \(\sigma^m a\) is to be compared with the limit in \(n\) of \(s(\sigma^m \text{id}, \text{id})_n\), shows that this map is a closed immersion. The arguments of loc. cit. also apply to the (easier) case of \(a_I\). \(\square\)

**Proposition 3.5.2.** — For all \(m \geq 1\) and \(n \geq c(\varphi, \tau)\), the following diagrams are Cartesian:

\[
\begin{array}{ccc}
\varpi^2 & \xrightarrow{\sigma^m a^F} & \mathcal{M}_\infty \\
\downarrow \sigma^m a^F & & \downarrow \sigma^m a^F \\
\mathcal{M}_\infty & \xrightarrow{\mathcal{M}_\infty} & \mathcal{M}_\infty \\
\end{array}
\quad \begin{array}{ccc}
\varpi^2 & \xrightarrow{\sigma^m a_E} & \mathcal{M}_\infty \\
\downarrow \sigma^m a_E & & \downarrow \sigma^m a_E \\
\mathcal{M}_\infty & \xrightarrow{\mathcal{M}_\infty} & \mathcal{M}_\infty \\
\end{array}
\quad \begin{array}{ccc}
\varpi^2 & \xrightarrow{\sigma^m a_I} & I_{\mathcal{M}_\infty} \\
\downarrow \sigma^m a_I & & \downarrow \sigma^m a_I \\
I_{\mathcal{M}_\infty} & \xrightarrow{I_{\mathcal{M}_\infty}} & I_{\mathcal{M}_\infty} \\
\end{array}
\]
Proof. — We verify the statement for the second diagram. The proof for the first diagram is similar and also a special case of [9, Proposition 3.8 (2)]. It suffices to verify that the diagram is Cartesian on \(A\)-valued points functorially in objects \(A\) of \(\mathfrak{g}_F\). We write the proof for the functors in infinite level where the idea is clearest. The \(A\)-point \(\alpha_0: \mathcal{O}_F^{1,2} \to \mathfrak{g}_F\) of \(\mathfrak{g}_F^0\) is a point of the Cartesian product of the diagram (which is a closed subscheme of \(\mathfrak{g}_F^0\) as \(f_i\) is a closed immersion) if and only if \(\sigma^m \alpha_0 = \gamma \circ \lambda\) factors through \(\lambda\) and, obviously, through \(\sigma^m\). As \(\text{Ker}(\lambda)\) is divisible, this is equivalent to \(\gamma = \sigma^m \gamma_0\) for some \(\gamma_0: \mathcal{O}_F^{1,2} \to \mathfrak{g}_F\), that is \(\alpha_0 = f_i(\gamma_0)\) for \(\gamma_0 \in \sigma_{\mathfrak{g}_F}\). \(\square\)

**Corollary 3.5.3.** — We have

\[
\begin{align*}
\mathcal{O}_{\mathcal{A}_{\mathcal{N}}+m} & = \mathcal{O}_{\mathfrak{g}_F^0}, \\
\mathcal{O}_{\mathcal{A}_{\mathcal{N}}+m} & = \mathcal{O}_{\mathfrak{g}_F^0}
\end{align*}
\]

We wish to compute the intersection multiplicity in \(\mathcal{M}_U\)

\[
m([Z(\varphi, \tau)]_U, [V(\lambda)]_U).
\]

By the definitions, this equals

\[
m([Z(\varphi, \tau)]_U, [V(\lambda)]_U) = d_\varphi^{-1} d_{\varphi^m} \cdot m([Z(\varphi, \tau)]_U, [V(\lambda)]_U).
\]

**4. Intersection numbers**

We wish to compute the intersection multiplicity in \(\mathcal{M}_U\)

\[
m([Z(\varphi, \tau)]_U, [V(\lambda)]_U).
\]

For all \(N\), then

\[
\sigma^m \mathcal{O}_{\mathcal{A}_{\mathcal{N}}+m} \phi (\mathcal{N}_\infty) = \sigma^m \mathcal{O}_{\mathcal{A}_{\mathcal{N}}} \phi (\mathcal{N}_\infty)
\]

where the second equality follows from Proposition 3.5.2 and the pullback-pushforward formula (Proposition A.2.1), and the last one follows from (3.5.4). This proves the first identity and the same argument works for the second one. \(\square\)

**4.1. Comparison.** — In the next proposition, we compare intersection multiplicities in Lubin–Tate spaces and in formal groups.

Define for all levels \(U\) and integers \(N:\)

\[
\begin{align*}
m^\sigma([Z(\varphi, \tau)]_U, [V(\lambda)]_U) & := \text{vol}(U)^{-1} \cdot m([Z(\varphi, \tau)]_U, [V(\lambda)]_U) \\
m^\sigma([Z(\varphi, \tau)]_N, [V(\lambda)]_N) & := d_N^2 \cdot m([Z(\varphi, \tau)]_N, [V(\lambda)]_N),
\end{align*}
\]

where \(\text{vol}(U)\) is normalised by \(\text{vol}(\text{GL}_2(\mathcal{O}_F)) = 1\). We note that for all \(N \geq 1,\)

\[
\text{vol}(U_N) = \zeta_F(1) \zeta_F(2) q^{-N}.
\]

**Proposition 4.1.1.** — 1. The limits

\[
\begin{align*}
m^\sigma([Z(\varphi, \tau)]_U, [V(\lambda)]_U) & := \lim_{U \to 1} m^\sigma([Z(\varphi, \tau)]_U, [V(\lambda)]_U) \\
m^\sigma([Z(\varphi, \tau)]_N, [V(\lambda)]_N) & := \lim_{N \to \infty} m^\sigma([Z(\varphi, \tau)]_N, [V(\lambda)]_N)
\end{align*}
\]

exist as limits of eventually constant sequences.
2. We have the following identity relating intersection multiplicities in $\mathcal{M}_\infty$, $(T\mathfrak{g}_F)^2$, and $\mathfrak{g}_F^2$:

$$m^\ast([Z(\varphi, \tau)]_U^\delta_2,(V(\lambda))]_U^\delta_2) = \frac{e_{\mathfrak{g}_F}(1)}{e_{\mathfrak{g}_F}(2)} \cdot m^\ast([Z(\varphi, \tau)]_U^\delta_2,(V(\lambda))]_U^\delta_2) = \frac{e_{\mathfrak{g}_F}(1)}{e_{\mathfrak{g}_F}(2)} \cdot m^\ast([Z(\varphi, \tau)](U),[V(\lambda)])_U^\delta_2).$$

**Proof.** — To establish the entirety of the proposition, it is enough to show that for all sufficiently large $N$ and $M$,

$$m^\ast([Z(\varphi, \tau)]_N^\delta_2,M,(V(\lambda))]_N^\delta_2) = m^\ast([Z(\varphi, \tau)]_N^\delta_2,M,(V(\lambda))]_N^\delta_2) = m^\ast([Z(\varphi, \tau)]_N^\delta_2,M,(V(\lambda))]_N^\delta_2),$$

where for a special cycle $Z$ in $\mathcal{M}_\infty$ we have denoted $[Z]^\delta_2 := [Z]^\delta_2_U$. The second equality follows trivially from the definitions and Lemma 3.4.3. The third one is proved exactly in the same way as [9, (4.8)]. We recall Li’s idea: the intersection number in $\mathfrak{g}_F^2$, as the length of a module supported at the closed point of $\mathfrak{g}_F^2$, is the same as the intersection number of the restrictions to a sufficiently thick Artinian thickening $\mathfrak{g}_F^2[N] = \text{Spf} \frac{\mathfrak{g}_F^2}{m^N_{\mathfrak{g}_F}}$ (this equality is [9, (4.11)]) of the closed point; here “sufficiently thick” means that $q^{2N}$ should be greater than the intersection number.

We now prove the first equality based on the argument to prove [9, (4.6), (6.7)], only writing the details which are different from loc. cit. By (the proof of) Corollary 3.5.3, for all sufficiently large $N$ and $M$ we have

$$d_{E,M} d_{R,M} d_{N} \cdot m((\sigma^M a_F)[(Z(\varphi, \tau)]_N^\delta_2,M,(\sigma^M a_F)(V(\lambda))]_N^\delta_2,M) = d_{N} \cdot m([Z(\varphi, \tau)]_N^\delta_2,M,(V(\lambda))]_N^\delta_2,M)$$

as intersections in $([\mathfrak{g}_F^2]^N)$. Then it suffices to observe that

$$d_{E,M} d_{R,M} d_{N} = e_{\mathfrak{g}_F}(1)^{-1} e_{\mathfrak{g}_F}(2)^{-1} \cdot \text{vol}(U)^{-1},$$

and to show that

$$m^\ast([Z(\varphi, \tau)]_N^\delta_2,M,(\sigma^M a_F)(V(\lambda))]_N^\delta_2,M) = m^\ast([Z(\varphi, \tau)]_N^\delta_2,M,(V(\lambda))]_N^\delta_2,M).$$

The argument on restrictions to sufficiently thick Artinian thickenings sketched above for the first equality in (4.1.3) also applies to prove (4.1.4): by [9, §4.4], $\sigma^M a_F$ identifies $\mathfrak{g}_F^2[N]$ with $\text{Spf} \frac{\mathfrak{g}_F^2[N]}{m^N_{\mathfrak{g}_F}} \subset \mathcal{M}_N^\ast$. □

**Corollary 4.1.2.** — We have

$$m^\ast([Z(\varphi, \tau)]_U^\delta_2,(V(\lambda))]_U^\delta_2) = \frac{e_{\mathfrak{g}_F}(1)}{e_{\mathfrak{g}_F}(2)} \cdot \text{vol}(U)^{-1} \cdot \int_U m^\ast([Z(\varphi, \tau)](U),[V(\lambda)])_U^\delta_2 \cdot d g.$$  

**Proof.** — By the proposition, we may replace the integrand with $m^\ast([Z(\varphi, \tau)]_U^\delta_2,(V(\lambda))]_U^\delta_2)$ for sufficiently small $U$. Then the result follows from the projection formula and the observation that

$$\pi^\ast_{U'/U}[V(\lambda)]_U = \pi^\ast_{U'/U} \pi^\ast_{U'/U} [V(\lambda)]_U = \sum_{g \in U/U'} [V(\lambda)]_U.$$  

□

### 4.2. Computation.

We first write CM (resp. vertical) cycles in $\mathfrak{g}_F^2$ as rational multiples of cycles which are image (resp. kernel) of a map in $\text{Hom}_{\mathfrak{g}_F}(\mathfrak{g}_F, \mathfrak{g}_F^2) = \mathcal{O}_B^2$ viewed as column vectors (resp. $\text{Hom}_{\mathfrak{g}_F}(\mathfrak{g}_F^2, \mathfrak{g}_F) = \mathcal{O}_B^2$ viewed as row vectors).

**Notation.** — By the identification $\mathcal{O}_E = \mathfrak{g}_F$ as $\mathcal{O}_B$-modules, we have an embedding

$$\mathcal{O}_E = \text{End}_{\mathcal{O}_B}(\mathcal{O}_E) \hookrightarrow \mathcal{O}_B = \text{End}_{\mathcal{O}_B}(\mathcal{O}_E).$$  

(4.2.1)
Then both $M_2(E)$ and $B$ are embedded in

\[(4.2.2)\quad M_2(B) = \operatorname{Hom}_{\mathcal{O}_E}(\mathcal{G}_E^2, \mathcal{G}_E^2) \otimes F.\]

We denote by $\text{nrd}$ the reduced norm of $\mathcal{O}_E$. Let $\mathcal{O}_E$ be an Iwasawa decomposition in $\GL_2$. Proposition 4.2.2

Proof. — The first equality is a special case of \[9, \text{Lemma 5.3} \]. The second one is clear. 

We may now compute the intersection in $\mathcal{G}_F^2$. Proposition 4.2.2. — With notation as in (4.2.3), (4.2.4), we have

\[m([Z^2(\varphi, \tau)], [V(\lambda)]) = |\text{nrd}(\varphi P_\tau)|^{-1} \cdot |\text{Im}(\varphi \Gamma_\tau \varphi^{-1}(\theta_1, 0))|^{-1}.\]

Proof. — By the previous lemma, $m([Z^2(\varphi, \tau)], [V(\lambda)])$ equals $|\text{nrd}(\varphi P_\tau)|^{-1}$ times the degree of the group scheme kernel of

\[(4.2.5)\quad (a \ b) \circ \varphi \Gamma_\tau \varphi^{-1}(\tau_\tau) : \mathcal{G}_E \to \mathcal{G}_F,\]

that is, the degree of the isogeny (4.2.5). By \[9, \text{Lemma 5.2} \], this is

\[|\text{nrd}(a \ b) \varphi \Gamma_\tau \varphi^{-1}(\theta_1, 0)|^{-1} = |\text{nrd}(a \ b) \varphi \Gamma_\tau(\theta_1, 0)|^{-1}.\]

Thanks to the decompositions

\[(\delta \tau \ | \ \overline{\delta \tau}) := \begin{pmatrix} \theta_1 & \overline{\theta}_2 \\ \theta_2 & \overline{\theta}_1 \end{pmatrix} = \begin{pmatrix} 1 & \theta_2 \theta_1 \theta_1 \\ \theta_2 & \overline{\theta}_1 \theta_2 \theta_1 \end{pmatrix} = \begin{pmatrix} \theta_1 \theta_2 \theta_1 \theta_2 \\ \theta_2 & \overline{\theta}_1 \theta_2 \theta_1 \end{pmatrix}\]

we may compute the elements $\Gamma_\tau, P_\tau$ from (4.2.4). We only write down the details if $v(\theta_1) \leq v(\theta_2)$; then the first decomposition is an Iwasawa decomposition, and

\[m([Z^2(\varphi, \tau)], [V(\lambda)]) = |\text{nrd}(\varphi \theta_1)|^{-1} |\text{nrd}(a \ b) (\theta_1, 1) \theta_1, 0) |^{-1} = |\text{nrd}(\varphi P_\tau)$
Inserting this result in Corollary 4.1.2 and removing the constants indicated by the superscripts ‘’, we obtain the following local analogue of Theorem B.

**Corollary 4.2.3.** — We have

\[ m([Z(\varphi, \tau)]_U, [V(\lambda)]_U) = \frac{\zeta_F(1)}{e\zeta_F'(2)} \cdot d_{\varphi^{-1}(U)} \cdot |\text{nrd}(\varphi)|^{-1} \cdot \int_U |\lambda| g \delta \bar{\tau}_{\lambda}^{-1} d g, \]

where \( d g \) is the Haar measure such that \( \text{vol}(GL_2(\mathcal{O}_F)) = 1 \).

It is a not unpleasant exercise to verify that, for \( U = GL_2(\mathcal{O}_F) \), this result agrees with the one given by the theory of quasicanonical liftings.

**4.3. Conclusion.** — We are now in position to complete the proof of the main theorems.

**Corollary 4.3.1.** — Let \( V \neq V' \) be irreducible components of \( \mathcal{M}_{U,k} \). The intersection multiplicity

\[ m(Z, V') \]

is bounded as \( Z \) varies among CM curves with generic fibre a point of \( \mathcal{S}_V \).

**Proof.** — Let us write \( V = V(\lambda)_U \), \( V' = V(\lambda')_U \). Suppose that \( z = z(\varphi, \tau) \in \mathcal{S}_V(\lambda)_U \) is a CM point of conductor \( s \) and write \( \delta \tau = (\theta_1, \theta_2) \) as in (4.2.3). Up to changing the choice of representatives \( \varphi \) and \( \tau \), we may assume that

\[ (4.3.1) \quad \mathcal{O}^2_{\tau} \subset \tau(\mathcal{O}_E) \subset \mathcal{O}^{-1} \mathcal{O}^2_{\tau} \]

optimally (that is, \( \mathcal{O}^{-1} \mathcal{O}^2_{\tau} \not\subset \tau(\mathcal{O}_E) \not\subset \mathcal{O}^{-1} \mathcal{O}^2_{\tau} \)), and \( \text{ht}(\varphi) = s \). We examine the terms of the multiplicity formula of Corollary 4.2.3.

We have \( |\text{nrd}(\varphi)|^{-1} = q^i \) and, as

\[ d_{\varphi^{-1}(U)} = e[\mathcal{O}^2_{\tau} : \varphi^{-1}(U)] = e\zeta_F(1)\zeta_F'(1)^{-1} \cdot q^i, \]

the integer \( d_{\varphi^{-1}(U)} \) is also bounded above (and below) by positive multiples of \( q^i \). Thus it suffices to show that the integrand \( |\lambda| g \delta \bar{\tau}_{\lambda}^{-1} \) is bounded by a positive multiple of \( q^{-2i} = |\mathcal{O}^{-1} \mathcal{O}^2_{\tau}|^{-1} \); more precisely, if \( n \geq 0 \) is such that \( U \subset U_n \), we claim that

\[ |\lambda| g \delta \bar{\tau}_{\lambda}^{-1} < q^{2n-2i}. \]

Let \( v \) be the valuation on \( E \) normalised by \( v(\sigma) = 1 \) (thus \( v \) may take half-integer values if \( E/F \) is ramified). We will prove the result by contradiction, showing that

\[ (4.3.2) \quad v(\lambda' g \delta \bar{\tau}) \geq n-s \quad \Rightarrow \quad z(\varphi, \tau) \in \mathcal{S}_V(\lambda')_U. \]

In fact \( v(\lambda' g \delta \bar{\tau}) \geq n-s \) if and only if \( v(\lambda' g \delta \bar{\tau}) \geq n-s \) for all \( t \in \mathcal{O}^{-1} \), that is if and only if \( \lambda' \) takes integral values on \( \text{Im}(\mathcal{O}^{-1} \delta \bar{\tau}) = \text{Im}(\mathcal{O}^{-i} \delta \bar{\tau}) \subset \mathcal{O}^{-1} \mathcal{O}^2_{\tau} / \mathcal{O}^2_{\tau} \). Equivalently, the first among the lines

\[ L_n(\tau) = \text{Im}(\mathcal{O}^{-i} \delta \bar{\tau}) / \mathcal{O}^2_{\tau}, \quad L_n(\lambda') = \text{Ker}((\mathcal{O}^{-i} \delta \bar{\tau} / \mathcal{O}^2_{\tau})^{\lambda'} / \mathcal{O}^{-i} \delta \bar{\tau} / \mathcal{O}^2_{\tau}) \subset (\mathcal{O}^{-i} \mathcal{O}^2_{\tau} / \mathcal{O}^2_{\tau}) \]

is contained in the second one, hence the two coincide. By definition, this means that \( z(\varphi, \tau) \in \mathcal{S}_V(\lambda')_U \).

**Proof of Theorems A and B.** — By Lemma 3.3.3, Corollary 4.2.3 implies Theorem B, and Corollary 4.3.1 implies Proposition 2.2.1.2. The conditional proof of Theorem A given at the end of §2 is now complete. 

\( \square \)
Appendix A. Cycles in formal schemes and their projective limits

A.1. Pro-Noetherian formal schemes. — Consider the category $\mathcal{F}$ of separated Noetherian finite-dimensional formal schemes and finite flat maps. We let $\mathcal{F}'$ be the category of pro-object of $\mathcal{F}$; that is, objects of $\mathcal{F}'$ are formal inverse limits

$$X = \lim_{i \in I} X_i$$

of filtered inverse systems of objects and maps in $\mathcal{F}$, and morphisms are defined by

$$(A.1.1) \quad \text{Hom}(X, Y) = \lim_{j} \lim_{i} \text{Hom}(X_i, Y_j).$$

Henceforth we will simply refer to objects of $\mathcal{F}$ (respectively $\mathcal{F}'$) as Noetherian formal schemes (respectively pro-Noetherian formal schemes) for short.

Definition A.1.1. — We say that a morphism $f : X \to Y$ in $\mathcal{F}'$ is

- of finite origin if there are indices $i \in I$, $j \in J$ and a morphism $f_{ij} : X_i \to Y_j$ such that for all $j' > j$ and sufficiently (depending on $j'$) large $i'$, the map $f_{ij'}$ factors through the Cartesian product

$$X_i \times_{Y_j} Y_{j'} \xrightarrow{\pi_i} X_i \xrightarrow{\pi_j} Y_{j'}.$$

In this case we say that $f$ originates from $f_{ij}$.

- a closed immersion of finite origin if it is of finite origin and it originates from a closed immersion in $\mathcal{F}$.

The image of a morphism $\lim f_{ij} : X \to Y$ as in (A.1.1) is the object $\lim f_j(X)$ with respect to the restrictions of the transition maps of $(Y_i)$, where $f_j(X) := f_{ij}(X_i)$ for any sufficiently large $i$.

If $f : X \to Z$, $g : Y \to Z$ are morphisms in $\mathcal{F}'$ with $g$ of finite origin, the Cartesian product $X \times_Z Y$ is (well-)defined as follows. Write $X = \lim_{i} X_i$, $Y = \lim_{j} Y_j$, $Z = \lim_{k} Z_k$, and assume that $g$ originates from $g_{ik} : Y_j \to Z_k$. Then

$$X \times_Z Y := \lim_{i} X_i \times_{Z_k} Y_j.$$

Remark A.1.2. — All the inverse limits under consideration in this paper exist in the category of formal schemes, but we will not need to use this fact.

A.2. Cycles, $K$-groups, intersections. — Let $X$ be a Noetherian formal scheme. We define the $K$-group of $X$ with $\mathbb{Q}$-coefficients

$$K(X)_{\mathbb{Q}}$$

in the usual way as the Grothendieck group (tensored with $\mathbb{Q}$) of the category of coherent sheaves on $X$. If $X = \lim_{i} X_i$ is a pro-Noetherian formal scheme, we define

$$K_\ast(X) := \lim_{i} K_\ast(X_i)_{\mathbb{Q}}$$

where the limits are with respect to pushforward.

It is easy to verify that $K_\ast(-)_{\mathbb{Q}}$ enjoys pushforward (respectively, pullback) functoriality with respect to arbitrary (respectively, finite-origin) morphisms in $\mathcal{F}'$.

Fundamental classes. — Let $X$ be a Noetherian formal scheme. Its fundamental class is

$$[X] := [\mathcal{O}_X] \in K(X)_{\mathbb{Q}}.$$
Let $X$ be a pro-Noetherian formal scheme with a morphism to a Noetherian formal scheme $B$; equivalently, for all sufficiently large $i$ we have a finite flat map $X_i \to B$ compatibly in the system. We define the normalised (relative to $B$) fundamental class of $X$ to be

$$[X]^\circ := \lim[ X_i]^\circ \in K_*(X)_Q, \quad [X_i]^\circ := \frac{1}{\deg(X_i/B)}[X_i] \in K_*(X_i)_Q.$$  

**Pullback and pushforward.** — The proof of the following proposition, generalising the usual pullback-pushforward formula for cycles, presents no difficulty and is left to reader.

**Proposition A.2.1.** — Let $f : X \to Z$, $g : Y \to Z$ be morphisms in $\mathcal{F}$ with $g$ of finite origin, and consider the Cartesian diagram

$$
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{f'} & Y \\
\downarrow g' & & \downarrow g \\
X & \xrightarrow{f} & Z.
\end{array}
$$

Let $C \in K_*(X)_Q$. Then

$$g^*f_*C = f'_*g'^*C.$$

**Intersection multiplicities.** — If $X$ is a Noetherian formal scheme and $x \in X$ is a regular point, the intersection multiplicity function

$$m_x : K(X)_Q \times K(X)_Q \to \mathbb{Q}$$

is defined via Serre’s Tor formula. It is a symmetric bilinear form restricting to the usual intersection multiplicity

$$m_\ast([Z_1],[Z_2]) = \dim \mathcal{O}_{X,x}/(\mathcal{I}_{Z_1,x} + \mathcal{I}_{Z_2,x})$$

whenever $X$ is 2-dimensional, $[Z_i]$ (for $i = 1, 2$) is the pushforward to $X$ of the fundamental class of the 1-dimensional irreducible subscheme defined by the ideal sheaf $\mathcal{I}_{Z_i,x}$, and the intersection $Z_1 \cap Z_2$ is proper at $x$. The subscript $x$ is omitted when $|X|$ consists of a single point.

**References**


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