p-ADIC EQUIDISTRIBUTION OF CM POINTS

by

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Abstract. — Let X be a modular curve and consider a sequence of Galois orbits of CM points in X, whose p-conductors tend to infinity. Its equidistribution properties in $X(\mathbf{C})$ and in the reductions of X modulo primes different from p are well understood. We study the equidistribution problem in the Berkovich analytification X_p^{an} of $X_{\mathbf{Q}_p}$.

We partition the set of CM points of sufficiently high conductor in $X_{\mathbf{Q}_p}$ into finitely many explicit *basins* B_V, indexed by the irreducible components V of the mod-p reduction of the canonical model of X. We prove that a sequence z_n of local Galois orbits of CM points with p-conductor going to infinity has a limit in X_p^{an} if and only if it is eventually supported in a single basin B_V. If so, the limit is the unique point of X_p^{an} whose mod-p reduction is the generic point of V.

The result is proved in the more general setting of Shimura curves over totally real fields. The proof combines Gross's theory of quasicanonical liftings with a new formula for the intersection numbers of CM curves and vertical components in a Lubin–Tate space.

Contents

1. Introduction and statements of the main results

The equidistribution properties of CM points have been studied from various points of view, with remarkable applications ranging from cases of the André–Oort conjecture to the non-triviality of Heegner points. The main results until recently have been archimedean or ℓ-adic.

To illustrate the situation, let $X_{/\mathbf{Q}}$ be a geometrically connected modular curve, and let

$$(z_n)_{n\in\mathbf{N}}$$

be a sequence of CM points on X with p-part of the conductor going to infinity. We view the z_n as scheme theoretic points of X, or equivalently as Galois orbits of geometric points.

Duke [Duk88], Clozel–Ullmo [CU04], Zhang [Zha05] and others considered the images of the z_n in $X(\mathbf{C})$, and proved the following archimedean equidistribution result: the sequence μ_n of averaged delta measures at the orbits z_n converges to the hyperbolic probability measure on $X(\mathbf{C})$. Cornut and Vatsal

This work was supported by BSF grant 2018250 and ISF grant 1963/20.

[CV05] considered the reduction modulo ℓ of the z_n, when $\ell \neq p$ is a prime nonsplit in the fields of complex multiplication (or more generally simultaneous reductions at a finite set of such primes); they proved that the images of the orbits z_n equidistribute to the counting probability measure on the finite set $\mathscr{X}^{\rm ss}_{\overline{\mathbf{F}}_\ell}$ of supersingular points. Very recently, Herrero–Menares–Rivera-Letelier [HMRL] described the set of accumulation measures of the μ_n on the Berkovich analytic curve attached to the base-change $X_{\mathbf{Q}_\ell}$.

The purpose of this paper is to prove a p-adic equidistribution result for sequences of CM points. Consider the Berkovich analytic curve $X_p^{\rm an}$ attached to $X_{\mathbf{Q}_p}$. It is a compact Hausdorff topological space, containing the set of closed points of $X_{\mathbf{Q}_p}$ and equipped with a reduction map to the special fibre $|\mathscr{X}_{\mathbf{F}_p}|$ of any model $\mathscr{X}/\mathbf{Z}_p$ of $X_{\mathbf{Q}_p}$. The generic point of any irreducible component V of $|\mathscr{X}_{\mathbf{F}_p}|$ is the reduction of a unique point $\zeta_V \in X_p^{\rm an}$, called the *Shilov point* of V. The study of our problem leads naturally to consider the canonical model defined by Katz–Mazur [KM85], whose special fibre is topologically a union of finitely many irreducible curves intersecting at the supersingular points.

Our main result is best described in terms of sequences (z_n) of CM points of $X_{\mathbf{Q}_p}$, equivalently local Galois orbits of geometric CM points. We explicitly partition the set of such CM points of sufficiently large p-adic conductor into finitely many *basins* B_V indexed by the irreducible components of the special fibre of $\mathscr{X}_{\mathbf{F}_p}$. Our main theorem is that the sequence of measures $\mu_n := \delta_{z_n}$ on $X_p^{\rm an}$ has a limit if and only if the sequence (z_n) is eventually supported in a single basin B_V, in which case the μ_n converge to the delta measure at ζ_V. Equivalently, the sequence (z_n) converges to ζ_V in the plain topological sense – so that we will only use the language of topology in what follows.

The theorem is proved in the more general context of Shimura curves over totally real fields. The rest of this introduction is dedicated to explaining its statement and the idea of proof, as well as the intersection formula that lies at its core.

For the modular curve of level 1, which has a single Shilov point, the equidistribution result was previously established in [HMRL20], as briefly discussed after Theorem A below.

1.1. CM points on Shimura curves and their integral models. —

Let F be a totally real field. Let D be a quaternion algebra over F whose ramification set Σ contains all the infinite places but one, which we denote by $\sigma\colon F \hookrightarrow \mathbf{R}$. We may attach to D a tower of Shimura curves X_U/F, where U runs over compact open subgroups of $D^\times_{\mathbf{A}^\infty} := (D\otimes_{\mathbf{Q}} \mathbf{A}^\infty)^\times$ (here $\mathbf{A}^\infty$ denotes the finite adèles of $\mathbf{Q}$, whereas $\mathbf{A}$ will denote the adèles). This curve and its canonical integral model were studied by Carayol [Car86], to which we refer for more details (see also [Zha01, §5] for a discussion of CM points).

The points of X_U over $\mathbf{C} \overset{\sigma}{\hookleftarrow} F$ can be identified with

$$X_U(\mathbf{C}) \cong D^\times \backslash \mathfrak{h}^\pm \times D^\times_{\mathbf{A}^\infty}/U \cup \{\text{cusps}\} \tag{1.1.1}$$

where $\mathfrak{h}^\pm \cong \mathbf{C}-\mathbf{R}$ is viewed as a quotient of $D^\times_\sigma \cong \mathrm{GL}_2(\mathbf{R})$, and the finite set '{cusps}' is non-empty only if $D = M_2(\mathbf{Q})$ (it plays no role in this work).

From now on we fix an arbitrary level $U \subset D^\times_{\mathbf{A}^\infty}$, assumed to be sufficiently small so that the $D^\times$-action in (1.1.1) has no nontrivial stabilisers. We let $X := X_U$.

CM points. — Let E be a CM quadratic extension of F, such that each finite $v \in \Sigma$ is nonsplit in E. Then the set of $F_{\mathbf{A}}$-algebra embeddings $\psi\colon E_{\mathbf{A}} \hookrightarrow D_{\mathbf{A}}$ is non-empty. For any such ψ, the group $\psi(E^\times) \subset D^\times_{\mathbf{A}^\infty}$ acts on the right on X. The fixed-point subscheme $X^{\psi(E^\times)}$, called the scheme of points with CM by (E,ψ), is isomorphic (as F-subscheme) to $\operatorname{Spec} E_{[\psi^{-1}(U)]}$ where E_C denotes the abelian extension of E with Galois group $C \subset E^\times \backslash E^\times_{\mathbf{A}^\infty}$. (This follows from the theory of complex multiplication as described in [Zha01, § 5.2].) The *CM (ind)-subscheme* of X_U is the union

$$X^{\rm CM} = \bigcup_{(E,\psi)} X^{\psi(E^\times)}.$$

Local integral models and their irreducible components. — Fix a finite place v of F not in Σ and an identification $D_v = M_2(F_v)$, and assume $U = U^v U_v$ with $U_v \subset D_v^\times$. Let $\mathscr{O}_{F_v} \subset F_v$ be the ring of integers, ϖ_v a uniformiser, $\mathbf{F}_v$ the residue field.

Carayol [Car86], generalising [KM85], defined a canonical integral model $\mathscr{X} = \mathscr{X}_U$ of X over $\mathscr{O}_{F_v}$. It carries a sheaf $\mathscr{G}$ of divisible $\mathscr{O}_{F_v}$-modules of rank 2 together with, if $U_v = U_{n,v} := \mathrm{Ker}(\mathrm{GL}_2(\mathscr{O}_{F_v}) \to \mathrm{GL}_2(\mathscr{O}_{F_v}/\varpi_v^n))$, a Drinfeld level structure(1)

$$\alpha\colon (\varpi_v^{-n}\mathscr{O}_{F_v}/\mathscr{O}_{F_v})^2 \to \mathscr{G}(\mathscr{X}).$$

In general, $\mathscr{X} = \mathscr{X}_U := \mathscr{X}_{U^v U_{n,v}}/(U_v/U_{n,v})$ for any $U_{n,v} \subset U_v$.

The special fibre $\mathscr{X}_{\mathbf{F}_v}$ (see [Car86, §9.4]) is a union of connected components permuted simply transitively by an action of $F^\times \backslash F^\times_{\mathbf{A}^\infty}/\mathscr{O}_{F_v}^\times \det U^v$. The (*supersingular*) locus $\mathscr{X}^{\mathrm{ss}}_{\mathbf{F}_v}$ where $\mathscr{G}$ is connected is finite; its complement $\mathscr{X}^{\mathrm{ord}}_{\mathbf{F}_v} := \mathscr{X}_{\mathbf{F}_v} - \mathscr{X}^{\mathrm{ss}}_{\mathbf{F}_v}$ (*ordinary* locus) is smooth. Each connected component $C \subset \mathscr{X}_{\mathbf{F}_v}$ is topologically a union of irreducible components, each intersecting all of the others in each of the finitely many supersingular points.

The irreducible components V within a given connected component C are canonically parametrised by(2)

$$U_v \backslash \mathbf{P}^1(F_v),$$

where we view $\mathbf{P}^1(F_v)$ as the space of $\mathscr{O}_{F_v}^\times$-classes of injections $\lambda\colon \mathscr{O}_{F_v} \to \mathscr{O}_{F_v}^2$ with saturated image; for any $\mathscr{O}_{F_v}$-module M, we will still denote by $\lambda\colon M \to M^2$ the induced injection. The parametrisation is simple to describe. Let

$$(-)^* := \mathrm{Hom}_{\mathscr{O}_{F_v}}(-, F_v/\mathscr{O}_{F_v}) = \varinjlim_n (-)^*_n, \qquad (-)^*_n := \mathrm{Hom}_{\mathscr{O}_{F_v}}(-, \varpi_v^{-n}\mathscr{O}_{F_v}/\mathscr{O}_{F_v}), \tag{1.1.2}$$

be the Pontryagin duality functors. Throughout this paper, we identify an irreducible component of a scheme (which by definition is an irreducible component of the underlying topological space) with the corresponding subscheme with the reduced structure. Then the component $V(\lambda)^{(C)}_U \subset C$ is the locus where the Drinfeld level structure α factors through any quotient

$$\lambda_n^*\colon (\varpi_v^{-n}\mathscr{O}_{F_v}/\mathscr{O}_{F_v})^2 \to \varpi_v^{-n}\mathscr{O}_{F_v}/\mathscr{O}_{F_v}$$

in the $\mathscr{O}_{F_v}^\times \times U_v/U_{n,v}$-class determined by λ_n^*.

CM points, their reductions, and basins. — We consider the base-change (ind-)scheme $X^{\mathrm{CM}}_{F_v} \subset X_{F_v}$ and refer to its points as the CM points in X_{F_v}. It is well known that if $z \in X^{\mathrm{CM}}_{F_v}$ is a point with CM by (E, ψ), its reduction is a supersingular point if and only if v is nonsplit in E. Assume that this is the case. The ring $\mathscr{O}_z := \mathrm{End}_{\mathscr{O}_F\text{-Mod}}(\mathscr{G}_z)$ is the $\mathscr{O}_{F_v}$-order in E_v with unit group $\psi_v^{-1}(U_{0,v})$. Such orders are classified: there is a unique integer $s \geq 0$, called the (v-)*conductor* of z, such that

$$\mathscr{O}_z = \mathscr{O}_{E_v}[s] := \mathscr{O}_{F_v} + \varpi_v^s \mathscr{O}_{E_v}.$$

By [Gro86, § 5], if z has v-conductor s there is an $\mathscr{O}^*_{E_v,s}$-linear isomorphism, unique up to $\mathscr{O}_{E_v}[s]^\times$,

$$\beta\colon \mathscr{O}_{E_v}[s]^* \to \mathscr{G}_z(\overline{F}_v), \tag{1.1.3}$$

where $(-)^* = (1.1.2)$.

1.2. Partition into basins and the main result. — Let $z \in X^{\mathrm{CM}}_{F_v}$ be a point with CM by E of v-conductor s, keep the assumption that E_v is nonsplit, and let β be an E-level structure on $\mathscr{G}_z$ as in (1.1.3). Let $\alpha\colon (F_v/\mathscr{O}_v)^2 \to \mathscr{G}_z(\overline{F_v})$ be any isomorphism extending the level structure α_z on $\mathscr{G}_z$, let β be as in

(1)This notion is recalled in §3.2 below. When $n = 0$, there is no level structure and the integral model was already constructed in [Mor81].

(2)In [Car86, §9.4], the discussion is over the algebraic closure $\overline{\mathbf{F}}_v$ (and the parameter is dual to our λ). However, as is clear from the defining condition, the irreducible components within each ($\mathbf{F}_v$-)connected component are already defined over $\mathbf{F}_v$.

(1.1.3), and let

$$\tau^* := \beta^{-1}\circ\alpha\colon (F_v/\mathscr{O}_{F_v})^2 \to \mathscr{O}_{E_v}[s]^*. \tag{1.2.1}$$

Let $\tau\colon E\to F^2$ be the F-linear extension of the Pontryagin-dual map to τ^*. The class

$$L(\tau) := U_v.L_s(\tau), \qquad \text{where} \quad L_s(\tau) := \tau(\mathscr{O}_{E_v}/\mathscr{O}_{E_v[s]}) \subset (\varpi_v^{-s}\mathscr{O}_{F_v}/\mathscr{O}_{F_v})^2,$$

is a well-defined U_v-orbit of lines in $(\varpi_v^{-s}\mathscr{O}_{F_v}/\mathscr{O}_{F_v})^2$.

If $U_v \supset U_{s,v}$, we may compare this with a corresponding invariant of an irreducible component $V(\lambda)_U^{(C)}$ of $\mathscr{X}_{U,\mathbf{F}_v}$, defined as

$$L(\lambda) := U_v L_s(\lambda), \qquad \text{where} \quad L_s(\lambda) := \lambda(\varpi_v^{-s}\mathscr{O}_{F_v}/\mathscr{O}_{F_v}) \subset (\varpi_v^{-s}\mathscr{O}_{F_v}/\mathscr{O}_{F_v})^2.$$

Definition. — Let $V = V(\lambda)_U^{(C)} \subset \mathscr{X}_{U,\mathbf{F}_v}$ be an irreducible component. The *basin*

$$B_V$$

of V is the set of CM points $z\in X_{F_v}^{\mathrm{CM}}$ such that either

- z has CM by E with E_v/F_v split, and the reduction of z belongs to $V\cap \mathscr{X}_{\mathbf{F}_v}^{\mathrm{ord}}$, or
- z has CM by E with E_v/F_v nonsplit, the conductor s of z is such that $U_{s,v}\subset U_v$, the reduction of z belongs to the connected component C, and with notation as above

$$L(\tau) = L(\lambda). \tag{1.2.2}$$

A geometric description of basins is given in Proposition 2.2.6.

The main theorem. — Let X_v^{an} be the Berkovich analytic space attached to X_{F_v}.

Theorem A. — *Let (z_n) be a sequence of points in $X_{F_v}^{\mathrm{CM}}$, and denote by the same name the image sequence in X_v^{an}. Assume that the v-conductor of z_n tends to infinity.*

The sequence (z_n) has a limit if and only if it is eventually supported in a single basin. If this is the case for the basin B_V of the irreducible component $V\subset \mathscr{X}_{\mathbf{F}_v}$, then

$$\lim_{n\to\infty} z_n = \zeta_V$$

in X_v^{an}.

It is perhaps remarkable that, while the setup of the theorem only involves the Berkovich space X_v^{an}, the geometry of the canonical integral model emerges to play an essential role in the conclusion.

Theorem A for the modular curve $X_0(1)$ was also independently, but earlier, obtained by Herrero–Menares–Rivera-Letelier [HMRL20, Theorem A]; see also [Ric18].(3) From the point of view of the present paper (see the next subsection), for trivial level at v the result is a relatively straightforward consequence of the theory of [Gro86], whereas more substantial work is needed to discern among the different Shilov points present in the case of higher level.

1.3. Idea of proof, intersection formula, and organisation of the paper. — To illustrate the proof of Theorem A, consider first a closed point $a\in|\mathscr{X}_{\mathbf{F}_v}|$ which is nonsingular in $\mathscr{X}_{\mathbf{F}_v}^{\mathrm{red}}$: this is the case if the v-level of X is minimal ($U_v\cong \mathrm{GL}_2(\mathscr{O}_{F_v})$) or a is ordinary. Then $\mathrm{red}^{-1}(a)\subset X_v^{\mathrm{an}}$ is a (twisted) analytic open unit disc, say with coordinate u, and Gross's theory of quasicanonical liftings [Gro86] shows that CM points of v-conductor s inside it lie in a circle $|u| = 1-\varepsilon_s$ with explicit $\varepsilon_s\to 0$ as $s\to\infty$. This immediately implies that any limit point of (z_n) is one of the points ζ_V. When the v-level of X is minimal, irreducible and connected components coincide and this argument is enough to conclude.

(3)The author conceived Theorem A and the argument described in § 1.3, in slightly different language, in the summer of 2018; soon thereafter he became aware of the 2016 thesis of S. Herrero, where [HMRL20, Theorem A] was first proved.

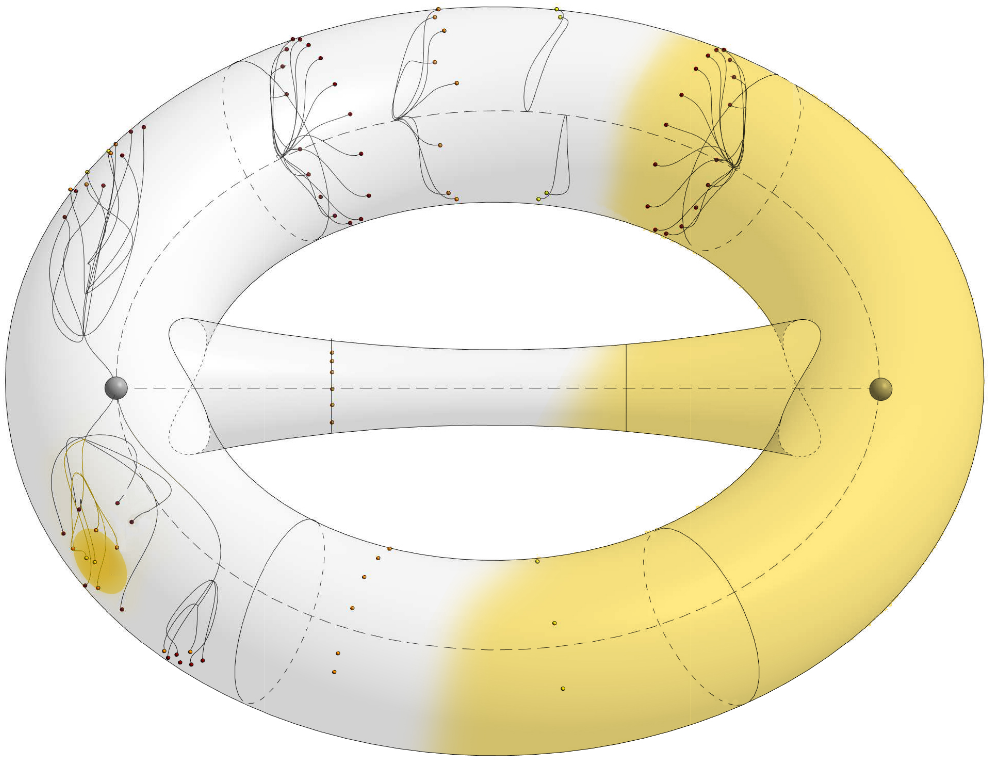

FIGURE 1. An illustration of Theorem A, in the case where X has genus 2, and $\mathscr{X}_{\mathbf{F}_v}$ consists of two rational irreducible components, intersecting in three supersingular points. The dual graph of $\mathscr{X}_{\mathbf{F}_v}$, which canonically embeds in X_v^{an} as a deformation retract [Ber90, §4], is drawn as the dashed lines in the interior of a 2-torus; the two thick points are the Shilov points of the components. The type-1 points of X_v^{an} are thought of as lying on the surface of the 2-torus (those with supersingular reduction are in the central part of the figure). In yellow, the complement of a typical open neighbourhood of the left Shilov point (cf. Proposition 2.1.4). In shades from yellow (lower conductor) to dark red (higher conductor), some CM points of X_{F_v}, drawn as Galois orbits of geometric points.

The above description of the distribution of CM points in nonsingular residue discs also implies the following: for each closed point $a \in |\mathscr{X}_{\mathbf{F}_v}|$, the intersection multiplicity at a between $\mathscr{X}_{\mathbf{F}_v}$ and the Zariski closure Z_n of those points z_n of the sequence that reduce to a tends to infinity with n. As the intersection multiplicity between different irreducible components is bounded, this means that for large n, exactly one of the irreducible components of $\mathscr{X}_{\mathbf{F}_v}$ intersects Z_n with high multiplicity – equivalently, the corresponding Shilov point is close to z_n. The argument described so far is developed in § 2.

We thus need to compute the intersection multiplicity at $a \in |\mathscr{X}_{\mathbf{F}_v}|$ between irreducible components of $\mathscr{X}_{\mathbf{F}_v}$ and Zariski closures of CM points, the nontrivial case being when a is supersingular. Hence the rest of the paper is dedicated to proving a new formula for this multiplicity.

We may state the formula using the following notation:

- we let $q := |\mathbf{F}_v|$, and denote by $|\cdot|_{E_v}$ the absolute value on E_v normalised by $|\varpi_v| = q^{-2}$;
- we choose any $\mathscr{O}_{E_v}$-generator $\delta^{-1} \in \mathscr{D}_{E_v}^{-1} := \mathrm{Hom}_{\mathscr{O}_{F_v}}(\mathscr{O}_{E_v}, \mathscr{O}_{F_v})$, and for $\tau \in \mathrm{Hom}_{F_v}(E_v, F_v^2)$ we view the element $\tau\delta \in \mathrm{Hom}_{\mathscr{O}_{F_v}}(\mathscr{O}_{E_v}, \mathscr{O}_{F_v}^2) \otimes_{\mathscr{O}_{E_v}} \mathscr{D}_{E_v} \otimes_{\mathscr{O}_{F_v}} F_v = E_v^2$ as a column vector over E_v;
- for $\lambda\colon \mathscr{O}_{F_v} \hookrightarrow \mathscr{O}_{F_v}^2$, we denote by $\lambda^\perp\colon \mathscr{O}_{F_v}^2 \twoheadrightarrow \mathscr{O}_{F_v}$ any surjection such that $\mathrm{Ker}(\lambda^\perp) = \mathrm{Im}(\lambda)$; we view $\lambda^\perp$ as a row vector in F_v^2;

– if $\mathscr{X}'$ is any 2-dimensional Noetherian formal scheme, and $x \in \mathscr{X}'$ is a regular closed point, we define the intersection multiplicity at x of two irreducible subschemes Z_1, Z_2 intersecting properly (that is, such that the quantity below is finite) by

$$m_x(Z_1, Z_2) = \mathrm{length}_{\mathscr{O}_{\mathscr{X}',x}} \mathscr{O}_{\mathscr{X}',x}/(\mathscr{I}_{Z_1,x} + \mathscr{I}_{Z_2,x}) \tag{1.3.1}$$

where $\mathscr{I}_{Z_i} \subset \mathscr{O}_{\mathscr{X}'}$ is the ideal sheaf defining Z_i.

Theorem B. — *Let $z \in X^{\mathrm{CM}}_{F_v}$ be a point with CM by (E, ψ) of v-conductor s, such that E_v/F_v is nonsplit. Let Z be the Zariski closure of z in $\mathscr{X}$, and let τ^* be attached to z as in* (1.2.1). *Let $V = V(\lambda)^{(C)}_U$ be an irreducible component through the reduction $\overline{z} \in \mathscr{X}_{\mathbf{F}_v}$ of z (endowed with the reduced structure).*

Then

$$\frac{m_{\overline{z}}(Z, V)}{[\mathscr{O}^\times_{E_v} : \psi_v^{-1}(U_v)] \cdot [\mathscr{O}^\times_{F_v} : U_v \cap \mathscr{O}^\times_{F_v}]} = [\mathbf{F}_v(\overline{z}) : \mathbf{F}_v] \cdot c_{E_v} \cdot q^s \cdot \int_{U_v} |\lambda^\perp g \tau \delta|_E^{-1} \, dg.$$

Here, the constant $c_{E_v} = 1$ if E_v/F_v is unramified and $c_{E_v} = 1 + q^{-1}$ if E_v/F_v is ramified, and dg is the Haar measure such that $\mathrm{vol}(\mathrm{GL}_2(\mathscr{O}_{F_v})) = 1$.

If we call *special cycles* in $\mathscr{X}$ those that are combinations of vertical components and Zariski closures of CM points, then our formula completes the calculations of intersections of special cycles in $\mathscr{X}$, where the case of intersections of cycles of the same type was treated respectively by Katz and Mazur ([KM85, § 13.8], for vertical cycles) and, very recently, Qirui Li ([Li], for CM cycles). It will be clear that our method also allows to recover (and find the most general form of) the Katz–Mazur intersection formula for vertical cycles.

The intersection problem in $\mathscr{X}$ is equivalent to one in a Lubin–Tate space $\mathscr{M}_{U_v}$ of deformations of the unique formal $\mathscr{O}_{F_v}$-module $\mathscr{G}_F$ of height 2 over $\overline{\mathbf{F}}_v$. We solve it by adapting the beautiful method devised by Li [Li] (and in turn inspired by a work of Weinstein [Wei16]): after passing to infinite level in the Lubin–Tate tower, our intersection problem can be compared to an easier intersection problem in the formal group $\mathscr{G}_F^2$.

The cycles in Lubin–Tate towers and formal groups are defined in §3. The computation of intersections and the completion of the proofs is in §4.

Related work, applications, and future directions. — A corollary of Theorem A is the following. Assume that the sequence (z_n) is convergent in X_v^{an}, and let ζ_V be its limit. Then the intersection multiplicity in $\mathscr{X}_v$ between a given horizontal divisor H and the Zariski closures of z_n is eventually equal to $m(H, V)$. A weak generalisation of this result is the key new ingredient in the recent proof of the p-adic Gross–Zagier formula at nonsplit primes [Dis], which was our main source of motivation.

Problems of equidistribution in Berkovich spaces have been considered in the context of arithmetic dynamics and Arakelov theory (see for example [CL06] and its citation orbit). In fact, sequences of CM points of increasing p-conductor may be obtained from the dynamics of some Hecke operator at p, for example the operator U_p on a modular curve of Iwahori level at p (more precisely, one should consider the forward-orbit of a given CM point). As far as we know the phenomenon of multiple, but finitely many, limit measures is exposed here for the first time, and it is a natural question to try and understand it within a more general framework.

On the other hand, the p-adic dynamics of *prime-to-p* Hecke operators on CM points, and the distribution of CM points whose p-conductors stay bounded, have been recently been studied by Goren–Kassaei [GK21] and in the aforementioned [HMRL], the latter work having some new arithmetic applications; it would be interesting to extend those results to our context.

Finally, it would also be very interesting to extend the intersection formula of Theorem B to higher-dimensional situations, and to explore its geometric implications and possible applications.

Acknowledgements. — I would like to thank Qirui Li and Sebastián Herrero for correspondence on their respective works, and Christian Johansson and Michael Temkin for correspondence on p-adic geometry. I am also grateful to Simone Dell'Ariccia for Figure 1, and to Francesco Maria Saettone and the referees for helpful comments.

2. Reduction to intersection multiplicities

In this section, we prove Theorem A in the case of minimal level, and reduce it to a statement on intersection multiplicities in general.

2.1. Geometry of Berkovich curves. — The standard reference on Berkovich spaces is [Ber90], see especially §§ 2-3.

Fix for this subsection a discretely valued field K, with uniformiser ϖ, ring of integers $\mathscr{O}_K$, and residue field k. We denote $q := |\varpi|^{-1}$.

Generalities. — Let X be a compact Hausdorff strictly K-analytic Berkovich space over K, generic fibre of a topologically finitely presented formal scheme $\mathscr{X}$ flat over $\mathrm{Spf}\,\mathscr{O}_K$. If $\mathscr{X} = \mathrm{Spf} A$ may be taken to be affine we say that X is affinoid. In this case, denoting $A_K = A \otimes_{\mathscr{O}_K} K$, the points $x \in X$ are in canonical bijection with the multiplicative seminorms, denoted

$$|\cdot(x)| \colon A_K \to \mathbf{R},$$

extending the absolute value of K. The set of maximal ideals $\mathrm{Max} A_K \subset \mathrm{Spec} A_K$ is embedded in X via the map $x \mapsto [\phi \mapsto |\phi(x)|]$, where $|\cdot|$ is the extension of K to $K(x)$. The image of $\mathrm{Max} A_K$ is called the set of type-1 points of X. It is dense in X. In general, if $X = \bigcup X_i$ is a cover of X by affinoids, the set of type-1 points is the union of the sets of type-1 points of the X_i; it is well-defined, and dense in X.

Reduction map and Shilov points. — There is a reduction map to the special fibre of $\mathscr{X}$,

$$\mathrm{red} \colon X \to |\mathscr{X}_k|,$$

sending type-1 points to closed points. If $V \subset \mathscr{X}_k$ is an irreducible component and $\xi_V \in \mathscr{X}_k$ is its generic point, then by [Ber90, Proposition 2.4.4] there is a unique point $\zeta_V \in X$ with

$$\mathrm{red}(\zeta_V) = \xi_V.$$

We call ζ_V the *Shilov point* of V.

Type-1 points on curves as intersection multiplicities. — Suppose for the rest of this subsection that X is a strictly K-analytic curve (that is X is as above and $\dim X = 1$), and that $\mathscr{X}$ is regular. The following notation will be used throughout the paper.

Notation 2.1.1. — If $z_?$ is a type-1 point of X (where '?' denotes an arbitrary decoration), we denote by

$$\mathrm{Z}_? \subset \mathscr{X}$$

the closed formal subscheme defined as follows. Let A be topologically finitely presented flat $\mathscr{O}_K$-algebra with an embedding $\mathrm{Spf} A \hookrightarrow \mathscr{X}$ such that $z_?$ corresponds to a point $\mathfrak{m}_? \in \mathrm{Max} A_K \subset X$; then $\mathrm{Z}_? \subset \mathrm{Spf} A \subset \mathscr{X}$ is the image of $\mathrm{Spf}\,\mathscr{O}_{K(z_?)}$ via the map $A \to \mathscr{O}_{K(z_?)}$ deduced from $A_K \to A_K/\mathfrak{m}_? = K(z_?)$. We denote by $\overline{z}_? \in \mathscr{X}_k$ the image of the closed point of $\mathrm{Spf}\,\mathscr{O}_{K(z)}$; by the definition of the reduction map in [Ber90, § 2.4], we have $\mathrm{red}(z_?) = \overline{z}_?$.

If X_0 is a proper smooth curve over $\mathrm{Spec} K$, $\mathscr{X}_0$ is a regular model over $\mathrm{Spec}\,\mathscr{O}_K$, and $z_{0,?}$ is a closed point of X_0, we denote by $\mathrm{Z}_{0,?}$ the Zariski closure of $z_{0,?}$ in $\mathscr{X}_0$, and by $\overline{z}_{0,?}$ the reduction of $z_{0,?}$ in $\mathscr{X}_{0,k}$. It is clear from the definitions that if $\mathscr{X}$ is the formal completion of $\mathscr{X}_0$ along the special fibre, $X = X_0^{\mathrm{an}}$, and $z_?$ is the type-1 point corresponding to $z_{0,?}$, then $Z_?$ is the formal completion of $Z_{0,?}$. For this reason we may think of $Z_?$ as the 'Zariski closure' of $z_?$.[(4)]

Lemma 2.1.2. — *Let $z \in X$ be a type-1 point, and assume that $Z \subset \mathscr{X}$ is normal. Let $\mathrm{Spf} A \subset \mathscr{X}$ be an affine neighbourhood of Z. Then for every $\phi \in A$ we have*

$$|\phi(z)| = q^{-m(Z,\mathrm{div}(\phi))/m(Z,\mathrm{div}(\varpi))},$$

where in the right-hand side, $m := m_{\overline{z}}$ is the intersection multiplicity at $\overline{z}$ as defined in (1.3.1).

Proof. — Write $Z = \mathrm{Spf} A/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$, and let $\overline{\mathfrak{m}} \supset \mathfrak{p}$ be the maximal ideal of A corresponding to $\overline{z}$. After applying the function $-\log_q$, the left-hand side equals

$$\frac{\mathrm{length}_{\mathscr{O}_{K(z)}} \mathscr{O}_{K(z)}/(\phi(z))}{\mathrm{length}_{\mathscr{O}_{K(z)}} \mathscr{O}_{K(z)}/(\varpi)},$$

and the right-hand side equals

$$\frac{\mathrm{length}_{A_{\overline{\mathfrak{m}}}} A_{\overline{\mathfrak{m}}}/(\mathfrak{p}+(\phi))}{\mathrm{length}_{A_{\overline{\mathfrak{m}}}} A_{\overline{\mathfrak{m}}}/(\mathfrak{p}+(\varpi))}.$$

Now $A_{\overline{\mathfrak{m}}}/\mathfrak{p} \hookrightarrow K(z)$ is an integrally closed $\mathscr{O}_K$-algebra with fraction field $K(z)$; it follows that $A_{\overline{\mathfrak{m}}}/\mathfrak{p} \cong \mathscr{O}_{K(z)}$, hence the two quantities above are equal. □

Residue discs and neighbourhoods of Shilov points. — If $a \in |\mathscr{X}_k|$ is a closed point, we denote by

$$\Delta(a, 1-\varepsilon), \qquad \varepsilon > 0,$$

an arbitrary increasing collection of compact subsets of $\mathrm{red}^{-1}(a)$ such that $\bigcup_{\varepsilon>0} \Delta(a, 1-\varepsilon) = \mathrm{red}^{-1}(a)$. (The reader may keep in mind the case where a is a nonsingular point and k is algebraically closed: then $\mathrm{red}^{-1}(a)$ is an open unit disc and one may take the $\Delta(a, 1-\varepsilon)$ to be discs of radius $1-\varepsilon$.)

We define a similar collection of compact subsets of X indexed by the open points of $\mathscr{X}_k$. Thus let V be an irreducible component of $\mathscr{X}_k$ and let ξ_V be its generic point. As $\mathscr{X}$ is regular, hence locally factorial, there is a finite open cover by flat affine formal schemes $\mathscr{X} = \bigcup_{\mathscr{Y} \in I} \mathscr{Y}$ such that in $\mathscr{O}(\mathscr{Y})$ we may factor

$$\varpi = \prod_V \phi_{V\cap\mathscr{Y}}^{e_{V\cap\mathscr{Y}}}, \tag{2.1.1}$$

where $\phi_{V\cap\mathscr{Y}}$ is a generator of the (height-1 prime, or unit) ideal $\mathfrak{p}_{V\cap\mathscr{Y}} \subset \mathscr{O}(\mathscr{Y})$ and $e_{V\cap\mathscr{Y}} \geq 0$. Let $Y \subset X$ denote the Berkovich generic fibre of $\mathscr{Y}$. We define

$$\begin{aligned} \Delta(\xi_{V\cap\mathscr{Y}}, 1-\varepsilon) &:= \{x : |\phi_{V\cap\mathscr{Y}}^{e_{V\cap\mathscr{Y}}}(x)| \leq 1-\varepsilon\} \subset Y \\ \Delta^\circ(\xi_{V\cap\mathscr{Y}}, 1-\varepsilon) &:= \{x : |\phi_{V\cap\mathscr{Y}}^{e_{V\cap\mathscr{Y}}}(x)| < 1-\varepsilon\} \subset Y. \end{aligned} \tag{2.1.2}$$

We denote by e_V the multiplicity of V in $\mathscr{X}_k$, that is, $e_V = e_{V\cap\mathscr{Y}}$ for any $\mathscr{Y}$ as above that is contained in the connected component of V in $\mathscr{X}_k$.

Lemma 2.1.3. — *For $\varepsilon \in (0,1)$ and $? \in \{\emptyset, \circ\}$,*[(5)] *there is a unique subset*

$$\Delta^?(\xi_V, 1-\varepsilon) \subset X$$

such that for all sufficiently small $\mathscr{O}_K$-flat affine open $\mathscr{Y} \subset \mathscr{X}$ with generic fibre Y,

$$\Delta^?(\xi_V, 1-\varepsilon) \cap Y = \Delta^?(\xi_{V\cap\mathscr{Y}}, 1-\varepsilon),$$

where $\Delta^?(\xi_{V\cap\mathscr{Y}}, 1-\varepsilon)$ is as in (2.1.2)*. Moreover:*

1. *if $z \in X$ reduces to a point $a \in \mathscr{X}_k - V$, then $z \notin \Delta(\xi_V, 1-\varepsilon)$ for any $\varepsilon > 0$;*
2. *for $r < q^{-1}$, the sets $\Delta(\xi_V, r)$ are empty;*

[(4)]This interpretation would acquire a more literal meaning if working in the context of adic spaces.

[(5)]Throughout this paper, we use the symbol '$\emptyset$' to denote 'no symbol' when referring to pieces of notation.

3. *the Shilov point ζ_V belongs to $\Delta(\xi_V, q^{-1})$.*

Proof. — For the first part, it suffices to observe the trivial facts that if $\mathscr{Y}'$ is a standard open affine subset of $\mathscr{Y}$ then the image in $\mathscr{O}(\mathscr{Y}')$ of a generator $\phi_{V\cap\mathscr{Y}}$ for $\mathfrak{p}_{V\cap\mathscr{Y}}$ (respectively of the factorisation (2.1.1)) is a generator for $\mathfrak{p}_{V\cap\mathscr{Y}'}$ (respectively a factorisation of the same form).

For the additional statements, we may replace X by any open affinoid Y as above such that $V\cap\mathscr{Y}\neq\emptyset$. For the first statement, we note that under our assumptions we have $|\phi_V(z)|=1$ by definition of the map red in [Ber90, §2.4]. It follows from this and (2.1.1) that $|\phi_V^{e_V}(\zeta_V)|\leq|\varpi(\zeta_V)|=q^{-1}$, proving the third statement. The second statement is also immediate from (2.1.1). □

Denote by $|\mathscr{X}_k|^0\subset|\mathscr{X}_k|$ the set of closed points.

Proposition 2.1.4. — *Let X be a compact strictly K-analytic Berkovich curve, generic fibre of a regular topologically finitely presented formal scheme $\mathscr{X}$ flat over* $\operatorname{Spf}\mathscr{O}_K$. *Let V be an irreducible component of $\mathscr{X}_k$ and let $\zeta_V\in X$ be the Shilov point of V. With notation as above, each of the systems of open subsets of X*

$$U(\zeta_V;A,\varepsilon):=X-\bigcup_{a\in A}\Delta(a,1-\varepsilon),\qquad A\subset|\mathscr{X}_k|-\{\xi_V\}\ \textit{finite},\ \varepsilon>0,$$

$$U'(\zeta_V;A',\varepsilon):=\Delta^\circ(\xi_V,q^{-1}+\varepsilon)-\bigcup_{a\in A'}\Delta(a,1-\varepsilon),\qquad A'\subset|\mathscr{X}_k|^0\ \textit{finite},\ \varepsilon>0,$$

forms a fundamental system of neighbourhoods of ζ_V.

Proof. — By construction, the intersection of all the open sets $U(\zeta_V;A,\varepsilon)$ contains no element in $\mathrm{red}^{-1}(a)$ for $a\in|\mathscr{X}_k|$ a closed point. By the previous lemma, it contains $\zeta_V=\mathrm{red}^{-1}(\xi_V)$ but no element $\zeta_{V'}=\mathrm{red}^{-1}(\xi_{V'})$ for $V'\neq V$. □

Lemma 2.1.5. — *The open subsets $\Delta^\circ(\xi_V,q^{-1/2})\subset X$ are pairwise disjoint as V varies among irreducible components of $\mathscr{X}_k$.*

Proof. — Let $V=\mathrm{div}(\phi)$, $V'=\mathrm{div}(\phi')$ be distinct irreducible components of $\mathscr{X}_k$, and let e_V, $e_{V'}$ be the corresponding multiplicities in $\mathscr{X}_k$ (as defined after (2.1.2)). If $z\in\Delta^\circ(\xi_V,q^{-1/2})\cap\Delta^\circ(\xi_{V'},q^{-1/2})$, we have

$$q^{-1}=|\varpi(z)|\leq|\phi(z)||\phi'(z)|<q^{-1},$$

a contradiction. □

2.2. Reduction. — In this subsection we prove a 'soft' geometric version of our main theorem, in terms of geometric basins (certain open subsets of X_v^{an}) rather than algebraic basins. In the case of minimal level $U_v=\mathrm{GL}_2(\mathscr{O}_{F_v})$, this version is already equivalent to Theorem A, as there is only one basin per connected component. In general, the soft variant is vivid but not explicit; however, it reduces Theorem A to the explicit comparison of algebraic and geometric basins, stated in Corollary 2.2.6.

We resume with the notation of the introduction. Thus let X_v^{an} be the Berkovich analytification of the Shimura curve $X_{F_v}=X_{U,F_v}$ as in the introduction, and let $\mathscr{X}$ be its regular model over $\operatorname{Spf}\mathscr{O}_{F_v}$ defined in [Car86]. We denote by e the ramification degree of E_v/F_v, and for K a local field with residue field $\varkappa$ we denote by $\zeta_K(s):=(1-|\varkappa|^{-s})^{-1}$ its zeta function.

Proposition 2.2.1. — *Let $(z_n)_{n\in\mathbf{N}}$ be a sequence of CM points in X_v^{an} with v-conductor going to ∞. The set of limit points of (z_n) is contained in the set of Shilov points of irreducible component of $\mathscr{X}_{\mathbf{F}_v}$.*

Proof. — According to Proposition 2.1.4, we need to show that for each closed point $a\in|\mathscr{X}_{\mathbf{F}_v}|$ and $\varepsilon>0$, the sequence z_n eventually leaves the compact set $\Delta(a,1-\varepsilon)$.

Let $\breve{F}_v$ be the completion of the maximal unramified extension of F_v, and denote by $\breve{X}_v^{\mathrm{an}}$ the analytification of $X_{\breve{F}_v}$. Its reduction is $\mathscr{X}_k$ where $k=\overline{\mathbf{F}}_v$.

As the transition maps in the tower $(X_U)_U$ are finite and the images of CM points are CM points, it suffices to consider the case of minimal level $U_v\cong\mathrm{GL}_2(\mathscr{O}_{F_v})$. By the 'Serre–Tate' theorem of [Car86, §6.6], for each closed point $a \in |\mathscr{X}_k|$ the inverse image $\mathrm{red}^{-1}(a)$ is canonically identified with the analytic generic fibre of the universal deformation space[(6)] of the formal group $\mathscr{G}_a$, which by [Dri74] is isomorphic to the analytic open unit disc $\mathbf{D}$ generic fibre of $\mathrm{Spf}\breve{\mathscr{O}}_{F,v}[\![u]\!]$. The CM points in $\mathrm{red}^{-1}(a)$ are explicitly described by Gross's theory of quasicanonical liftings (see [Gro86], [Wew07, Meu07]). Namely, if $z_n \in X_{F_v}^{\mathrm{CM}}$ has conductor $s = s(n) \geq 1$, then any point of $X_{\breve{F}_v}$ over z_n has residue field $\breve{E}_{v,s}$, a specific totally ramified abelian (respectively dihedral if v ramifies in E) extension of $\breve{F}_v$ of degree

$$m_{s(n)} := e\zeta_{F_v}(1)\zeta_{E_v}(1)^{-1}q^{s(n)}. \tag{2.2.1}$$

The inclusion of the point, spread out to the formal models, is dual to a map $\breve{\mathscr{O}}_{F_v}[\![u]\!] \to \mathscr{O}_{\breve{E}_{v,s}}$ sending u to a uniformiser, hence z_n lies in the circle $|u| = |\varpi|^{1/m_{s(n)}}$. As $m_{s(n)} \to 0$ when $n \to \infty$, any compact disc in $\mathrm{red}^{-1}(a)$ contains only finitely many CM points of $X_{\breve{F}_v}$. Therefore all points in the limit set of the sequence (z_n) in X_v^{an} reduce to a non-closed point of $\mathscr{X}_{\mathbf{F}_v}$. □

Remark 2.2.2. — From the theory of quasicanonical liftings just discussed, it also follows that if $U_v = \mathrm{GL}_2(\mathscr{O}_{F_v})$ and $z \in X_v^{\mathrm{an}}$ is a CM point of v-conductor s with reduction $\overline{z}$, then

$$m_{\overline{z}}(Z, \mathscr{X}_{\mathbf{F}_v})/[\mathbf{F}_v(\overline{z}) : \mathbf{F}_v] = m_s = e\zeta_{F_v}(1)\zeta_{E_v}(1)^{-1}q^s. \tag{2.2.2}$$

We may now define the geometric counterpart of the basins defined in the introduction.

Definition 2.2.3. — Let $V \subset \mathscr{X}_{\mathbf{F}_v}$ be an irreducible component. The *geometric basin* of V is the open subset

$$\mathscr{B}_V := \Delta^\circ(\xi_V, q^{-1/2}) \subset X_v^{\mathrm{an}}.$$

By Lemma 2.1.5, the geometric basins are disjoint open subsets of X_v^{an}.
The following is the 'soft' version of theorem A.

Theorem 2.2.4. — *Let $(z_n)_{n\in\mathbf{N}}$ be a sequence of CM points in X_v^{an}, with v-conductor going to ∞. The sequence has a limit if and only if it is eventually supported in a single* geometric *basin $\mathscr{B}_V$. If this is the case for the geometric basin of the irreducible component $V \subset \mathscr{X}_{\mathbf{F}_v}$, then*

$$\lim_{m\to\infty} z_n = \zeta_V.$$

Proof. — By Proposition 2.1.4, the neighbourhoods $\Delta^\circ(\xi_V, q^{-1}+\varepsilon)$ of ζ_V are contained in the elements of a fundamental system of neighbourhoods. Thus if the sequence (z_n) has a limit ζ_V, it is eventually supported in $\Delta^\circ(\xi_V, q^{-1/2}) = \mathscr{B}_V$. Conversely, if z_n is eventually supported in $\mathscr{B}_V$, by Lemma 2.1.5 it eventually leaves $\mathscr{B}_{V'}$ for all $V' \neq V$, so that $\zeta_{V'}$ cannot be a limit point. Together with the previous result and the sequential compactness of X_v^{an}, this shows that $z_n \to \zeta_V$. □

In view of Theorem 2.2.4, Theorem A is reduced to Corollary 2.2.6 below, a consequence of the following.

Proposition 2.2.5. — *Let V be an irreducible component of $\mathscr{X}_{\mathbf{F}_v}$. Let $\overline{z} \in \mathscr{X}_{\mathbf{F}_v}$ be a supersingular point and let $V' \neq V$ be an irreducible component of $\mathscr{X}_{\mathbf{F}_v}$ through $\overline{z}$. The intersection multiplicity*

$$m_{\overline{z}}(Z, V')$$

is bounded as z varies among CM points of X_{F_v} in B_V reducing to $\overline{z}$.

The proof of Proposition 2.2.5, given at the very end of the paper (after Corollary 4.3.1), will be a corollary of the formula of Theorem B, which computes geometric intersection multiplicities in terms of algebraic data.

(6) More general versions of this object and of the Serre–Tate theorem will be precisely described in § 3.2 below.

Corollary 2.2.6. — *Let $V \subset \mathscr{X}_{\mathbf{F}_v}$ be an irreducible component. For all $z \in X_v^{\mathrm{CM}}$ that is either of ordinary reduction, or of sufficiently high conductor (depending only on the level U_v), we have*

$$z \in B_V \Longleftrightarrow z \in \mathscr{B}_V.$$

Proof of Corollary 2.2.6 and of Theorem A, assuming Proposition 2.2.5.. — The corollary is obvious for points of ordinary reduction. Let $z \in B_V$ have supersingular reduction and conductor s. We have

$$e_V \cdot m(Z,V)/m(Z,\mathscr{X}_{\mathbf{F}_v}) = 1 - \sum_{V' \neq V} e_{V'} m(Z,V')/m(Z,\mathscr{X}_{\mathbf{F}_v}).$$

By (2.2.2) (which remains valid for $\mathscr{X}$ of arbitrary level by the projection formula) and Proposition 2.2.5, the right-hand side is close to 1 if s is sufficiently large. In particular, by Lemma 2.1.2 and the definition, the point z belongs to $\mathscr{B}_V$. Since the B_V form a complete partition of the set of CM points of sufficiently high conductor, this also proves the reverse implication. □

The following basic result will be established at the end of § 3.3, based on a modular interpretation of CM curves.

Lemma 2.2.7. — *For any $z \in X_{F_v}^{\mathrm{CM}}$, its Zariski closure of $Z \subset \mathscr{X}$ is a regular curve.*

Proof of Corollary 2.2.6 and of Theorem A, assuming Lemma 2.2.7 and Proposition 2.2.5.

The corollary is obvious for points of ordinary reduction. Let $z \in B_V$ have supersingular reduction and conductor s. We have

$$e_V \cdot m(Z,V)/m(Z,\mathscr{X}_{\mathbf{F}_v}) = 1 - \sum_{V' \neq V} e_{V'} m(Z,V')/m(Z,\mathscr{X}_{\mathbf{F}_v}).$$

By (2.2.2) (which remains valid for $\mathscr{X}$ of arbitrary level by the projection formula) and Proposition 2.2.5, the right-hand side is close to 1 if s is sufficiently large. Therefore, by Lemma 2.2.7, Lemma 2.1.2 (applied to the formal completion of $\mathscr{X}$ along the special fibre), and the definitions, the point z belongs to $\mathscr{B}_V$. Since the B_V form a complete partition of the set of CM points of sufficiently high conductor, this also proves the reverse implication of Corollary 2.2.6. As noted above, Theorem A is then implied by Theorem 2.2.4. □

3. Special cycles in Lubin–Tate spaces and in formal groups

The goal of the rest of the paper is to establish an intersection multiplicity formula that will imply Proposition 2.2.5. The formal completion of $\mathscr{X}$ at a supersingular geometric point admits a purely local description as a Lubin–Tate space $\mathscr{M}$. After introducing some notation, in this section we define $\mathscr{M}$ and its collection of special curves, comparing those with the special curves in $\mathscr{X}$. Then we define companion cycles in formal groups; the definitions are a special case of those in [Li] for CM curves, and are new for vertical curves.

3.1. Notation. — The notation introduced here will be used throughout the rest of the paper unless otherwise noted.

Valued fields. — We fix a prime v of F nonsplit in E and work in a purely local setting, dropping the subscript v from the notation; thus F and E will denote the local fields previously denoted by F_v and E_v respectively. We denote by ϖ a fixed uniformiser in F, by v the valuation on $F^\times$ (so $v(\varpi) = 1$), and by $|\cdot| := q^{-v(\cdot)}$ the standard absolute value on F.

If K is a finite extension of F, we denote by $\breve{K}$ the completion of the maximal unramified extension of K, by $\breve{\mathscr{O}}_K$ its ring of integers, and by k the residue field of $\breve{K}$ (which is independent of the choice of K).

Functors on $\mathscr{O}_F$-modules. — We denote by

$$M_{[n]} := M/\varpi^n M$$

the truncation of $\mathscr{O}_F$-modules, and by

$$M^* := \mathrm{Hom}_{\mathscr{O}_F}(M, F/\mathscr{O}_F), \qquad M^\vee := \mathrm{Hom}_{\mathscr{O}_F}(M, \mathscr{O}_F)$$

the Pontryagin and, respectively, plain dualities on $\mathscr{O}_F$-modules. We stipulate the following convention on the order of reading of symbols:

$$\mathscr{O}_{K,[N]}^* := (\mathscr{O}_{K,[N]})^*, \qquad \mathscr{O}_{K,[N]}^\vee := (\mathscr{O}_K^\vee)_{[N]}.$$

We let

$$\mathscr{D}_K^{-1} := \mathscr{O}_K^\vee = \mathrm{Hom}_{\mathscr{O}_F}(\mathscr{O}_K, \mathscr{O}_F)$$

denote the relative inverse different of K/F, an invertible $\mathscr{O}_K$-module.

Integers. — For compact open subgroups $C_K \subset K^\times$, we define integers

$$d_{C_K} = [\mathscr{O}_K^\times : C_K]. \tag{3.1.1}$$

If E/F is a quadratic extension, we also define $d_0 = d_{\mathrm{Ig},0} = d_{E,0} = 1$ and, for $n \geq 1$,

$$d_n := q^{2n,} \quad d_{E,n} = \zeta_E(1)^{-1} q^{2n}, \quad d_{\mathrm{Ig},n} := \zeta_F(1)^{-1} q^{2n}.$$

Miscellaneaous. — We denote by $\mathscr{C}_K$ the category of complete Noetherian local $\breve{\mathscr{O}}_K$-algebras with residue field k, and by $\mathscr{C}_k$ its subcategory consisting of objects that are k-algebras.

Finally, we will freely use the language and notation of the Appendix, which the reader is now invited to skim through.

3.2. Lubin–Tate spaces. — For basic references, see [Dri74], or [Car86, Appendice].

Let K be a finite extension of F. Let $\mathscr{G}$ be a formal $\mathscr{O}_K$-module of dimension 1 and height h over an object A of $\mathscr{C}_F$.[7] A *Drinfeld structure* on $\mathscr{G}$ of level n is an $\mathscr{O}_K$-module map

$$\alpha \colon \mathscr{O}_{K,[n]}^{h,*} \to \mathrm{Hom}_{\mathrm{Spf}A}(\mathrm{Spf}A, \mathscr{G})$$

such that

$$\alpha_*(\mathscr{O}_{K,[n]}^{h,*}) := \sum_{x \in \mathscr{O}_{K,[n]}^{h,*}} [\alpha(x)] = \mathscr{G}[\varpi^n]$$

as Cartier divisors. A Drinfeld structure of infinite level on $\mathscr{G}$ is a map $\mathscr{O}_K^{h,*} \to \mathscr{G}(A) = \mathrm{Hom}_{\mathrm{Spf}A}(\mathrm{Spf}A, \mathscr{G})$ whose restriction to $\mathscr{O}_{K,[n]}^{h,*}$ is a Drinfeld structure of level n for all n.

Let $\mathscr{G}_{h,K}$ be a formal $\mathscr{O}_K$-module of height h over k, which is unique up to isomorphism. Consider the functor $\mathscr{M}_{h,K,n}$ on $\mathscr{C}_F$ that associates to an object A the set of equivalence classes of triples

$$[\mathscr{G}, \iota, \alpha], \tag{3.2.1}$$

where $\mathscr{G}$ is a formal $\mathscr{O}_K$-module over A of height h, $\iota \colon \mathscr{G}_{h,K} \to \mathscr{G} \times_{\mathrm{Spf}A} \mathrm{Spf}\,k$ is a quasi-isogeny of height 0, and $\alpha \colon \mathscr{O}_{K,[n]}^{h,*} \to \mathscr{G}(A)$ is a Drinfeld structure of level n. A theorem of Drinfeld [Dri74] asserts that $\mathscr{M}_{h,K,n}$ is representable by a regular formal scheme finite flat over $\mathscr{M}_{h,K,0}$, and that $\mathscr{M}_{h,K,0} \cong_{/\breve{\mathscr{O}}_F} \mathrm{Spf}\,\breve{\mathscr{O}}_K[\![u_1, \ldots, u_{h-1}]\!]$.

If $U \subset \mathrm{GL}_h(\mathscr{O}_K)$ is the open compact subgroup

$$U = U_n^{(h,K)} := \mathrm{Ker}(\mathrm{GL}_h(\mathscr{O}_K) \to \mathrm{GL}_h(\mathscr{O}_K/\varpi^n)),^{(8)}$$

[7] Note as in [Li, § 2.2] that the $\mathscr{O}_K$-action on $\mathrm{Lie}\,\mathscr{G} \cong A$ gives A an $\mathscr{O}_K$-algebra structure, so that A is also an object of $\mathscr{C}_K$.

[8] We will simply write U_n for $U_n^{(h,K)}$ when h, K are clear from context.

we define $\mathscr{M}_{h,K,U} := \mathscr{M}_{h,K,n}$. If $U \subset \mathrm{GL}_h(\mathscr{O}_K)$ is any compact subgroup containing U_n, we define $\mathscr{M}_{h,K,U} := \mathscr{M}_{h,K,n}/(U/U_n)$.

The formal scheme (representing the functors) $\mathscr{M}_{h,K,U}$ are called *Lubin–Tate spaces*. In what follows, we will denote

$$\mathscr{G}_E := \mathscr{G}_{1,K}, \qquad \mathscr{G}_F := \mathscr{G}_{2,F} \tag{3.2.2}$$

and identify $\mathscr{G}_F$ with the image of $\mathscr{G}_E$ under the forgetful functor. The endomorphism ring of $\mathscr{G}_E$ is $\mathscr{O}_E$; the endomorphism ring of $\mathscr{G}_F$ is the ring of integers (or elements with integral norms) of the unique division algebra D of rank 4 over F.

We will also denote

$$\mathscr{M}_U := \mathscr{M}_{2,F,U},\ \mathscr{M}_n := \mathscr{M}_{U_n^{(2,F)}}, \qquad \mathscr{N}_C := \mathscr{M}_{1,E,C},\ \mathscr{N}_n := \mathscr{N}_{U_n^{(1,E)}}.$$

The space $\mathscr{N}_C$ is isomorphic to $\mathrm{Spf}\,\breve{\mathscr{O}}_{E,C}$, the ring of integers in the abelian extension of $\breve{E}$ with Galois group $\mathscr{O}_E^\times/C$.

Finally, we denote by

$$\mathscr{M}_\infty := \varprojlim_n \mathscr{M}_n, \qquad \mathscr{N}_\infty := \varprojlim_n \mathscr{N}_n,$$

as pro-objects of a category $\mathscr{F}$ of Noetherian formal schemes (cf. the Appendix), and as the functors on $\mathscr{C}_F$ of deformations of $\mathscr{G}_F$ (respectively $\mathscr{G}_E$) together with a Drinfeld structure of infinite level.

We fix for the rest of the paper a level $U \subset \mathrm{GL}_2(\mathscr{O}_F)$ and write $\mathscr{M} := \mathscr{M}_U$. We denote by M the Berkovich generic fibre of $\mathscr{M}$. Let $\mathscr{G} \to \mathscr{X}_U$ be Carayol's sheaf of divisible $\mathscr{O}_{F_v}$-modules recalled in § 1.1, and for a $z \in \mathscr{X}_U$, denote by $\mathscr{G}_z$ the fibre of $\mathscr{G}$ at z.

A Serre–Tate theorem. — We restore the subscript v just for the following proposition.

***Proposition 3.2.1* (Carayol [Car86, §7.4]).** — *Let $\overline{z} \in \mathscr{X}_{U,k}$ be a closed point.*

1. *If $\overline{z}$ is supersingular, then, after identifying $\mathscr{G}_F = \mathscr{G}_{\overline{z}}$, the formal completion of $\mathscr{X}_{\breve{\mathscr{O}}_{F_v}}$ at $\overline{z}$ is canonically $\breve{\mathscr{O}}_{F_v}$-isomorphic to the Lubin–Tate space $\mathscr{M}_{U_v}$, via the map sending an A-valued point z to $\mathscr{G}_z$ with its level structure α_z.*
2. *If $\overline{z}$ is ordinary, the formal completion of $\mathscr{X}_{\breve{\mathscr{O}}_{F_v}}$ at $\overline{z}$ is canonically $\breve{\mathscr{O}}_{F_v}$-isomorphic to the deformation space of the $\mathscr{O}_{F_v}$-divisible module $\mathscr{G}_{F_v}^{(1)} \oplus F_v/\mathscr{O}_{F_v}$ over k, and non-canonically to* $\mathrm{Spf}\,\breve{\mathscr{O}}_{F_v}[\![u]\!]$*; in particular, it is smooth over $\breve{\mathscr{O}}_{F_v}$*

3.3. Special curves in Lubin–Tate spaces. — We consider two special classes of curves in $\mathscr{M}$, the irreducible components of the special fibre, that we call *vertical curves*, and the "Zariski closures" of CM points, which we call *CM curves*. Then we compare them with the corresponding objects in the Shimura curve $\mathscr{X}$.

Vertical curves. — Let A be a complete Noetherian k-algebra and let $\mathscr{G}$ be a formal $\mathscr{O}_F$-module of height 2 over A. An *Igusa structure* of level n on $\mathscr{G}$ is an $\mathscr{O}_F$-module map

$$\gamma\colon \mathscr{O}_{F,[n]}^* \to \mathrm{Hom}_{\mathrm{Spf}A}(\mathrm{Spf}A, \mathscr{G})$$

such that $q^n \cdot \gamma_*(\mathscr{O}_{F,[n]}^*) = \mathscr{G}[\varpi^n]$ as Cartier divisors.[9] The functor on $\mathscr{C}_k$ sending an object A to the set of isomorphism classes of triples $[\mathscr{G}, \iota, \gamma]$, where $[\mathscr{G}, \iota]$ is a deformation of $\mathscr{G}_F$ as in (3.2.1), and γ is an Igusa structure of level n, is representable by a formally smooth curve

$$\mathrm{Ig}_n \to \mathrm{Spec}\, k$$

[9]Our Ig_n should be compared with the 'exotic Igusa curve' $\mathrm{ExIg}(p^n, n)$ of [KM85, (12.10.5.1)]; cf. *ibid.* Proposition 13.7.5.

called the $n^{\rm th}$ Igusa curve. It is of degree $d_{\rm Ig,n}$ over $\mathrm{Ig}_0 = \mathscr{M}_{U_0,k} \cong \mathrm{Spf}\, k[\![u]\!]$. We define $\mathrm{Ig}_\infty := \varprojlim_n \mathrm{Ig}_n$ and

$$[\mathrm{Ig}_\infty]^\circ := \varprojlim [\mathrm{Ig}_n]^\circ \in K_0(\mathrm{Ig}_\infty)_{\mathbf{Q}}, \qquad [\mathrm{Ig}_n]^\circ := d_{\mathrm{Ig},n}^{-1} \cdot [\mathrm{Ig}_n].$$

Let

$$\lambda \in \mathbf{P}^1(F) \subset \mathscr{O}_F^\times \backslash \mathrm{Hom}_{\mathscr{O}_F}(\mathscr{O}_F, \mathscr{O}_F^2).$$

We attach to λ the morphism

$$\begin{aligned} f_\lambda \colon \mathrm{Ig}_\infty &\to \mathscr{M}_{U,k} \hookrightarrow \mathscr{M}_U \\ [\mathscr{G}, \iota, \gamma] &\mapsto [\mathscr{G}, \iota, \gamma \circ \lambda^*] \end{aligned}$$

(where $\mathscr{M}_{U,k} \hookrightarrow \mathscr{M}_U$ is the natural map base-change of $k \hookrightarrow \mathrm{Spf}\, \breve{\mathscr{O}}_F$) and the irreducible and reduced curve

$$f_\lambda(\mathrm{Ig}_\infty) = V(\lambda)_U \subset \mathscr{M}_{U,k} \tag{3.3.1}$$

defined by the closed condition that any extension to $\mathscr{O}_F^{*,2}$ of the level structure α of a geometric point $[\mathscr{G}, \iota, \alpha]$ belonging to $V(\lambda)_U$ should factor through $\lambda^* g \colon \mathscr{O}_F^{*,2} \to \mathscr{O}_F^*$ for some $g \in U$. The $V(\lambda)_U$, for $\lambda \in \mathbf{P}^1(F)/U$, are the irreducible components of $\mathscr{M}_{U,k}$ and they all meet at the closed point [Car86, Appendice § 8, and § 9.4].

Definition 3.3.1. — The *vertical curve* attached to λ in $\mathscr{M}_\infty$, respectively $\mathscr{M}_U$, is the curve

$$V(\lambda) := f_\lambda(\mathrm{Ig}_\infty) \subset \mathscr{M}_\infty, \qquad V(\lambda)_U := f_{\lambda,U}(\mathrm{Ig}_\infty) \subset \mathscr{M}_U.$$

The corresponding normalised *vertical cycle* in $K_0(\mathscr{M}_\infty)_{\mathbf{Q}}$, respectively $K_0(\mathscr{M}_U)_{\mathbf{Q}}$), is

$$[V(\lambda)]^\circ := f_{\lambda,*}[\mathrm{Ig}_\infty]^\circ = \varprojlim_U [V(\lambda)_U]^\circ, \qquad [V(\lambda)_U]^\circ = d_{U\cap\mathscr{O}_F^\times}^{-1} \cdot [V(\lambda)_U].$$

CM curves. — We consider a local analogue of CM points and of their Zariski closures in integral models. These will be images of maps of Lubin–Tate towers and we start by defining the data parametrising them. The definition is slightly streamlined version of the one in [Li, § 2.5].

Consider an element

$$(\varphi, \tau) \in \mathrm{LT}(E,F) := \mathrm{QIsog}(\mathscr{G}_E, \mathscr{G}_F) \times_{E^\times} \mathrm{Isom}_F(E, F^2)$$

where the notation means that φ is a quasi-isogeny of formal $\mathscr{O}_F$-modules, τ is an F-linear isomorphism, and we consider the quotient of the product by the relation $(\varphi \circ t, \tau \circ t) \sim (\varphi, \tau)$, $t \in E^\times$. There exists a shortest integer interval $[r,s]$ such that $\varpi^{-r}\mathscr{O}_F^2 \subset \tau(\mathscr{O}_E) \subset \varpi^{-s}\mathscr{O}_F^2$; this depends on the choice of representative τ.

We define two functions, *conductor* and *height*, on $\mathrm{LT}(E,F)$ by

$$\begin{aligned} c(\varphi, \tau) &:= s - r, \\ \mathrm{ht}(\varphi, \tau) &:= -\mathrm{ht}(\varphi) + \log_q [\tau(\mathscr{O}_E) : \mathscr{O}_F^2], \end{aligned} \tag{3.3.2}$$

where $\mathrm{ht}(\varphi)$ is the usual $\mathscr{O}_F$-height of a quasi-isogeny of $\mathscr{O}_F$-modules, and $[:]$ denotes generalised index (that is, for two lattices Λ, $\Lambda' \subset F^2$, we have $[\Lambda : \Lambda'] := \frac{|\Lambda/\Lambda\cap\Lambda'|}{|\Lambda'/\Lambda\cap\Lambda'|}$). We denote by $\mathrm{LT}(E,F)^0$ the subspace of pairs of height 0.

From now on, the representatives (φ, τ) of elements of $\mathrm{LT}(E,F)^0$ *will always be chosen to satisfy* $r = 0$.[10] In this case $c(\varphi, \tau) = s = \mathrm{ht}(\varphi)$.

[10] The reader unwilling to make this assumption will simply need to interpret some maps of formal $\mathscr{O}_F$-modules in the diagrams to follow as quasi-isogenies.

To an element $(\varphi,\tau)\in \mathrm{LT}(E,F)^0$ we attach a morphism $f_{(\varphi,\tau)}$ of Lubin–Tate towers, given by the following morphism of functors on $\mathscr{C}_F$ (which are representable when restricted to finite levels):

$$\begin{aligned} f_{(\varphi,\tau)}\colon \mathscr{N}_\infty &\to \mathscr{M}_\infty \\ [\mathscr{G},\iota,\beta] &\mapsto [\mathscr{G}/\beta_*K, \overline{\varphi}_\beta\circ\iota\circ\varphi^{-1}, \beta'\circ\tau^*], \end{aligned} \tag{3.3.3}$$

where the notation is according to the following commutative diagrams:

$$\begin{tikzcd} & K:=(\mathscr{O}_E/\tau^{-1}\mathscr{O}_F^2)^* \arrow[r] \arrow[d] & \beta_*K \arrow[d] \\ & \mathscr{O}_E^* \arrow[r, "\beta"] \arrow[d] & \mathscr{G} \arrow[d, "\varphi_\beta"] \\ \mathscr{O}_F^{2,*} \arrow[r, "\tau^*"] & \tau^*(\mathscr{O}_F^{2,*}) \arrow[r, "\beta'"] & \mathscr{G}/\beta_*K \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathscr{G}} \arrow[d, "\overline{\varphi}_\beta"] & \mathscr{G}_E \arrow[l, "\iota"'] \arrow[r, "\varphi"] & \mathscr{G}_F \arrow[dll, dotted] \\ \overline{\mathscr{G}}/\overline{\beta}_*K. & & \end{tikzcd} \tag{3.3.4}$$

In the above diagrams, the columns are exact, the bars denote reduction modulo the maximal ideal of $\breve{\mathscr{O}}_E$, and β_*K is the subgroup scheme of $\mathscr{G}$ defined by the product of ideal sheaves of the β_*x, $x\in K$.

Let $\psi\colon E\hookrightarrow M_2(F)$ be the F-algebra morphism such that $\psi(t)\tau(x)=\tau(tx)$ for all $t,x\in E$. The composition to level U factors as

$$f_{(\varphi,\tau),U}:=\mathrm{p}_U\circ f_{(\varphi,\tau)}\colon \mathscr{N}_\infty\to\mathscr{N}_C\to\mathscr{M}_U \tag{3.3.5}$$

where $C:=\psi^{-1}(U)$ and the second map is a finite monomorphism, hence a closed immersion. In particular, the morphism of functors defined in infinite level does define a morphism in the pro-category of Noetherian formal schemes $\widehat{\mathscr{F}}$ defined in the Appendix.

If $U=U_0$ and (φ,τ) has conductor s, then $C=(\mathscr{O}_F+\varpi^s\mathscr{O}_E)^\times$, and the image of $\mathscr{N}_C$ is a quasicanonical lift of level s.

Definition 3.3.2. — The *CM curve* in $\mathscr{M}_\infty$ (respectively $\mathscr{M}_U$) attached to (φ,τ) is

$$\mathrm{Z}(\varphi,\tau):=f_{(\varphi,\tau)}(\mathscr{N}_\infty)\subset\mathscr{M}_\infty,\qquad \mathrm{Z}(\varphi,\tau)_U:=f_{(\varphi,\tau),U}(\mathscr{N}_\infty)\subset\mathscr{M}_U.$$

where $f_{(\varphi,\tau)}$ (respectively $f_{(\varphi,\tau),U}$) is the morphism defined by (3.3.3) (respectively (3.3.5)).

The local CM point $z(\varphi,\tau)$ in the Berkovich generic fibre M of $\mathscr{M}$ is the generic fibre of $\mathrm{Z}(\varphi,\tau)$; equivalently, it is the image under $f_{(\varphi,\tau)}$ of the single point in the generic fibre of $\mathscr{N}_n$ for any sufficiently large n; thus the notation is consistent with Notation 2.1.1.

The normalised *CM cycle* attached to (φ,τ) in $K_0(\mathscr{M}_\infty)_{\mathbf{Q}}$ (respectively $K_0(\mathscr{M}_U)_{\mathbf{Q}}$) is

$$[\mathrm{Z}(\varphi,\tau)]^\circ:=f_{(\varphi,\tau),*}[\mathscr{N}_\infty]^\circ=\varprojlim_U[\mathrm{Z}(\varphi,\tau)_U]^\circ,\qquad [\mathrm{Z}(\varphi,\tau)_U]^\circ=\frac{1}{d_{\psi^{-1}(U)}}\cdot[\mathrm{Z}(\varphi,\tau)_U].$$

Definition 3.3.3. — Let $V=V(\lambda)_U\subset\mathscr{M}_k$ be the irreducible component parametrised by the U-class of λ, and let n be minimal such that $U\supset U_n$. The *basin* $B_{V(\lambda)_U}$ of V is the set of those CM points $z=z(\varphi,\tau)\in M$ of conductor $s\geq n$, such that if (φ,τ) is chosen to satisfy $\mathscr{O}_F^2\subset\tau(\mathscr{O}_E)\subset\varpi^{-s}\mathscr{O}_F^2$, the lines

$$L_s(\tau):=\tau(\mathscr{O}_E)/\mathscr{O}_F^2,\qquad L_s(\lambda):=\lambda(\varpi^{-s}\mathscr{O}_F/\mathscr{O}_F)\qquad\subset\quad(\varpi^{-s}\mathscr{O}_F/\mathscr{O}_F)^2 \tag{3.3.6}$$

are in the same orbit for the left action of U.

Comparison of local and global objects. — We temporarily restore the subscripts v for local objects. We use the notation of the introduction for CM points on our fixed Shimura curve $X=X_U$, and we abbreviate $\mathscr{M}=\mathscr{M}_{U_v}$, as well as $M=M_{U_v}$ for the generic fibre of $\mathscr{M}$.

Lemma 3.3.4. — *Let $z \in X^{\mathrm{CM}}_{F_v}$ be a point with complex multiplication by (E,ψ), with E_v/F_v nonsplit. Let Z be the Zariski closure of z in $\mathcal{X}$. Let $\tau\colon E_v \to F_v^2$ be such that $\psi_v(t)\tau(x) = \tau(tx)$ for all $t, x \in E_v$. Let $\overline{z} \in \mathcal{X}_{\mathbf{F}_v}$ be the reduction of z and let $C \subset \mathcal{X}_{\mathbf{F}_v}$ be the connected component containing $\overline{z}$. Let $\lambda \in \mathbf{P}^1(F_v)/U_v$, let*

$$V = V(\lambda)_U^{(C)} \subset C \subset \mathcal{X}_{\mathbf{F}_v}$$

be the corresponding irreducible component.

Let $\overline{z}' \in \mathcal{X}_k$ be any point over $\overline{z}$. Let $z' \in M$ be any point mapping to $z \in X_v^{\mathrm{an}}$ under the identification of M with $\mathrm{red}^{-1}(\overline{z}) \subset X^{\mathrm{an}}_{\breve{F}_v}$ given by Proposition 3.2.1. Let $V' \subset \mathcal{M}_k$ be any irreducible component mapping to $V \subset \mathcal{X}_{\mathbf{F}_v}$.

1. *$V' = V(\lambda)_{U_v} \subset \mathcal{M}_{U_v,k}$.*
2. *There exist $(\varphi,\tau) \in \mathrm{LT}(E,F)^0$, whose second component is equal (up to rescaling by $E_v^\times$) to the isomorphism $\tau\colon E_v \to F_v^2$ attached to z, such that $z' = z(\varphi,\tau)$.*
3. *The CM point z belongs to B_V if and only if z' belongs to $B_{V'}$.*
4. *The images of $Z' := Z(\varphi,\tau)$ and of Z in the formal completion $\hat{\mathcal{X}}_{\overline{z}}$ of $\mathcal{X}$ at $\overline{z}$ (via the map of Proposition 3.2.1 and the completion, respectively) are equal.*
5. *We have the following relation between intersection multiplicities in $\mathcal{X}$ and $\mathcal{M}_{U_v}$:*

$$m_{\overline{z}}(Z,V) = [\mathbf{F}_v(\overline{z}) : \mathbf{F}_v] \cdot m(Z',V').$$

Proof. — The first thee statements are clear from the definitions. For the fourth statement, let $\mathrm{Spf} A \subset \hat{\mathcal{X}}_{\overline{z}}$ be a neighbourhood of $\overline{z}$ with A a topologically finitely presented flat $\mathcal{O}_{F_v}$-algebra. Then the images of Z and Z' are both induced by the unique map $A \to \mathcal{O}_{F_v(z)}$ lifting the map $A_{F_v} \to F_v(z)$ corresponding to z.

Finally for the last statement, observe that the right hand side is the sum of the intersection multiplicities of all the $\mathrm{Gal}(\mathbf{F}_v(\overline{z})/\mathbf{F}_v)$-conjugates of the pair (Z',V') (as a pair of cycles in the completion of $\mathcal{X}_{\breve{\mathcal{O}}_{F_v}}$ at $\overline{z}'$), and all those multiplicities are equal. □

Proof of Lemma 2.2.7. — By Lemma 3.3.4, it suffices to prove the regularity of the formal scheme $Z' = Z(\varphi,\tau)_{U_v}$; using the notation of (3.3.5), this follows from the fact that Z' is the image of the regular formal scheme $\mathcal{N}_C$ under a closed immersion. □

3.4. Special cycles in formal groups and their Tate modules. — We define special cycles, analogous to those defined above in $\mathcal{M}_\infty$, inside $\mathcal{G}_F^2$, its truncations or Tate modules. The definitions are variants of those of [Li, § 3.1, 3.3]. Later in §4.1, we will reduce the intersection problem in Lubin–Tate spaces to a similar problem in $\mathcal{G}_F^2$.

Let $K = F$ or E and let $\mathcal{G}_K = $ (3.2.2). For $? \in \{\emptyset, [N]\}$ and $\prime \in \{*, \vee\}$, let

$$'\mathcal{G}_{K,?} := \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}'_{K,?}, \mathcal{G}_K) \tag{3.4.1}$$

Then we have identifications and maps

$$\begin{array}{c} {}^*\mathcal{G}_K \cong \mathcal{D}_K^{-1} \otimes_{\mathcal{O}_K} T\mathcal{G}_K \\ \Big\downarrow{\scriptstyle \pi_N} \\ {}^*\mathcal{G}_{K,[N]} \cong \mathcal{D}_K^{-1} \otimes_{\mathcal{O}_K} \mathcal{G}_K[\varpi^N] = {}^\vee\mathcal{G}_{K,[N]} \xhookrightarrow{j_N} {}^\vee\mathcal{G}_K = \mathcal{D}_K^{-1} \otimes_{\mathcal{O}_K} \mathcal{G}_K, \end{array} \tag{3.4.2}$$

where the *Tate module* $T\mathcal{G}_K$ is the $\mathcal{O}_K$-module scheme

$$T\mathcal{G}_K := \varprojlim_n \mathcal{G}_K[\varpi^n]$$

with transition maps given by multiplication by ϖ. (The isomorphisms denoted by $\cong$ depend on the choice of ϖ, which allows to compatibly identify $\mathcal{O}_{F,[N]} \cong \mathcal{O}^*_{F,[N]}$. This dependence will be negligible for our purposes .)

Vertical cycles. — Let $\lambda \in \mathbf{P}^1(F)$. We attach to λ the morphisms, all denoted by the same name,

$$\begin{aligned} f_\lambda^\flat \colon {}'\mathscr{G}_{F,?} &\to {}'\mathscr{G}_{F,?}^2 \\ x &\mapsto x \circ \lambda', \end{aligned}$$

where $? \in \{\emptyset, [N]\}$ and $' \in \{*, \vee\}$.

Definition 3.4.1. — Let $? \in \{\emptyset, [N]\}$, let $' \in \{*, \vee\}$, and let $\bullet = \emptyset$ if $' = \vee$, $\bullet = \circ$ if $' = *$. The *vertical cycle* attached to λ is

$$[V^\flat(\lambda)]_?^\bullet := f^\flat_{\lambda,*}[{}'\mathscr{G}_{F,?}]^\bullet \in K_0({}'\mathscr{G}^2_{F,?})_{\mathbf{Q}},$$

where in the case of $' = *$, the normalisation is made viewing $\mathscr{G}_{F,[N]}$ as a scheme (of degree d_N) over the base k.

(The symbol $'$ is omitted from the name of the cycle, as it is determined by the superscript $\bullet$.)

CM cycles. — Let $(\varphi, \tau) \in \mathrm{LT}(E, F)^0$. For $n \geq c(\varphi, \tau)$, consider the morphism

$$\begin{aligned} \varpi^n f^\flat_{(\varphi,\tau)} \colon {}'\mathscr{G}_{E,?} &\to {}'\mathscr{G}^2_{F,?} \\ y &\mapsto \varphi \circ y \circ \varpi^n \tau'. \end{aligned} \tag{3.4.3}$$

(Its name is meant to suggest the idea that $\varpi^n f^\flat_{(\varphi,\tau)}$ may be thought of as the composition of $[\varpi^n]$ and an, in general non-existent, morphism $f^\flat_{(\varphi,\tau)}$.)

Definition 3.4.2. — Let $? \in \{\emptyset, [N]\}$, let $' \in \{*, \vee\}$, and let $\bullet = \emptyset$ if $' = \vee$, $\bullet = \circ$ if $' = *$. The *CM cycle* attached to (φ, τ) is

$$[Z^\flat(\varphi,\tau)]_?^\bullet := d_n^{-1} \cdot (\varpi^n f^\flat_{(\varphi,\tau)})_*[{}'\mathscr{G}_{E,?}]^\bullet \qquad \in K_0({}'\mathscr{G}^2_{F,?})_{\mathbf{Q}},$$

where in the case of $' = *$, the normalisation is made viewing ${}^*\mathscr{G}_{E,[N]}$ as a scheme of degree ed_N over the base k.

It is easy to verify that the definition is independent of $n \geq c(\varphi, \tau)$.

For $Z^\flat = Z^\flat(\varphi,\tau), V^\flat(\lambda)$, we lighten the notation by $[Z^\flat]_N^\bullet := [Z^\flat]_{[N]}^\bullet$.

Lemma 3.4.3. — *With reference to the maps in* (3.4.2)*, for* $Z^\flat = Z^\flat(\varphi,\tau)$*,* $V^\flat(\lambda)$ *we have*

$$\pi_{N,*}[Z^\flat]^\circ = [Z^\flat]_N^\circ = d_N^{-1} \cdot [Z^\flat]_N = d_N^{-1} \cdot j_N^*[Z^\flat].$$

Proof. — Clear from the definitions. □

Remark 3.4.4. — We compare our setup and our CM cycles with those of [Li] (specialised to the case $K = E$, $h = 1$). Let $\mathrm{t} := \mathrm{tr}_{E/F} \in \mathrm{Hom}_{\mathscr{O}_F}(\mathscr{O}_E, \mathscr{O}_F) =: \mathscr{D}_E^{-1}$, and recall from the introduction that δ^{-1} denotes an $\mathscr{O}_E$-generator of $\mathscr{D}_E^{-1}$. Denote by $(-)^{\vee_L}$ the duality defined in [Li, first paragraph of § 2 and paragraph after (2.3)], and denoted by $(-)^\vee$ there. For $\tau \in \mathrm{Isom}_F(E, F^2)$, let

$$\tau_L := \tau\delta\mathrm{t} \qquad \in \mathrm{Isom}_F(E,F^2) \otimes_{\mathscr{O}_E} \mathscr{D}_E \otimes_{\mathscr{O}_E} \mathscr{D}_E^{-1} = \mathrm{Isom}_F(E,F^2).$$

Then $\tau_L^{\vee_L} = \mathrm{t}^{-1}\tau_L^\vee = \delta\tau^\vee$. Moreover, by comparing (3.3.2) and [Li, (2.13)], we see that $(\varphi,\tau) \in \mathrm{LT}(E,F)^0$ if and only if (φ, τ_L) belongs to set $\mathrm{Equi}_1(E/F)$ of [Li, Definition 2.19]. Finally, it follows from the respective definitions that our cycle $[Z^\flat(\varphi,\tau)]$ equals the cycle $Z_\infty(\varphi, \tau_L)$ of [Li, Definition 3.10].

3.5. Approximation. — We show that the special cycles in $T\mathscr{G}_F^2 = {}^*\mathscr{G}_F^2$ approximate those in $\mathscr{M}_\infty$. The comparison is done via the following maps:
(3.5.1)
$$\begin{array}{ccc} {}^*\mathscr{G}_F^2 \xrightarrow{\varpi^m a_F} \mathscr{M}_\infty & {}^*\mathscr{G}_E \xrightarrow{\varpi^m a_E} \mathscr{N}_\infty & {}^*\mathscr{G}_F \xrightarrow{\varpi^m a_{\mathrm{Ig}}} \mathrm{Ig}_\infty \\ \\ \alpha \longmapsto [\mathscr{G}_F, \mathrm{id}, \varpi^m\alpha], & \beta \longmapsto [\mathscr{G}_E, \mathrm{id}, \varpi^m\beta], & \gamma \longmapsto [\mathscr{G}_F, \mathrm{id}, \varpi^m\gamma]. \end{array}$$

(The notation is meant to evoke non-existent maps a_F, a_E, a_{Ig}.)

Lemma 3.5.1. — *For all $m \geq 1$, the maps $\varpi^m a_F$, $\varpi^m a_E$, $\varpi^m a_{\mathrm{Ig}}$ are well-defined closed immersions of finite origin (Definition A.1.1). More precisely, for any N, those maps respectively originate from the closed immersions*

$$\varpi^m a_F\colon {}^*\mathscr{G}^2_{F,[N]} \hookrightarrow \mathscr{M}_{N+m}, \quad \varpi^m a_E\colon {}^*\mathscr{G}_{E,[N]} \hookrightarrow \mathscr{N}_{N+m}, \quad \varpi^m a_{\mathrm{Ig}}\colon {}^*\mathscr{G}_{F,[N]} \hookrightarrow \mathrm{Ig}_{N+m}. \tag{3.5.2}$$

For $N=0$, the maps (3.5.2) are simply the inclusion of the closed point $\mathrm{Spec}\, k$.

Proof. — The well-definedness follows from [Li, Lemma 3.5]. The rest of the statement, at least for $N=0$ (which suffices) and for a_F and a_E, is a special case of [Li, Proposition 3.9 (1)] – in whose notation (to which we add a subscript 'L' for clarity) we should take $h_L = 2, K_L = F_L = F$ (for a_F) or $h_L = 1, K_L = F_L = E$ (for a_E), and $\varphi_L = \tau_L = \mathrm{id}$. The arguments of *loc. cit.* also apply to the (easier) case of a_{Ig}. □

Proposition 3.5.2. — *For all $m \geq 1$ and $n \geq c(\varphi,\tau)$, the following diagrams are Cartesian:*

$$\begin{tikzcd} {}^*\mathscr{G}_E \arrow[r, "\varpi^n f^\flat_{(\varphi,\tau)}"] \arrow[d, "\varpi^{m+n} a_E"'] & {}^*\mathscr{G}_F^2 \arrow[d, "\varpi^m a_F"] \\ \mathscr{N}_\infty \arrow[r, "f_{(\varphi,\tau)}"] & \mathscr{M}_\infty, \end{tikzcd} \qquad \begin{tikzcd} {}^*\mathscr{G}_F \arrow[r, "f^\flat_\lambda"] \arrow[d, "\varpi^m a_{\mathrm{Ig}}"'] & {}^*\mathscr{G}_F^2 \arrow[d, "\varpi^m a_F"] \\ \mathrm{Ig}_\infty \arrow[r, "f_\lambda"] & \mathscr{M}_\infty. \end{tikzcd}$$

Proof. — We verify the statement for the second diagram. The proof for the first diagram is similar and also a special case of [Li, Proposition 3.9 (2)]. It suffices to verify that the diagram is Cartesian on A-valued points functorially in objects A of $\mathscr{C}_k$. We write the proof for the functors in infinite level where the idea is clearest. The A-point $\alpha_0\colon \mathscr{O}_F^{*,2} \to \mathscr{G}_F$ of ${}^*\mathscr{G}_F^2$ is a point of the Cartesian product of the diagram (which is a closed subscheme of ${}^*\mathscr{G}_F^2$ as f_λ is a closed immersion) if and only if $\varpi^m\alpha_0 = \gamma\circ\lambda$ factors through λ and, obviously, through $[\varpi^m]$. As $\mathrm{Ker}(\lambda)$ is divisible, this is equivalent to $\gamma = \varpi^m\gamma_0$ for some $\gamma_0\colon \mathscr{O}_F^* \to \mathscr{G}_F$, that is $\alpha_0 = f_\lambda(\gamma_0)$ for $\gamma_0 \in {}^*\mathscr{G}_F$. □

Corollary 3.5.3. — *We have*

$$\begin{aligned} d_{E,m}\cdot(\varpi^m a_F)^*[Z(\varphi,\tau)]^\circ &= [Z^\flat(\varphi,\tau)]^\circ, \\ d_{\mathrm{Ig},m}\cdot(\varpi^m a_F)^*[V(\lambda)]^\circ &= [V^\flat(\lambda)]^\circ. \end{aligned} \tag{3.5.3}$$

in $K_0({}^*\mathscr{G}_F^2)_{\mathbf{Q}}$.

Proof. — By Lemma 3.5.1, we have

$$(\varpi^m a_E)^*\mathscr{O}_{\mathscr{N}_{N+m}} = \mathscr{O}_{\mathscr{G}_{E,[N]}}, \qquad (\varpi^m a_{\mathrm{Ig}})^*\mathscr{O}_{\mathrm{Ig}_{N+m}} = \mathscr{O}_{\mathscr{G}_{F,[N]}} \tag{3.5.4}$$

for all N. Then,

$$\begin{aligned} (\varpi^m a_F)^*[Z(\varphi,\tau)]^\circ &= (\varpi^m a_F)^* f_{(\varphi,\tau),*}[\mathscr{N}_\infty]^\circ \\ &= (\varpi^n f^\flat_{(\varphi,\tau)})_*(\varpi^{m+n} a_E)^*[\mathscr{N}_\infty]^\circ = d^{-1}_{E,m+n}\cdot(\varpi^n f^\flat_{(\varphi,\tau)})_*[{}^*\mathscr{G}_E]^\circ, \end{aligned}$$

where the second equality follows from Proposition 3.5.2 and the pullback-pushforward formula (Proposition A.2.1), and the last one follows from (3.5.4). This proves the first identity and the same argument works for the second one. □

4. Intersection numbers

We wish to compute the intersection multiplicity

$$m([Z(\varphi,\tau)_U],[V(\lambda)_U]) = d_{\psi_\tau^{-1}(U)} d_{U\cap\mathcal{O}_F^\times} \cdot m([Z(\varphi,\tau)_U]^\circ,[V(\lambda)_U]^\circ)$$

in $\mathcal{M}_U$ (here, the constants $d_?$ are as in (3.1.1)). Similarly to what is done in [Li] for intersections of CM cycles, we first compare this with an intersection multiplicity in formal groups, then compute the latter.

4.1. Comparison. — In the next proposition, we compare intersection multiplicities in Lubin–Tate spaces and in formal groups.

Define for all levels U and integers N:

$$\begin{aligned} m^\circ([Z(\varphi,\tau)_U]^\circ,[V(\lambda)_U]^\circ) &:= \mathrm{vol}(U)^{-1}\cdot m([Z(\varphi,\tau)_U]^\circ,[V(\lambda)_U]^\circ) \\ m^\circ([Z^\flat(\varphi,\tau)]_N^\circ,[V^\flat(\lambda)]_N^\circ) &:= d_N^2\cdot m([Z^\flat(\varphi,\tau)]_N^\circ,[V^\flat(\lambda)]_N^\circ), \end{aligned} \tag{4.1.1}$$

where $\mathrm{vol}(U)$ is normalised by $\mathrm{vol}(\mathrm{GL}_2(\mathcal{O}_F)) = 1$. We note that for all $N \geq 1$,

$$\mathrm{vol}(U_N) = \zeta_F(1)\zeta_F(2)q^{-4N}.$$

Proposition 4.1.1. — *1. The limits*

$$\begin{aligned} m^\circ([Z(\varphi,\tau)]^\circ,[V(\lambda)]^\circ) &:= \lim_{U\to\{1\}} m^\circ([Z(\varphi,\tau)_U]^\circ,[V(\lambda)_U]^\circ) \\ m^\circ([Z^\flat(\varphi,\tau)]^\circ,[V^\flat(\lambda)]^\circ) &:= \lim_{N\to\infty} m^\circ([Z^\flat(\varphi,\tau)]_N^\circ,[V^\flat(\lambda)]_N^\circ) \end{aligned} \tag{4.1.2}$$

exist as limits of eventually constant sequences.

2. We have the following identity relating intersection multiplicities in $\mathcal{M}_\infty$, $(T\mathcal{G}_F)^2$, and $\mathcal{G}_F^2$:

$$m^\circ([Z(\varphi,\tau)]^\circ,[V(\lambda)]^\circ) = \frac{\zeta_E(1)}{\zeta_F(2)}\cdot m^\circ([Z^\flat(\varphi,\tau)]^\circ,[V^\flat(\lambda)]^\circ) = \frac{\zeta_E(1)}{e\zeta_F(2)}\cdot m([Z^\flat(\varphi,\tau)],[V^\flat(\lambda)]).$$

Proof. — To establish the entirety of the proposition, it is enough to show that for all sufficiently large N and M,

$$\begin{aligned} &\frac{\zeta_F(2)}{\zeta_E(1)}\cdot m^\circ([Z(\varphi,\tau)]_{N+M}^\circ,[V(\lambda)]_{N+M}^\circ) = m^\circ([Z^\flat(\varphi,\tau)]_N^\circ,[V^\flat(\lambda)]_N^\circ) \\ &\quad = m([Z^\flat(\varphi,\tau)]_{N+M},[V^\flat(\lambda)]_{N+M}) = m([Z^\flat(\varphi,\tau)],[V^\flat(\lambda)]), \end{aligned} \tag{4.1.3}$$

where for a special cycle Z in $\mathcal{M}_\infty$ we have denoted $[Z]_N^\circ := [Z_{U_N}^\circ]$.

The second equality follows trivially from the definitions and Lemma 3.4.3. The third one is proved exactly in the same way as [Li, (4.10)]. We recall Li's idea: the intersection number in $\mathcal{G}_F^2$, as the length of a module supported at the closed point of $\mathcal{G}_F^2$, is the same as the intersection number of the restrictions to a sufficiently thick Artinian thickening

$$\mathrm{Spf}\,\mathcal{O}_{\mathcal{G}_F^2}/\mathfrak{m}_{\mathcal{G}_F^2}^{q^{2N}} \overset{\iota_N}{\hookrightarrow} \mathcal{G}_{F,[N]}^2 \subset \mathcal{G}_F^2,$$

where the natural inclusion ι_N is the map denoted by j_M in [Li, (4.16)] (with the M of *loc. cit.* being equal to our N). Here, "sufficiently thick" means that q^{2N} should be greater than the intersection number.

We now prove the first equality based on the argument to prove [Li, (4.8), (4.9)], only writing the details that are different from *loc. cit.* By (the proof of) Corollary 3.5.3, for all sufficiently large N and M we have

$$d_{E,M}d_{\mathrm{Ig},M}d_{2N}\cdot m\left((\varpi^M a_F)^*[Z(\varphi,\tau)]_{N+M}^\circ,(\varpi^M a_F)^*[V(\lambda)]_{N+M}^\circ\right) = d_{2N}\cdot m\left([Z^\flat(\varphi,\tau)]_N^\circ,[V^\flat(\lambda)]_N^\circ\right)$$

as intersections in $(\mathscr{G}_{F,[N]})^2$. Thus it suffices to observe that

$$d_{E,M}d_{\mathrm{Ig},M}d_{2N}=\zeta_E(1)^{-1}\zeta_F(1)^{-1}\cdot q^{4(N+M)}=\zeta_E(1)^{-1}\zeta_F(2)\cdot\mathrm{vol}(U_{N+M})^{-1},$$

and to show that

$$m((\varpi^M a_F)^*[Z(\varphi,\tau)]^\circ_{N+M},(\varpi^M a_F)^*[V(\lambda)]^\circ_{N+M})=m([Z(\varphi,\tau)]^\circ_{N+M},[V(\lambda)]^\circ_{N+M}). \tag{4.1.4}$$

The argument on restrictions to sufficiently thick Artinian thickenings sketched above for the first equality in (4.1.3) also applies to prove (4.1.4): by [Li, §4.4], $\varpi^M a_F\circ\iota_N$ identifies $\mathrm{Spf}\,\mathscr{O}_{\mathscr{M}_{N+M}}/\mathfrak{m}^{q^{2N}}_{\mathscr{M}_{N+M}}\subset\mathscr{M}_{N+M}$ with $\mathrm{Spf}\,\mathscr{O}_{\mathscr{G}_F^2}/\mathfrak{m}^{q^{2N}}_{\mathscr{G}_F^2}\subset\mathscr{G}_F^2$. □

Corollary 4.1.2. — *We have*

$$m^\circ([Z(\varphi,\tau)_U]^\circ,[V(\lambda)_U]^\circ)=\frac{\zeta_E(1)}{\zeta_F(2)}\cdot\mathrm{vol}(U)^{-1}\cdot\int_U m([Z^\flat(\varphi,\tau)],[V^\flat(\lambda g)])\,dg.$$

Proof. — By the proposition, we may replace the integrand with $m^\circ([Z(\varphi,\tau)_{U'}]^\circ,[V(\lambda g)_{U'}]^\circ)$ for sufficiently small U'. Then the result follows from the projection formula and the observation that

$$\pi^*_{U'U}[V(\lambda)_U]^\circ=\pi^*_{U'U}\pi_{U'U,*}[V(\lambda)_{U'}]^\circ=\sum_{g\in U/U'}[V(\lambda g)_{U'}]^\circ.$$

□

4.2. Computation. — We first write CM (respectively vertical) cycles in $\mathscr{G}_F^2$ as rational multiples of cycles that are image (respectively kernel) of a map in $\mathrm{Hom}_{\mathscr{O}_F}(\mathscr{G}_F,\mathscr{G}_F^2)=\mathscr{O}_D^2$ viewed as column vectors (respectively $\mathrm{Hom}_{\mathscr{O}_F}(\mathscr{G}_F^2,\mathscr{G}_F)=\mathscr{O}_D^2$ viewed as row vectors).

Notation. — By the identification $\mathscr{G}_E=\mathscr{G}_F$ as $\mathscr{O}_F$-modules, we have an embedding

$$\mathscr{O}_E=\mathrm{End}_{\mathscr{O}_E\text{-Mod}}(\mathscr{G}_E)\hookrightarrow\mathscr{O}_D=\mathrm{End}_{\mathscr{O}_F\text{-Mod}}(\mathscr{G}_F). \tag{4.2.1}$$

Therefore both $M_2(E)$ and D are embedded in

$$M_2(D)=\mathrm{Hom}_{\mathscr{O}_F}(\mathscr{G}_F^2,\mathscr{G}_F^2)\otimes F.$$

We denote by nrd the reduced norm of D, which restricts to the relative norm $N_{E/F}$ of E/F under (4.2.1).

Recall from the introduction that we denote by $\lambda^\perp\colon\mathscr{O}_F^2\twoheadrightarrow\mathscr{O}_F$ a map such that $\mathrm{Ker}(\lambda^\perp)=\mathrm{Im}(\lambda)$, and by $\delta^{-1}\in\mathscr{D}_E^{-1}=\mathrm{Hom}_{\mathscr{O}_F}(\mathscr{O}_E,\mathscr{O}_F)$ a generator. For $\tau\in\mathrm{Isom}_F(E,F^2)$, write

$$\tau\delta=\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\in\mathrm{Hom}(\mathscr{O}_E,\mathscr{O}_F^2)\otimes_{\mathscr{O}_E}\mathscr{D}_E\otimes_{\mathscr{O}_F}F=E^2,\qquad\lambda^\perp=(a,b)\in(\mathscr{O}_F^2)^\vee. \tag{4.2.2}$$

Let

$$(\tau\delta\mid\overline{\tau\delta})=\Gamma_\tau\begin{pmatrix}P_\tau & *\\ & Q_\tau\end{pmatrix},\qquad\Gamma_\tau\in\mathrm{GL}_2(\mathscr{O}_E), \tag{4.2.3}$$

be an Iwasawa decomposition in $\mathrm{GL}_2(E)$.

Lemma 4.2.1. — *With notation as in* (4.2.2), (4.2.3), *we have*

$$[V^\flat(\lambda)]=[\mathrm{Ker}((a\ b))],$$
$$[Z^\flat(\varphi,\tau)]=|\mathrm{nrd}(\varphi P_\tau)|^{-1}\cdot[\mathrm{Im}(\varphi\Gamma_\tau\varphi^{-1}\begin{pmatrix}1\\0\end{pmatrix})],$$

where the map $\varphi\Gamma_\tau\varphi^{-1}\binom{1}{0}\colon\mathscr{G}_E\to\mathscr{G}_F^2$ is a closed immersion.

Proof. — The first equality is clear. The second one is a special case of [Li, (5.3)]; under the comparison of setups from Remark 3.4.4, our vector $\binom{\theta_1}{\theta_2}$ corresponds to the vector M_{τ_L} of [Li, Definition 5.4]. □

We may now compute the intersection in $\mathscr{G}_F^2$.

Proposition 4.2.2. — *With notation as in* (4.2.2), (4.2.3), *we have*

$$m([Z^\flat(\varphi,\tau)],[V^\flat(\lambda)]) = |\mathrm{nrd}(\varphi)|^{-1}\cdot|\lambda^\perp\tau\delta|_E^{-1}.$$

Proof. — By the previous lemma, $m([Z^\flat(\varphi,\tau)],[V^\flat(\lambda)])$ equals $|\mathrm{nrd}(\varphi P_\tau)|^{-1}$ times the degree of the group scheme kernel of

$$(a\ b)\circ\varphi\Gamma_\tau\varphi^{-1}\begin{pmatrix}1\\0\end{pmatrix}\colon \mathscr{G}_F\to\mathscr{G}_F, \tag{4.2.4}$$

that is, the degree of the isogeny (4.2.4). By [Li, Lemma 5.7], this is

$$|\mathrm{nrd}((a\ b)\varphi\Gamma_\tau\varphi^{-1}\begin{pmatrix}1\\0\end{pmatrix})|^{-1} = |\mathrm{nrd}((a\ b)\Gamma_\tau\begin{pmatrix}1\\0\end{pmatrix})|^{-1}.$$

Thanks to the decompositions

$$\begin{aligned}(\tau\delta\ \ \overline{\tau\delta}) := \begin{pmatrix}\theta_1 & \overline{\theta}_1\\ \theta_2 & \overline{\theta}_2\end{pmatrix} &= \begin{pmatrix}1 & \\ \theta_2/\theta_1 & 1\end{pmatrix}\begin{pmatrix}\theta_1 & \overline{\theta}_1\\ & \overline{\theta}_2-\overline{\theta}_1\theta_2/\theta_1\end{pmatrix}\\ &= \begin{pmatrix}\theta_1/\theta_2 & 1\\ 1 & \end{pmatrix}\begin{pmatrix}\theta_2 & \overline{\theta}_2\\ & \overline{\theta}_1-\theta_1\overline{\theta}_2/\theta_2\end{pmatrix}\end{aligned}$$

we may compute the elements Γ_τ, P_τ from (4.2.3). We only write down the details if $v(\theta_1)\le v(\theta_2)$: then the first decomposition is an Iwasawa decomposition, and

$$\begin{aligned}m([Z^\flat(\varphi,\tau)],[V^\flat(\lambda)]) &= |\mathrm{nrd}(\varphi\theta_1)|^{-1}|\mathrm{nrd}((a\ b)\left(\begin{smallmatrix}1 & \\ \theta_2/\theta_1 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))|^{-1}\\ &= |\mathrm{nrd}(\varphi)|^{-1}\cdot|N_{E/F}(a\theta_1+b\theta_2)|^{-1},\end{aligned}$$

which is the desired formula. □

Inserting this result into Corollary 4.1.2 and removing the constants indicated by the superscripts '∘', we obtain the following local analogue of Theorem B.

Corollary 4.2.3. — *We have*

$$m([Z(\varphi,\tau)_U],[V(\lambda)_U]) = \frac{\zeta_E(1)}{\zeta_F(2)}\cdot d_{\psi_\tau^{-1}(U)}d_{U\cap\mathscr{O}_F^\times}\cdot|\mathrm{nrd}(\varphi)|^{-1}\cdot\int_U|\lambda^\perp g\tau\delta|_E^{-1}\,dg,$$

where dg is the Haar measure such that $\mathrm{vol}(\mathrm{GL}_2(\mathscr{O}_F))=1$.

It is a not unpleasant exercise to verify that, for $U=\mathrm{GL}_2(\mathscr{O}_F)$, this formula agrees with (2.2.2).

4.3. Conclusion. — We are now in a position to complete the proof of the main theorems.

Corollary 4.3.1. — *Let $V\neq V'$ be irreducible components of $\mathscr{M}_{U,k}$. The intersection multiplicity $m(Z,V')$ is bounded as Z varies among CM curves with generic fibre a point of B_V.*

A qualitatively similar result from a somewhat different point of view can be deduced from [Dis, Proposition 4.3.3]; the relation '$z\in B_V$' should be compared to the relation '$V(z)=V$' in the notation of *loc. cit.*

Proof. — Let n be such that $U\supset U_n$, let $V=V(\lambda)_U$, $V'=V(\lambda')_U$, and let $z=z(\varphi,\tau)_U\in B_V$ have conductor $s\ge n$. Up to changing the choice of representatives φ and τ, we may assume that $\mathscr{O}_F^2\subset\tau(\mathscr{O}_E)\subset\varpi^{-s}\mathscr{O}_F^2$, and $\mathrm{ht}(\varphi)=s$.

Suppose that $z=z(\varphi,\tau)\in B_{V(\lambda)_U}$ is a CM point of conductor s and write $\tau\delta=\left(\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right)$ as in (4.2.2). Up to changing the choice of representatives φ and τ, we may assume that

$$\mathscr{O}_F^2\subset\tau(\mathscr{O}_E)\subset\varpi^{-s}\mathscr{O}_F^2$$

optimally (that is, $\varpi^{-1}\mathscr{O}_F^2 \not\subset \tau(\mathscr{O}_E) \not\subset \varpi^{1-s}\mathscr{O}_F^2$), and $\mathrm{ht}(\varphi)=s$. We examine the terms of the multiplicity formula of Corollary 4.2.3.

We have $|\mathrm{nrd}(\varphi)|^{-1}=q^s$ and

$$d_{\psi_\tau^{-1}(U)} \leq d_{\psi_\tau^{-1}(U_0)} = [\mathscr{O}_E^\times : \psi_\tau^{-1}(U_0)] = e\zeta_E(1)\zeta_F(1)^{-1}\cdot q^s.$$

Thus it suffices to show that the integrand $|\lambda'^{\perp}g\tau\delta|_E^{-1}$ is bounded by a constant multiple of $q^{-2s} = |\varpi^{-s}|_E^{-1}$. Let v be the valuation on E normalised by $v(\varpi)=1$ (thus v may take half-integer values if E/F is ramified). We will prove more precisely that

$$v(\lambda'^{\perp}g\tau\delta)\geq n-s \quad\Longrightarrow\quad z(\varphi,\tau)\in B_{V(\lambda')_U}. \tag{4.3.1}$$

In fact $v(\lambda'^{\perp}g\tau\delta)\geq n-s$ if and only if $\lambda'^{\perp}g$ takes integral values on $\mathrm{Im}(\varpi^{s-n}\tau)\subset \varpi^{-n}\mathscr{O}_F^2/\mathscr{O}_F^2$. Equivalently, the first among the lines

$$L_n(\tau):=\mathrm{Im}(\varpi^{s-n}\tau)/\mathscr{O}_F^2,\quad L_n(g\lambda'):=\mathrm{Ker}[\varpi^{-n}\mathscr{O}_F/\mathscr{O}_F)^2 \stackrel{\lambda'^{\perp}g}{\to} \varpi^{-n}\mathscr{O}_F/\mathscr{O}_F] \quad \subset (\varpi^{-n}\mathscr{O}_F/\mathscr{O}_F)^2$$

is contained in the second one, hence the two coincide. Up to changing g by a $g'\in U_n\subset U$, the analogous lines $L_s(\tau)$ and $L_s(g\lambda')$ also coincide; by definition (see (1.2.2)), this means that $z(\varphi,\tau)\in B_{V(\lambda')_U}$. □

Proof of Theorems A and B. — By Lemma 3.3.4, Corollary 4.2.3 implies Theorem B (whose constant $c_{E_v}=\zeta_{E_v}(1)/\zeta_{F_v}(2)$), and Corollary 4.3.1 implies Proposition 2.2.5. As shown in § 2.2, the latter implies Theorem A. □

Appendix A. Cycles in formal schemes and their projective limits

A.1. Pro-Noetherian formal schemes. — Consider the category $\mathscr{F}$ of separated Noetherian finite-dimensional formal schemes and finite flat maps. We let $\widehat{\mathscr{F}}$ be the pro-category of $\mathscr{F}$; that is, objects of $\widehat{\mathscr{F}}$ are formal inverse limits

$$X=\varprojlim_{i\in I} X_i$$

of filtered inverse systems of objects and maps in $\mathscr{F}$, and morphisms are defined by

$$\mathrm{Hom}(X,Y)=\varprojlim_j \varinjlim_i \mathrm{Hom}(X_i,Y_j). \tag{A.1.1}$$

Henceforth we will simply refer to objects of $\mathscr{F}$ (respectively $\widehat{\mathscr{F}}$) as Noetherian formal schemes (respectively pro-Noetherian formal schemes) for short.

Definition A.1.1. — We say that a morphism $f\colon X\to Y=\varprojlim_{j\in J} Y_j$ in $\widehat{\mathscr{F}}$ is

- *of finite origin* if there exist a map $f_0\colon X_0\to Y_0$ in $\mathscr{F}$, a map $Y\to Y_0$ in $\widehat{\mathscr{F}}$, and an isomorphism
$$X\cong \varprojlim_{j\in J} X_0\times_{Y_0} Y_j$$
over Y. In this case we say that f *originates from* f_0.
- *a closed immersion* of finite origin if it is of finite origin and it originates from a closed immersion in $\mathscr{F}$.

The *image* of a morphism $\lim f_{ij}\colon X\to Y$ as in (A.1.1) is the object $\varprojlim f_j(X)$ with respect to the restrictions of the transition maps of (Y_j), where $f_j(X):=f_{ij}(X_i)$ for any sufficiently large i.

If $f\colon X\to Z$, $g\colon Y\to Z$ are morphisms in $\widehat{\mathscr{F}}$ with g of finite origin, the Cartesian product $X\times_Z Y$ is (well-)defined, up to isomorphism, as follows. Assume that g originates from $g_0\colon Y_0\to Z_0$, and write $X=\varprojlim X_i$. Then

$$X\times_Z Y:=\varprojlim_i X_i\times_{Z_0} Y_0.$$

A.2. Cycles, K-groups, intersections. — Let X be a Noetherian formal scheme. We define the K-group of X with $\mathbf{Q}$-coefficients

$$K_0(X)_{\mathbf{Q}}$$

in the usual way as the Grothendieck group (tensored with $\mathbf{Q}$) of the category of coherent sheaves on X (cf. [Ful98, §15.1]). If $X = \varprojlim_i X_i$ is a pro-Noetherian formal scheme, we define

$$K_0(X)_{\mathbf{Q}} := \varprojlim_i K_0(X_i)_{\mathbf{Q}}$$

where the limits are with respect to pushforward.

Fundamental classes. — Let X be a Noetherian formal scheme. Its fundamental class is

$$[X] := [\mathscr{O}_X] \in K_0(X)_{\mathbf{Q}}.$$

Let X be a pro-Noetherian formal scheme with a morphism to a Noetherian formal scheme B; equivalently, for all sufficiently large i we have a finite flat map $X_i \to B$ compatibly in the system. We define the normalised (relative to B) fundamental class of X to be

$$[X]^\circ := \lim [X_i]^\circ \in K_0(X)_{\mathbf{Q}}, \qquad [X_i]^\circ := \frac{1}{\deg(X_i/B)}[X_i] \qquad \in K_0(X_i)_{\mathbf{Q}}.$$

Pullback and pushfoward. — It is easy to verify that $K_0(-)_{\mathbf{Q}}$ enjoys pushforward (respectively, pullback) functoriality with respect to arbitrary (respectively, finite-origin) morphisms in $\widehat{\mathscr{F}}$.

Proposition A.2.1. — *Let $f : X \to Z$, $g : Y \to Z$ be morphisms in $\widehat{\mathscr{F}}$ with g of finite origin, and consider the Cartesian diagram*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z. \end{tikzcd}$$

For $C \in K_0(X)_{\mathbf{Q}}$, we have

$$g^* f_* C = f'_* g'^* C.$$

Proof. — Write $X = \varprojlim_i X_i$, $C = \varprojlim C_i$, and assume that g originates from $g_0 : Y_0 \to Z_0$. For sufficiently large i, let $f_i : X_i \to Z_0$ be the relevant component of $X \xrightarrow{f} Z \to Z_0$, and consider the Cartesian diagram

$$\begin{tikzcd} X_i \times_{Z_0} Y_0 \arrow[r, "f_i'"] \arrow[d, "g_0'"] & Y_0 \arrow[d, "g_0"] \\ X_i \arrow[r, "f_i"] & Z_0. \end{tikzcd}$$

By the definitions, the desired formula reduces to $g_0^* f_{i,*} C_i = f'_{i,*} g_0'^* C_i$, which is just the usual pullback-pushforward formula in $\mathscr{F}$. □

Intersection multiplicities. — If X is a Noetherian formal scheme and $x \in X$ is a regular point, the intersection multiplicity function

$$m_x : K_0(X)_{\mathbf{Q}} \times K_0(X)_{\mathbf{Q}} \to \mathbf{Q}$$

is defined via Serre's Tor formula [Ful98, §20.4]. It is a symmetric bilinear form such that if $X = \mathscr{X}'$ and Z_1, Z_2 are as in (1.3.1), then

$$m_x([Z_1], [Z_2]) = m_x(Z_1, Z_2)$$

where the right-hand side is the plain intersection multiplicity defined in (1.3.1). The subscript x is omitted when $|X|$ consists of a a single point.

References

[Ber90] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990. MR1070709 ↑5, 7, 9

[Car86] Henri Carayol, *Sur la mauvaise réduction des courbes de Shimura*, Compositio Math. **59** (1986), no. 2, 151–230 (French). MR860139 (88a:11058) ↑2, 3, 9, 12, 13, 14

[CL06] Antoine Chambert-Loir, *Mesures et équidistribution sur les espaces de Berkovich*, J. Reine Angew. Math. **595** (2006), 215–235, DOI 10.1515/CRELLE.2006.049 (French). MR2244803 ↑6

[CU04] Laurent Clozel and Emmanuel Ullmo, *Équidistribution des points de Hecke*, Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 193–254 (French). MR2058609 ↑1

[CV05] C. Cornut and V. Vatsal, *CM points and quaternion algebras*, Doc. Math. **10** (2005), 263–309. MR2148077 ↑2

[Dis] Daniel Disegni, *The p-adic Gross–Zagier formula on Shimura curves, II: nonsplit primes*, J. Inst. Math. Jussieu, to appear, DOI 10.1017/S1474748021000608. ↑6, 21

[Dri74] V. G. Drinfel′d, *Elliptic modules*, Mat. Sb. (N.S.) **94(136)** (1974), 594–627, 656 (Russian). MR0384707 ↑10, 12

[Duk88] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms*, Invent. Math. **92** (1988), no. 1, 73–90, DOI 10.1007/BF01393993. MR931205 ↑1

[Ful98] William Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998. MR1644323 ↑23

[GK21] Eyal Goren and Payman Kassaei, *p-adic dynamics of Hecke operators on modular curves*, J. Théor. Nombres Bordeaux **33** (2021), no. 2, 387–431. ↑6

[Gro86] Benedict H. Gross, *On canonical and quasicanonical liftings*, Invent. Math. **84** (1986), no. 2, 321–326, DOI 10.1007/BF01388810. MR833193 (87g:14051) ↑3, 4, 10

[HMRL20] Sebastián Herrero, Ricardo Menares, and Juan Rivera-Letelier, *p-adic distribution of CM points and Hecke orbits I: Convergence towards the Gauss point*, Algebra Number Theory **14** (2020), no. 5, 1239–1290, DOI 10.2140/ant.2020.14.1239. MR4129386 ↑2, 4

[HMRL] ———, *p-adic distribution of CM points and Hecke orbits. II: Linnik equidistribution on the supersingular locus*, preprint. ↑2, 6

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985. MR772569 ↑2, 3, 6, 13

[Li] Qirui Li, *An intersection formula for CM cycles on Lubin–Tate spaces*, Duke Math. J., to appear; available at arXiv:1803.07553v4. ↑6, 11, 12, 14, 16, 17, 18, 19, 20, 21

[Mor81] Yasuo Morita, *Reduction modulo* $\mathfrak{P}$ *of Shimura curves*, Hokkaido Math. J. **10** (1981), no. 2, 209–238, DOI 10.14492/hokmj/1381758092. MR625590 ↑3

[Meu07] Volker Meusers, *Canonical and quasi-canonical liftings in the split case*, Astérisque **312** (2007), 87–98 (English, with English and French summaries). MR2340373 ↑10

[Ric18] Rodolphe Richard, *Répartition galoisienne ultramétrique d'une classe d'isogénie de courbes elliptiques: le cas de la mauvaise réduction. Application aux hauteurs locales*, J. Théor. Nombres Bordeaux **30** (2018), no. 1, 1–18 (French, with English and French summaries). MR3809706 ↑4

[Wei16] Jared Weinstein, *Semistable models for modular curves of arbitrary level*, Invent. Math. **205** (2016), no. 2, 459–526, DOI 10.1007/s00222-015-0641-5. MR3529120 ↑6

[Wew07] Stefan Wewers, *Canonical and quasi-canonical liftings*, Astérisque **312** (2007), 67–86 (English, with English and French summaries). MR2340372 ↑10

[Zha01] Shou-Wu Zhang, *Gross-Zagier formula for* GL_2, Asian J. Math. **5** (2001), no. 2, 183–290. MR1868935 (2003k:11101) ↑2

[Zha05] ———, *Equidistribution of CM-points on quaternion Shimura varieties*, Int. Math. Res. Not. **59** (2005), 3657–3689, DOI 10.1155/IMRN.2005.3657. MR2200081 ↑1

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