

Euler systems for conjugate-symplectic motives

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The Birch and Swinnerton-Dyer conjecture

E/\mathbf{Q} elliptic curve, $N = N_E$. BSD principle:

$$(|E(\mathbf{F}_p)|)_{p \nmid N} + \text{analysis} \quad \rightsquigarrow \quad r_E = \dim E(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Let $a_p := p + 1 - |E(\mathbf{F}_p)|$ and

$$L(E, s) := \prod_{p \nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1} \cdot L_N(E, s), \quad \Re(s) > 3/2.$$

Hecke, Wiles et al.: $L(E, s) = L(\varphi_E, s)$ has *entire* continuation, f. equation $s \leftrightarrow 2 - s$,
for $\varphi_E(\tau) = \sum_{n \geq 1} a_n q^n \in S_2(N)$.

Conjecture (BSD)

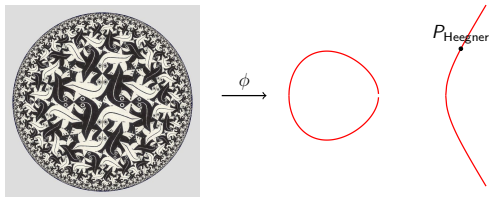
1. $\text{ord}_{s=1} L(E, s) = r \implies r_E = r$.
2. $\kappa: E(\mathbf{Q}) \otimes \mathbf{Q}_p \rightarrow \text{Sel}_p(E)$ is isomorphism $\forall p$ ($\Leftarrow \text{Sha}(E)$ is finite),
where

$$\text{Sel}_p(E) = \left(\varinjlim_n \text{Sel}_{p^n}(E) \right) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \stackrel{\text{Bloch-Kato}}{=} H_f^1(\mathcal{G}_{\mathbf{Q}}, V_p E) \subset H^1(\mathcal{G}_{\mathbf{Q}}, V_p E)$$

The case $r = 1$: Heegner points

BSD is known for $r \leq 1$. If $r = 1$, for a suitable $K = K_D = \mathbf{Q}(\sqrt{-D})$:

- ▶ let $z_{K,1} :=$ CM-by- K point in $X_0(N)(\mathbf{C}) = \Gamma_0(N) \backslash \mathfrak{h}^*$.
Theory of Complex Multiplication: $z_{K,1} \in X_0(N)(H_K)$, so
 $z_K := \text{Tr}_{H_K/K}(z_{K,1}) - [H_K : K] \cdot \infty \in \text{Div}^0(X_0(N)_K) \rightarrow \text{Jac}(X_0(N))(K)$.
- ▶ Modularity (Wiles et al., '93): there is $f : X_0(N) = X_{\text{GL}_2, U_0(N)} \rightarrow E$.
- ▶ Heegner ('52): consider $z_K(f) := f_*(z_K) \in E(K)$.



- ▶ Gross–Zagier ('86): $\text{ord}_{s=1} L(E_K, s) = 1 \iff z_K(f)$ non-torsion.
- ▶ Kolyvagin ('90): if $z_K(f)$ is non-torsion, then it spans $E(K)_{\mathbf{Q}}$ and $H_f^1(K, V_p E)$.

Heegner points/2

A modularity result can eliminate $K = K_D$ and yield $\text{BSD}_{r=1}$ for E/\mathbf{Q} .

- ▶ Define $Z := \sum_D \text{Tr}_{K_D/\mathbf{Q}}(z_{K_D}) q^D \in \text{Jac}(X_0(N))(\mathbf{Q}) \otimes \mathbf{Z}[[q]]$
- ▶ Gross–Kohnen–Zagier's modularity: $Z \in \text{Jac}(X_0(N))(\mathbf{Q}) \otimes M_{3/2}(4N)$.
- ▶ Kohnen, Shimura, Waldspurger: $S_2(N) \leftrightarrow S_{3/2}(4N)^+$.
- ▶ Let $\varphi \in S_{3/2}(4N)^+$ correspond to $\varphi_E \in S_2(N)$, and let

$$\begin{aligned} z(\varphi) &:= \Omega^{-1} \cdot \langle \varphi, Z \rangle_{\text{Pet}} && \in \text{Jac}(X_0(N))(\mathbf{Q})_{\mathbf{Q}}, \\ z(f, \varphi) &:= f_* z(\varphi) && \in E(\mathbf{Q})_{\mathbf{Q}}. \end{aligned}$$

This is a canonical point in $E(\mathbf{Q})_{\mathbf{Q}}$ and

$$\text{ord}_{s=1} L(E, s) = 1 \stackrel{\text{GKZ}}{\iff} z(f, \varphi) \neq 0 \iff r_E = 1 \text{ and } |\text{Sha}(E)| < \infty.$$

Beilinson–Bloch–Kato Conjecture on algebraic cycles

Let K number field, X/K proper smooth variety of *odd* dimension $n - 1$. Consider

- ▶ $\mathrm{Ch}^{n/2}(X) := \mathbf{Q}[Y \subset (n/2) X]_{/\sim}, \quad \supset \quad \mathrm{Ch}^{n/2}(X)^0 := \mathrm{Ker}(\mathrm{cl});$
- ▶ $V = V_p X := H_{\acute{e}t}^{n-1}(X_{\bar{K}}, \mathbf{Q}_p(\frac{n}{2})) \circlearrowleft \mathcal{G}_K,$
 $\rightsquigarrow L(V, s) := \prod_{v \nmid p} \det(1 - \mathrm{Fr}_v \cdot q_v^{-s} | V^{I_v})^{-1} L_{(p)}(V, s).$

Conjecture (BBK). – $L(V, s)$ has entire continuation, f.e.q. $s \leftrightarrow -s$, and

1. $\mathrm{ord}_{s=0} L(V, s) = r \implies \dim \mathrm{Ch}^{n/2}(X)^0 = r;$
2. $\mathrm{AJ}_p: \mathrm{Ch}^{n/2}(X)_{\mathbf{Q}_p}^0 \rightarrow H_f^1(K, V_p X)$ is an isomorphism.

Variant: if V irreducible, geometric p -adic \mathcal{G}_K -representation of weight -1 , let

$$H_f^1(K, V)^{\mathrm{mot}} := \sum_{(X, f: V_p X \rightarrow V)} f_* \mathrm{AJ}_p(\mathrm{Ch}^{n/2}(X)_{\mathbf{Q}_p}^0) \subset H_f^1(K, V).$$

Conjecture (BBK'). – $\mathrm{ord}_{s=0} L(V, s) = \dim H_f^1(K, V)^{\mathrm{mot}} = \dim H_f^1(K, V).$

Main result

Among n -dimensional \mathcal{G}_K -representations as above:

- (1) 'Easier': K is TR field, symplectic \mathcal{G}_K -pairing $V \times V \rightarrow \mathbf{Q}_p(1)$ (e.g. $V_p X$), Hodge–Tate weights are distinct (e.g. $V \doteq \text{Sym}^{n-1} V_p E$).
- (2) 'Easiest': K is CM field, conjugate-symplectic pairing $V \times V^c \rightarrow \mathbf{Q}_p(1)$, (e.g. $V = \text{Ind}_F^K V^{(1)}$, $F = K^+$), HT weights are distinct and consecutive.

Conjecture: $V = V_\Pi$ for cuspidal automorphic $\Pi \circlearrowleft \mathbf{GL}_n(\mathbf{A}_K)$. If so, $L(V, s) \checkmark$.

Theorem A.** – Let V be irreducible, automorphic, (2). Assume (ES), (Mod), for simplicity: $F \neq \mathbf{Q}$, n even. Then for an explicit

$$\Theta \in H_f^1(K, V)^{\text{mot}},$$

$$\text{ord}_{s=0} L(V, s) = 1 \stackrel{?}{\iff} \Theta \neq 0 \implies \dim H_f^1(K, V)^{\text{mot}} = \dim H_f^1(K, V) = 1.$$

Li–Liu ('21): if AJ_p is injective + some assumptions, then \implies holds.

Previous results: low rank V [many!]; $V^{(2)} = V_n \otimes V_{n+1}$ [LTXZZ]; $V^{(1)}$ symplectic [Cornut]; $V^{(2)}$ c -symplectic + symplectic, any regular HT type [Graham–Shah].

Construction of Θ

As $V = V_{\Pi}$ c -symplectic $\implies \Pi = \text{BC}(\sigma)$ for $\sigma \circ \mathbf{G}(\mathbf{A}_F)$, $\mathbf{G} = \mathbf{U}(\frac{n}{2}, \frac{n}{2})/F$;

if $\varepsilon(V) = -1$ (at least) $\implies \Pi = \text{BC}(\pi)$ for $\pi \circ \mathbf{H}(\mathbf{A}_F)$, $\mathbf{H} = \mathbf{U}(W)/F$,

$$\text{sig}(W_{\infty}) = \{(n-1, 1); (n, 0), \dots, (n, 0)\} \quad [\text{GRS, Mok, KMSW}].$$

In fact V appears in the cohomology of the Shimura variety $X = (X_{\mathbf{H}, U})_U/K$:

$$X_{\mathbf{H}, U}(\mathbf{C}) = \mathbf{H}(F) \backslash \mathbf{H}(\mathbf{A}_F^{\infty}) \times \mathbf{B}_{\mathbf{C}}^{(n-1)}, \quad V_p X_{\mathbf{H}, U} = \bigoplus_{\pi' \in \mathcal{A}(\mathbf{H})^{\vee}} \pi'^{V, \infty, U} \otimes V_{\pi'} \quad [\text{Kisin-Shin-Zhu}].$$

- (0) For $\underline{x} \in W^{n/2}$ with $T(\underline{x}) = (x_i, x_j)_{ij} > 0$, let $W(\underline{x}) := \text{Span}(x_i)^{\perp} \subset {}^{(n/2)}W$,
 let $\mathbf{H}(\underline{x}) := \begin{pmatrix} \mathbf{U}(W(\underline{x})) \\ 1 \end{pmatrix} \subset \mathbf{H}$ and $U(\underline{x}) := U \cap \mathbf{H}(\underline{x})$
 \rightsquigarrow sub-unitary Shimura variety $Z(\underline{x})_U := [X_{\mathbf{H}(\underline{x}), U(\underline{x})}] \in \text{Ch}^{n/2}(X_{\mathbf{H}, U})$.

- (1) Kudla: form

$$\Theta(\phi) := \sum_{\underline{x} \in U \backslash W^{n/2} \otimes \mathbf{A}^{\infty} : T(\underline{x}) \geq 0} \phi(\underline{x}) \cdot Z(\underline{x}) \mathbf{q}_{\tau}^{T(\underline{x})}, \quad \tau \in \mathfrak{h}_{\mathbf{G}}, \phi \in \mathcal{S}(W_{\mathbf{A}^{\infty}}^{n/2}).$$

Conjecture (Mod): $\Theta(\phi) \in \text{Ch}^{n/2}(X) \otimes S_{\mathbf{G}}$.

Assume this. *Known: orthogonal Shimura varieties [Zhang, Bruinier–W.Raum];
 if $n = 2$ [Liu]; ‘formally’ [Liu]; for some $[K : \mathbf{Q}] = 2$ [J. Xia].*

Construction of Θ / continuation

(1) Kudla: $\Theta(\phi) := \sum_{T(\underline{x}) \geq 0} \phi(\underline{x}) \cdot Z(\underline{x}) \mathbf{q}^{T(\underline{x})} \in^{\text{(Mod)}} \text{Ch}^{n/2}(X_U) \otimes S_{\mathbf{G}}$.

(2) Liu: for $\varphi \in \sigma^{\vee}$ antiholomorphic, consider the *arithmetic theta lift*

$$\Theta(\varphi, \phi) := \Omega^{-1} \cdot \langle \varphi, \Theta(\phi) \rangle_{\text{Pet}} \in \text{Ch}^{n/2}(X_U)^0 \xrightarrow{\text{AJ}_R} H_f^1(K, \bigoplus_{V_{\pi'}=V} \pi'^{\vee, \infty, U} \otimes V).$$

(3) for $f \in \pi^{\infty, U}$, consider

$$\Theta(f, \varphi, \phi) := f_* \text{AJ}_p \Theta(\varphi, \phi) \in H_f^1(K, V)^{\text{mot}}.$$

This defines

$$\Theta \in \text{Hom}_{(\mathbf{H} \times \mathbf{G})(\mathbf{A}_F^{\infty})}(\pi^{\infty} \otimes \sigma^{\vee, \infty} \otimes S_{\text{Weil}}(W_{\mathbf{A}_K^{\infty}}^{n/2}), \mathbf{Q}_p) \otimes H_f^1(K, V)^{\text{mot}}.$$

The space $\theta(\pi, \sigma) := \nearrow$ is 1D. Our cycle is $\Theta := \Theta(f^1, \varphi^1, \phi^1)$.

Theta dichotomy (correcting half-lies) [Harris-Kudla-Sweet, Gan-Ichino]:

given Π thus σ , there is a unique pair $(W, \pi \circlearrowleft \mathbf{H}_W(\mathbf{A}_F))$ such that

$\dim \theta(\pi, \sigma) = 1 > 0$. We pick this pair.

Euler systems according to Jechev–Nekovář–Skinner

We'll construct an *Euler system* based on Θ , i.e. companion classes $(\Theta_m)_{m \in R}$ satisfying compatibility conditions related to the Euler product of $L(V, s)$.

Theorem* (JNS). – Let $\rho: \mathcal{G}_K \rightarrow \text{Aut}_{\mathbf{Q}_p}(V)$ be abs. irreducible, c -symplectic.

Assume (ES): 1. there is $\gamma_1 \in \mathcal{G}_K$ fixing $K[1](\mathcal{O}_F^{\times, 1/p^\infty})$, such that $\dim V^{\gamma_1} = 1$.

2. there is $\gamma_2 \in \mathcal{G}_K$ fixing $K[1](\mathcal{O}_K^{\times, 1/p^\infty})$, such that $V^{\gamma_2} = 0$.

Let $S := \{\text{split primes } v = w\bar{w} \nmid p \text{ of } F \text{ such that } V \circlearrowleft \mathcal{G}_{K_w} \text{ is unramified}\}$.

Let $R := \{\text{squarefree products of primes in } S\}$.

For $m \in R$, let $K[m] \subset K^{\text{ab}}$ with $\text{Gal}(K[m]/K) = \frac{\text{Cl}(\mathcal{O}_F + m\mathcal{O}_K)}{\text{Cl}(\mathcal{O}_F)} =: C[m]$.

Assume given a collection $z_m \in H_f^1(K[m], V_{z_p})$, $m \in R$, satisfying:

$$\text{Tr}_{K[mv]/K[m]} z_{mv} = P_w(\text{Fr}_w) z_m, \quad P_w(T) := \det(1 - \text{Fr}_w T | V)$$

for all $m \in R$, $v \in S$, $v \nmid m$.

Then

$$z := \text{Tr}_{K[1]/K} z_1 \neq 0 \quad \implies \quad H_f^1(K, V) = \mathbf{Q}_p \cdot z.$$

* statement may not be entirely correct, any errors mine.

Remark: there is enhanced notion of ES when V is P.-ordinary, with classes “ z_{mp^s} ”,
 \implies finer Iwasawa-theoretic consequences.

The Euler system of theta cycles

Theorem B. – Let V be as in Theorem A. There is an Euler system $(\Theta_m)_{m \in \mathbb{R}}$ with $\mathrm{Tr}_{K[1]/K} \Theta_1 = \Theta$. If V is p -ordinary, there is a p -enhanced Euler system.

Let's explain the construction. Recall that

$$\Theta := \Theta(f^1, \varphi^1, \phi^1) = f_*^1 \mathrm{AJ}_p(\langle \varphi^1, \Theta(\phi^1) \rangle), \quad \Theta(\phi^1) = \sum_{T(\underline{x}) \geq 0} \phi^1(\underline{x}) \cdot Z(\underline{x}) \mathbf{q}^{T(\underline{x})}.$$

First, $\Theta_1 \xrightarrow{\mathrm{Tr}_{K[1]/K}} \Theta$ will be a “connected component of the cycle Θ ”.

Let $\mathbf{T} = \mathbf{U}(1)$; for $C \subset \mathbf{T}(\mathbf{A}_F^\infty)$, we have $X_{\mathbf{T}, C} \cong \mathrm{Spec} K_C$ (class field theory).

Fix a basepoint $y_C \in X_{\mathbf{T}, C}(K_C) = \bullet \circ \bullet \bullet$. By Shimura and Deligne,

$\det: X_{\mathbf{H}', U'} \rightarrow X_{\mathbf{T}, \det(U')}$ induces a bijection on π_0 . Let $C(\underline{x}) := \det U(\underline{x})$, and let

$$Z(\underline{x})^C := \begin{cases} [\det_{\mathbf{H}(\underline{x})}^{-1}(y_C)] \subset [X_{\mathbf{H}(\underline{x}), U(\underline{x}), K_C}] & \text{if } C(\underline{x}) \stackrel{\subset}{=} C \\ |C(\underline{x})/C|^{-1} \cdot [\det_{\mathbf{H}(\underline{x})}^{-1}(y_{C(\underline{x})})] & \text{if } C(\underline{x}) \supsetneq C \end{cases} \xrightarrow{\mathrm{Tr}_{K_C/K}} Z(\underline{x}).$$

Let $\Theta(\phi)^C := \sum_{T(\underline{x}) \geq 0} \phi(\underline{x}) \cdot Z(\underline{x})^C \mathbf{q}^{T(\underline{x})}$, still modular if (Mod) holds.

Define

$$H_f^1(K[1], V_{\mathbf{Z}_p}) \ni \Theta_1 := f_{1,*} \mathrm{AJ}_p(\langle \varphi^1, \Theta(\phi^1)^{C[1]} \rangle) \xrightarrow{\mathrm{Tr}_{K[1]/K}} \Theta.$$

The Euler system of theta cycles / 2: construction of Θ_v

For $\chi \in \widehat{C[1]}$, $\kappa \in H_f^1(K[1], V)$, let $\kappa(\chi) := \sum_{\tau \in C[1]} \chi^{-1}(\tau) \kappa^\tau \in H_f^1(K, V(\chi))$.

Now, for $v \in S$, we look for $\Theta_v \in H_f^1(K[v], V_{Z_p})$ satisfying

$$\begin{aligned} \mathrm{Tr}_{K[v]/K[1]} \Theta_v &=: \Theta_v^{K[1]} \stackrel{\heartsuit}{=} P_w(\mathrm{Fr}_w) \Theta_1 \quad \forall \chi \in \widehat{C[1]} \\ &\quad \Theta_v^{K[1]}(\chi) \stackrel{\heartsuit}{=} P_w(\mathrm{Fr}_w) \Theta_1(\chi) \stackrel{\mathrm{rec}}{=} L(V_w, \chi_w, 0)^{-1} \cdot \Theta_1(\chi). \end{aligned}$$

For $\chi = 1$, want: $\Theta_v^{K[1]}(\mathbf{1}) := \Theta(f^1, \varphi^1, \phi^v) \stackrel{\heartsuit}{=} L(V_w, 0)^{-1} \cdot \Theta(f^1, \varphi^1, \phi^1)$.

Only change ϕ at v . By multiplicity 1 (cf. [YZZ, LSZ]), can replace Θ with any

$$0 \neq \theta_v: \pi_v \otimes \sigma_v^\vee \otimes S_{\mathrm{Weil}}(W_v^{n/2}) \rightarrow \mathbf{C}$$

that is equivariant for $\mathbf{H}(F_v) \times \mathbf{G}(H_v) \cong \mathbf{GL}_n(F_v) \times \mathbf{GL}_n(F_v)$. Note: $\sigma_v \cong \pi_v$.

Fact: there is $\mathcal{F}: S_{\mathrm{Weil}}(W_v^{n/2}) \rightarrow S_{\mathrm{lin}}(M_n(F_v))$. So take $\theta_v = \zeta_{\mathrm{GJ}} \circ \mathcal{F}^{-1}$ for

$$\zeta_{\mathrm{GJ}}(f', \varphi', \phi') = \frac{1}{L(\pi_v, \frac{1}{2})} \int_{\mathbf{GL}_n(F_v)} (g.f', \varphi') \phi'(g) dg = \begin{cases} 1 & \text{if unramified} \\ L(\pi_v, \frac{1}{2})^{-1} & \phi' = \mathbf{1}_{\mathbf{GL}_n(\mathcal{O}_{F_v})} \end{cases}.$$

The Euler system of theta cycles / 3: conclusion

Summing up: for $\chi = \mathbf{1}$, let $\Theta_v^{K[1]}(\chi) = \Theta(f^1, \varphi^1, \phi^v)$ with $\phi_v^v := \mathcal{F}^{-1}(\mathbf{1}_{\mathrm{GL}_n(\mathcal{O}_{F_v})})$.

In fact, this $\checkmark \forall \chi \in \widehat{C[1]} \rightsquigarrow \Theta_v^{K[1]} \in H_f^1(K[1], V_{\mathbf{Z}_p})$.

Is there $\Theta_v = \mathrm{Tr}_{K[v]/K[1]}^{-1}(\Theta_v^{K[1]}) \in H_f^1(K[v], V_{\mathbf{Z}_p})$?

Easy: just do as above, take $\Theta_v := f_*^1 \mathrm{AJ}_\rho(\Theta(\varphi, \phi^v)^{C[v]})$.

Not so easy: we may lose integrality of Θ_v ! (Recall $Z(\underline{x})^{C[v]}$ may have denominators.)

So we have to show that for every $\underline{x} \in \mathrm{Spt}(\phi^v)$:

- ▶ either $Z(\underline{x})$ has “Galois-level” $\preceq C[v]$ (i.e. $\det U(\underline{x}) \subset C[v]$),
- ▶ or $\phi^v(\underline{x}) \in |C[1]/C[v]| \cdot \mathbf{Z}$, killing denominators.

It works by explicit computation.

Thanks!