EULER SYSTEMS FOR CONJUGATE-SYMPLECTIC MOTIVES

by

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Abstract. — Consider a conjugate-symplectic geometric representation ρ of the Galois group of a CM field. Under the assumption that ρ is automorphic, even-dimensional, and of minimal regular Hodge–Tate type, we construct an Euler system for ρ in the sense of forthcoming work of Jetchev–Nekovář–Skinner. The construction is based on Theta cycles as introduced in a previous paper, following works of Kudla and Liu on arithmetic theta series on unitary Shimura varieties; it relies on a certain modularity hypothesis for those theta series.

Under some ordinariness assumptions, one can attach to ρ a p-adic L-function. By recent results of Liu and the author, and the theory of Jetchev–Nekovář–Skinner, we deduce the following (unconditional) result under mild assumptions: if the p-adic L-function of ρ vanishes to order 1 at the centre, then the Selmer group of ρ has rank 1, generated by the class of an algebraic cycle. This confirms a case of the p-adic Beilinson–Bloch–Kato conjecture.

Contents

1. Introduction

A remarkable construction of Kolyvagin shows that if a Heegner point is non-torsion, then the Mordell–Weil and Selmer groups of a (modular) elliptic curve both have rank one [Kol88]. Combined with the formulas of Gross–Zagier and Perrin-Riou [GZ86, PR87], which relate heights of Heegner points and derivatives of L-functions in complex or p-adic coefficients, Kolyvagin's work provides important evidence for the Birch and Swinnerton-Dyer conjecture and its p-adic analogue.

We are interested in analogous pictures for higher-rank motives, or more simply geometric[(1)] Galois representations, of weight -1 . The most accessible ones are arguably those over a CM field that are

[(1)] This and other unexplained notions will be defined in the main body of the paper.

conjugate-symplectic. For those Galois representations, Jetchev–Nekovář–Skinner have recently theorised a variant of Kolyvagin's method based on the notion of (what we propose to call) a *JNS* Euler system; this is still a system of Selmer classes satisfying certain compatibility conditions. The purpose of this work is to construct such an Euler system, for those representations as above that are automorphic and even-dimensional of minimal regular Hodge–Tate type.

The companion formulas of Gross–Zagier/Perrin-Riou type were recently proved in [LL21, LL22] and [DL24], which allows to obtain various applications to the analogues (by Beilinson, Bloch, Kato, and Perrin-Riou) of the Birch and Swinnerton-Dyer conjecture.(2)

In the rest of this introduction, we briefly state our main result and consequence, and the idea of its proof. For an overview on the context and history of the constructions, and statements of other arithmetic applications, we refer to [Dis].

1.1. Main result. — Let E be a CM field with absolute Galois group G_E and maximal totally real subfield F, and let $\mathrm{c}\in \mathrm{Gal}(E/F)$ be the complex conjugation.

Let $n=2r$ be an even positive integer and let

$$\rho\colon G_E\to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

be an irreducible continuous representation, that is geometric in the sense of [FM95, I, §1]. We denote by $\rho^{\mathrm{c}}\colon G_E\to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ the representation defined by $\rho^{\mathrm{c}}(g)=\rho(cgc^{-1})$, where $c\in G_E$ is any fixed lift of c. (A different choice of lift would yield an isomorphic representation.)

Suppose that the following conditions are satisfied:

1. ρ is *conjugate-symplectic* in the sense that there exists a perfect pairing
$$\rho\otimes_{\overline{\mathbf{Q}}_p}\rho^{\mathrm{c}}\to \overline{\mathbf{Q}}_p(1)$$
such that for the induced map $u\colon \rho^c\to\rho^*(1)$ (where * denotes the linear dual) and its conjugate-dual $u^*(1)^{\mathrm{c}}\colon \rho^{\mathrm{c}}\to\rho^{\mathrm{c},*}(1)^{\mathrm{c}}=\rho^*(1)$, we have $u=-u^*(1)^{\mathrm{c}}$;
2. for every place $w|p$ of E and every embedding $\jmath\colon E_w\hookrightarrow \mathbf{C}_p$, the $\jmath$-Hodge–Tate weights(3) of ρ are the n integers $\{-r,-r+1,\ldots,r-1\}$;
3. ρ is *automorphic* in the sense that for each $\iota\colon\overline{\mathbf{Q}}_p\hookrightarrow\mathbf{C}$, there is a cuspidal automorphic representation Π^ι of $\mathrm{GL}_n(\mathbf{A}_E)$ such that $L_\iota(\rho,s)=L(\Pi^\iota,s+1/2)$;

For a place v of F, denote by ρ_v the restriction of ρ to $G_{E_v}:=\prod_{w|v}G_{E_w}$ (where the product ranges over the one or two places of E above v). For each ideal $m\subset\mathcal{O}_E$, we have a ring class field $E[m]\supset E$; we also put $E[0]:=E$. We denote by $H^1_f(E[m],\rho)$ the Bloch–Kato Selmer groups [BK90].

Let $\mathscr{M}_1$ be a set consisting of all but finitely many of the places v of F that are split in E and at which ρ is unramified, and let $\mathscr{M}$ be the set of finite subsets of $\mathscr{M}_1$, which we identify with a set of squarefree ideals in $\mathcal{O}_F$. Fix a set $\wp$ of p-adic places of F that are split in E, such that for each $v\in\wp$, the representation ρ_v is Panchishkin-ordinary (Definition 3.3.1) and crystalline,(4) and let $\mathscr{M}[\wp]$ be the set of ideals of the form $m\prod_{v\in\wp}v^{s_v}$ with $s=(s_v)\in\mathbf{Z}_{\geqslant 0}^{\wp}$.

Theorem A. *— Let ρ be a representation satisfying conditions 1., 2., 3. above, and let $\mathscr{M}_1,\mathscr{M},\wp,\mathscr{M}[\wp]$ be as above. Assume that the root number $\varepsilon(\rho)=-1$, that $F\neq\mathbf{Q}$ or $n=2$, and that the Modularity Hypothesis 2.2.5 holds.*

(2)In a future version, we plan to include applications to anticyclotomic Iwasawa theory.
(3)Our convention is that the cyclotomic character has weight -1.
(4)In fact the crystalline condition can be replaced by the considerably weaker condition of (3.3.4) and even entirely removed, see Remark 3.3.6.

The system of classes

$$\Theta_m \in H^1_f(E[m],\rho), \qquad m \in \mathscr{M}[\wp]\cup\{0\},$$

of Definition 2.3.2 forms a JNS Euler system.

For the definition of JNS Euler systems[5] and the precise statement of the theorem, see Theorem 2.3.3.

Remark 1.1.1. — Hypothesis 2.2.5 concerns the modularity of a certain generating series of Selmer classes coming from cycles in unitary Shimura varieties, for which the evidence is discussed in [Dis, Remark 4.4] and references therein. We also rely on a description of part of the cohomology of those varieties, Hypothesis 2.1.1, expected to be confirmed in a sequel to [KSZ].

The assumption on the root number is natural in the sense that, by (1.1.1) below and the Beilinson–Bloch–Kato conjecture (e.g. [Dis, Conjecture 2.2]), in the complementary case $\varepsilon(\rho) = +1$ every JNS Euler system is expected to be zero.

The main result of the work of Jetchev–Nekovář–Skinner (see [Ski] or [ACR23, § 8]) implies that, under mild conditions on the image of ρ, we have

$$\Theta_0 \neq 0 \qquad \Longrightarrow \qquad H^1_f(E,\rho) = \overline{\mathbf{Q}}_p\Theta_0. \tag{1.1.1}$$

Thus Theorem A demands a nonvanishing criterion for Θ_0. Under some ramification restrictions:

- Li and Liu have proved a nonvanishing criterion in terms of derivatives of L-functions [LL21, LL22], conditionally on some standard conjectures on Abel–Jacobi map;
- Liu and the author, under the further assumption that one can take $\wp = \{\text{all places of } F \text{ above } p\}$, have proved an unconditional nonvansihing criterion in terms of p-adic L-functions, and confirmed the Modularity Hypothesis in that context [DL24].

For the precise statements cast into the setup of the present paper, and their consequences towards the complex and p-adic Beilinson–Bloch–Kato conjectures,[6] see [Dis, Theorem A].

In particular, we restate the following result from *loc. cit.*, which appears to be the first complete result towards the Beilinson–Bloch–Kato conjectures in analytic rank 1 for high dimensional representations. It follows from combining [DL24, Theorems 1.7, 1.8], Theorem A above, and the theory of Jetchev–Nekovář–Skinner as in [ACR23, Theorem 8.3].

Corollary. — *Suppose further that E/F is totally split above* 2 *and p, that $p > n$, that places of F ramified in E are unramified over* $\mathbf{Q}$*, and that the representation ρ is:*

- *Panchishkin-ordinary and crystalline at all p-adic places of E;*
- *of 'large image' in the sense that it satisfies the analogue of* [ACR23, Hypothesis (HW) in § 8.1]*;*
- *'mildly ramified' in the sense that the associated automorphic representation π (§ 3.2) satisfies* [DL24, Assumption 1.6 (1)-(2)-(3)].

Denote by $\mathscr{X}_F$ the $\overline{\mathbf{Q}}_p$-scheme of continuous p-adic characters of G_F that are unramified outside p, by $\mathfrak{m} \subset \mathscr{O}(\mathscr{X}_F)$ the ideal of functions vanishing at 1*, and by $L_p(\rho) \in \mathscr{O}(\mathscr{X}_F)$ the p-adic L-function of ρ from* [DL24] *(see* [Dis, Proposition 5.2]*).*

Then

$$\mathrm{ord}_{\mathfrak{m}} L_p(\rho) = 1 \quad \Longrightarrow \quad \dim_{\overline{\mathbf{Q}}_p} H^1_f(E,\rho) = 1.$$

[5]Some of the (young) literature on the subject calls this notion "split anticylcotomic Euler system".
[6]See [Dis, Conjecture 2.2] for a formulation

Remark 1.1.2. — To the author's knowledge, the vast literature on Euler systems contains only three other constructions for high rank motives: one by Liu–Tian–Xiao–Zhang–Zhu [LTX+22] for conjugate-symplectic Rankin–Selberg motives, which is of a type introduced by Bertolini–Darmon in [BD05]; one by Cornut [Cor], for base-changes of some symplectic motives, of a type similar to the one of [Kol88]; and one, of JNS type, by Graham–Shah [GS23], for conjugate-symplectic motives that are also symplectic, valid for an infinite range of Hodge–Tate weights.

1.2. Idea of the proof. — The construction of the *Euler system of Theta cycles* starts from the arithmetic theta lifts on unitary Shimura varieties introduced by Liu in [Liu11] (partly based on a construction of Kudla [Kud97]); as in [Dis], we recast them as trilinear forms valued in $H^1_f(E,\rho)$. The higher layers of the system are given by taking connected components of the special cycles arising in the constructions, and varying the input data in a well-chosen way.

To prove that the constructed classes indeed form an Euler system we need to establish that they are integral and that they are bound up by certain norm relations ('horizontal' and 'vertical', i.e. at non-p-adic and p-adic places). Following an idea pioneered in [YZZ12] and developed in the context of Euler systems in [LSZ22], we prove the horizontal norm relations based on the fact that the space of (scalar-valued) trilinear forms appearing in the constructions decomposes into a product of local spaces, each of dimension 1 by the theory of the local theta correspondence. Then some equivalent relations may be established in any models of these local spaces: in our context, an explicit one is given by the zeta integrals used by Godement–Jacquet to construct the standard L-functions for GL_n, where the desired identity is easy to prove. This model in fact guides the choice of input data away from p; the integrality relations are then established by explicit computation. At p-adic places, we use a variant of choices of data from [DL24], and prove its local nontriviality again by a computation in the Godement–Jacquet model.

In § 2, we construct the system and reduce its fundamental properties to local statements. In § 3, we prove those statements.

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2. The Euler system of Theta cycles

2.1. Setup. — We briefly review the setup for the construction of Theta cycles, referring to [Dis] and references therein for the details.

2.1.1. *Notation.* — Suppose for the rest of this paper that E is a CM field with totally real subfield F. We denote by $\mathrm{c}\in \mathrm{Gal}(E/F)$ the complex conjugation, and by $\eta\colon F^\times\backslash \mathbf{A}^\times \to \{\pm 1\}$ the quadratic character attached to E/F. We denote by $\mathbf{A}$ the adèles of F; if S is a finite set of places of F, we denote by $\mathbf{A}^S$ the adèles of F away from S. If G is a group over F and v is a place of F, we write $G_v := \mathrm{G}(F_v)$; if S a finite set of places of F, we write $G_S := \prod_{v\in S} G(F_S)$. (For notational purposes, we will identify a place of $\mathbf{Q}$ with the set of places of F above it.) We denote by $\psi\colon F\backslash \mathbf{A} \to \mathbf{C}^\times$ the standard additive character with $\psi_\infty(x) = e^{2\pi i \mathrm{Tr}_{F_\infty/\mathbf{R}} x}$, and we set $\psi_E := \psi\circ \mathrm{Tr}_{E/F}$.

We fix a rational prime p and denote by $\mathbf{Q}^\circ \subset \overline{\mathbf{Q}}_p$ the extension of $\mathbf{Q}$ generated by all roots of unity.

We fix an embedding $\iota^\circ\colon \mathbf{Q}^\circ \hookrightarrow \mathbf{C}$, by which we view $\psi_{|\mathbf{A}^\infty}$ as valued in $\mathbf{Q}^\circ$. We denote by $\mathscr{O}$ the integral closure of $\mathbf{Z}_p$ in $\overline{\mathbf{Q}}_p$.

2.1.2. *Quasisplit unitary group.* — Let $W = E^n = W^+ \oplus W^-$ where $W^+ = \mathrm{Span}(e_1, \ldots, e_r)$, $W^- = \mathrm{Span}(e_{r+1}, \ldots, e_{2r})$, equipped with the skew-hermitian form $\langle\ ,\ \rangle_W$ with matrix $\left(\begin{smallmatrix} & 1_r \\ -1_r & \end{smallmatrix}\right)$ (here 1_r is the identity matrix of size r). We denote by $\mathrm{G} = \mathrm{U}(W)$ its unitary group, which we may view as a subgroup of $\mathrm{Res}_{E/F}\mathrm{GL}_n$. Denote by $\mathrm{P} \subset \mathrm{G}$ the parabolic subgroup stabilizing W^-, and by Herm_r the space of hermitian $r \times r$ matrices.

We have:

- a Weyl element
$$w = \begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix} \in \mathrm{G};$$
- a homomorphism
$$m\colon \mathrm{Res}_{E/F}\mathrm{GL}_r \to \mathrm{P} \subset \mathrm{G}$$
$$a \mapsto m(a) := \begin{pmatrix} a & \\ & {}^{\mathrm{t}}a^{\mathrm{c},-1}. \end{pmatrix}$$
whose image is a Levi factor of P;
- a homomorphism
$$n\colon \mathrm{Herm} \to \mathrm{P} \subset \mathrm{G}$$
$$b \mapsto n(b) := \begin{pmatrix} 1_r & b \\ & 1_r \end{pmatrix},$$
whose image is the unipotent radical of P. Here, we denote by Herm the space of hermitian matrices; we will also denote by $\mathrm{Herm}(F)^+ \subset \mathrm{Herm}(F)$ the subspace consisting of totally positive semidefinite matrices.

Attached to G, we have:

- a $\overline{\mathbf{Q}}_p$-vector space $\mathscr{H}_{\overline{\mathbf{Q}}_p}$ of modular forms (see [DL24, § 2.2], [Dis, § 4.3]);
- for any ring R, the space $\underline{\mathrm{SF}}_R$ of those formal (Siegel–Fourier) expansions
$$\sum_{T \in \mathrm{Herm}_r(F)^+} c_T(a) q^T, \qquad c_T \in C^\infty(\mathrm{GL}_r(\mathbf{A}_E^\infty), R)$$
satisfying $c_{{}^{\mathrm{t}}a^{\mathrm{c}}Ta}(y) = c_T(ay)$ for all $a \in \mathrm{GL}_r(E)$;
- an injective p-adic q-expansion map

$$\underline{\mathbf{q}}\colon \mathscr{H}_{\overline{\mathbf{Q}}_p} \longrightarrow \underline{\mathrm{SF}}_{\overline{\mathbf{Q}}_p}, \tag{2.1.1}$$

denoted by $\underline{\mathbf{q}}_p$ in [Dis, § 4.2].

2.1.3. *Incoherent unitary groups.* — Let V be an incoherent, totally positive definite E/F-hermitian space; this is simply a collection of E_v/F_v-hermitian spaces V_v of the same dimension, indexed by the places of F_v, which is not isomorphic to one of the form $(V_0 \otimes_F F_v)_v$ for some hermitian space V_0 over E, and such that V_v is positive definite for all $v|\infty$. If S is a finite set of places of F, we put $V_{\mathbf{A}^S} = \otimes_{v \notin S} V_v$. For $x_1, \ldots, x_r \in V_v$, we have the moment matrix
$$T(x) := ((x_i, x_j)_{V_v})_{ij} \in \mathrm{Herm}_r(F_v).$$
We denote by H_V the incoherent unitary group associated with V in the sense of [Dis].

Attached to H_V we have a tower of Shimura varieties
$$(X_{\mathrm{H}_V, K})_{K \subset \mathrm{H}_V(\mathbf{A}^\infty)}$$
of dimension $\dim V - 1$ over E as in [Dis, § 4.2].

2.1.4. *Weil representation.* — Let v be a finite place of F, and let V_v be an E_v/F_v-hermitian space of dimension n. The basis $\{e_1, \ldots, e_r\}$ of W^+ identifies $V_v \otimes_{E_v} W_v^+ = V_v^r$. We have a representation $\omega_v := \omega_{V_v}$ of $G_v \times H_{V_v}$ on the Schwartz space $\mathscr{S}(V_v \otimes_{E_v} W_v^+, \overline{\mathbf{Q}}_p)$, characterized by the property that for $\phi \in \mathscr{S}(V_v \otimes_{E_v} W_v^+, \overline{\mathbf{Q}}_p)$:

- for $h \in H_{V_v}$, we have
$$\omega_v(h)\phi(x) = \phi(h^{-1}x);$$
- for $a \in \mathrm{GL}_r(E_v)$ and $b \in \mathrm{Herm}_r(F_v)$, we have
$$\omega_v(m(a))\phi(x) = |\det a|_E^r \cdot \phi(xa),$$
$$\omega_v(n(b))\phi(x) = \psi_v(\mathrm{Tr}\, b\, T(x))\phi(x),$$
$$\omega_v(w_r)\phi(x) = \gamma_{V_v,\psi_v}^r \cdot \widehat{\phi}(x),$$
where $\gamma_{V_v,\psi_v} \in \{\pm 1\}$ is the Weil constant of V_v with respect to ψ_v, and $\widehat{\phi}$ denotes the Fourier transform
$$\widehat{\phi}(x) := \int_{V_v^r} \phi(y)\psi_{E,v}\left(\sum_{i=1}^r (x_i, y_i)_V\right) \mathrm{d}y$$
for the $\psi_{E,v}$-self-dual Haar measure $\mathrm{d}y$ on V_v^r.

For an incoherent E/F-hermitian space V, we put $\omega = \omega_V = \otimes_v \omega_{V_v}$, the product running over all finite places of F; it is a representation of $\mathrm{G}(\mathbf{A}^\infty) \times \mathrm{H}_V(\mathbf{A}^\infty)$ on the space $\mathscr{S}(V_{\mathbf{A}^\infty} \otimes_E W^+, \overline{\mathbf{Q}}_p)$.

2.1.5. *p-adic automorphic representations.* — Given our Galois representation ρ, we choose a relevant p-adic automoprhic representation of $\mathrm{G}(\mathbf{A})$ over $\overline{\mathbf{Q}}_p$ (in the sense of [Dis, Definition 3.2])
$$\pi \subset \mathscr{H}_{\overline{\mathbf{Q}}_p}$$
whose base-change to $\mathrm{GL}_n(\mathbf{A}_E)$ is Π_ρ, which exists by [Dis, Proposition 3.4]. (The representation π is not uniquely determined by this condition, although as noted in [Dis, Remark 3.5], there is a 'standard' choice.)

We enforce from now on the assumption that $\varepsilon(\rho) = -1$. Then by [Dis, Proposition 3.8], attached to ρ and π we have a pair consisting of

- an incoherent totally definite E/F-hermitian space V of dimension n, and
- a relevant p-adic automorphic representation σ of $\mathrm{H}(\mathbf{A})$ over $\overline{\mathbf{Q}}_p$ (in the sense of [Dis, Definition 3.2])

uniquely characterised (up to isomorphism) by the condition that the space of coinvariants
$$\Lambda_\rho := (\pi^\vee \otimes \omega \otimes \sigma)_{\mathrm{G}(\mathbf{A}^\infty)\times \mathrm{H}_V(\mathbf{A}^\infty)}$$
is nonzero. The space Λ_ρ is then in fact 1-dimensional over $\overline{\mathbf{Q}}_p$.

Henceforth, we write $\mathrm{H} = \mathrm{H}_V$, $\omega = \omega_V$, $X = X_{\mathrm{H}_V}$.

2.1.6. *Realisation of σ in cohomology.* — From now on we assume that $F \neq \mathbf{Q}$ or $n = 2$, which implies that the varieties $X_{\mathrm{H}_V,K}$ are projective – except in a case related to modular curves where $X_{\mathrm{H}_V,K}$ can be canonically compactified by adding finitely many cusps; in that case, we replace $X_{\mathrm{H}_V,K}$ by its compactification.

For each open compact $K \subset \mathrm{H}(\mathbf{A}^\infty)$, let
$$\sigma_\rho^K := \mathrm{Hom}_{\overline{\mathbf{Q}}_p[G_E]}(H^{2r-1}_{\text{ét}}(X_{\mathrm{H}_V,K,\overline{E}}, \overline{\mathbf{Q}}_p(r)), \rho).$$

We will assume the following hypothesis (a variant of [Dis, Hypothesis 4.1]); it is known for $n=2$, and it is expected to be confirmed in general in a sequel to [KSZ].

Hypothesis 2.1.1. — *For each open compact subgroup $K \subset \mathrm{H}(\mathbf{A}^\infty)$, we have an isomorphism of $\overline{\mathbf{Q}}_p[K\backslash \mathrm{H}_V(\mathbf{A}^\infty)/K]$-modules*

$$\sigma_\rho^K \cong \bigoplus_{\sigma'} \sigma'^K, \tag{2.1.2}$$

where the direct sum runs over the isomorphism classes of relevant p-adic automorphic representation (in the sense of [Dis, Definition 3.2]*) σ' of $\mathrm{H}_V(\mathbf{A})$ with $\mathrm{BC}(\sigma')=\Pi$.*

We put $M_{\sigma,K} := \sigma^{\vee,K} \otimes \rho$, which we identify with a subspace of $M_{\rho,K} := \sigma_\rho^{\vee,K}\otimes\rho \subset H^{2r-1}_{\text{ét}}(X_{K,\overline{E}}, \overline{\mathbf{Q}}_p(r))$. We then identify

$$\sigma = \varinjlim_K \mathrm{Hom}_{\overline{\mathbf{Q}}_p[K\backslash \mathrm{H}_V(\mathbf{A}^\infty)/K]}(M_{\sigma,K},\rho) \subset \varinjlim_K \mathrm{Hom}_{\overline{\mathbf{Q}}_p[K\backslash \mathrm{H}_V(\mathbf{A}^\infty)/K]}(M_{\rho,K},\rho) = \varinjlim_K \sigma_{\rho,K}.$$

2.2. Special cycles and generating series over ring class fields

2.2.1. *Connected components of unitary Shimura varieties.* — Let T be the unitary group of E with the form induced by the norm $N_{E/F}$ (as an algebraic group over F), and denote

$$\mathscr{C} := \{\text{open compact subgroups } C \subset \mathrm{T}(\mathbf{A}^\infty)\} \cup \{\mathrm{T}(\mathbf{A}^\infty)\}.$$

(Elements $C\in\mathscr{C}$ will often appear as sub/superscript, omitted when $C=\mathrm{T}(\mathbf{A}^\infty)$.) For each $C\in\mathscr{C}$, let E_C be the abelian extension of E with

$$\mathrm{Gal}(E_C/E) =: \Gamma_{/C} := \mathrm{T}(F)\backslash \mathrm{T}(\mathbf{A}^\infty)/C$$

under the class field theory isomorphism (which we will view as an identification). Let $\Gamma := \varprojlim_C \Gamma_{/C}$. For any profinite group Γ', we will denote by $\widehat{\Gamma} := \varinjlim_{C'} \mathrm{Spec}\,\overline{\mathbf{Q}}_p[\Gamma'/C']$ the space of locally constant $\overline{\mathbf{Q}}_p$-valued characters of Γ' (where the limit ranges over finite-index subgroups).

Let X_T be the tower of 0-dimensional Shimura varieties over E associated with . For every coherent or incoherent unitary group H', we denote by $\nu_{\mathrm{H}'}\colon \mathrm{H}'\to\mathrm{T}$ the determinant character (the subscript will be omitted when understood from the context). The tower $X_{\mathrm{H}'} = (X_{\mathrm{H}',K'})_{K'}$ of Shimura varieties maps to the tower X_T via surjective morphisms still denoted

$$\nu\colon X_{\mathrm{H},K'} \to X_{\mathrm{T},\nu(K')}.$$

These induce bijections on the set of geometrically connected components.

Fix an identification of Γ-sets $X_\mathrm{T}(E^{\mathrm{ab}})\cong\Gamma$. A subset $S\subset\mathrm{T}(\mathbf{A}^\infty)$ is said to be of *level* $C\in\mathscr{C}$ if C is minimal for the property that S is a union of C-cosets. If $S\subset\mathrm{T}(\mathbf{A}^\infty)$ is of level $C\supset C'$, let

$$X^S_{\mathrm{T},C'} \subset X_{\mathrm{T},C',E_C}$$

be the E_C-subscheme whose set of E^{ab}-points is identified with the image of S in $\Gamma_{/C'}$. If $S\subset\mathrm{T}(\mathbf{A}^\infty)$ is of level $C\supset\nu(K)$, let

$$X^S_{\mathrm{H},K} := \nu^{-1}(X^{(S)}_{\mathrm{T},\nu(K)}) \subset X_{\mathrm{H},K,E_C}.$$

Then for each i,j, each $C\in\mathscr{C}$, and each $t\in\Gamma_{/C}$ we have a direct $\mathbf{Q}_p[G_{E_C}]$-module summand $H^i_{\text{ét}}(X^{tC}_{K,\overline{E}},\mathbf{Q}_p(j)) \subset H^i_{\text{ét}}(X_{K,\overline{E}},\mathbf{Q}_p(j))$, and we denote by

$$\mathrm{r}^{tC}\colon H^i_{\text{ét}}(X_{K,\overline{E}},\mathbf{Q}_p(j)) \to H^i_{\text{ét}}(X^{tC}_{K,\overline{E}},\mathbf{Q}_p(j)) \tag{2.2.1}$$

the projection induced by the inclusion $X^{tC}_K \hookrightarrow X_{K,E_C}$.

2.2.2. *Special cycles.* — From now on, we abbreviate $\mathrm{H}=\mathrm{H}_V$ and $X_? := X_{\mathrm{H},?} = X_{\mathrm{H}_V,?}$ (for any decoration '?'). For a $C\in\mathscr{C}$ and a compact open subgroup $K'\subset \mathrm{H}'(\mathbf{A}^\infty)$ (for some unitary group $\mathrm{H}'\xrightarrow{\nu}\mathrm{T}$), we write $K'^C := K'\cap \nu^{-1}(C)$.

Let $x\in V_{\mathbf{A}^\infty}\otimes_E W^+ = V^r_{\mathbf{A}^\infty}$.

– Suppose that

$$T(x) := ((x_i,x_j)_V)_{ij}\in \mathrm{Herm}_r(F)^+ \quad \text{and} \quad V(x):=\mathrm{Span}_E(x_1,\ldots,x_r) \text{ is positive-definite.} \tag{2.2.2}$$

For any compact open subgroup $K\subset \mathrm{G}(\mathbf{A}^\infty)$, any $C\in\mathscr{C}$, and any $t\in\mathrm{T}(\mathbf{A}^\infty)$, we have a cycle $\mathrm{Z}^{tC}(x)_K$ defined as follows (see [Liu11, § 3A] or [LL21, § 4] for more details when $tC=\mathrm{T}(\mathbf{A}^\infty)$).

Pick an embedding $\iota\colon E\hookrightarrow \mathbf{C}$, and let V^ι be the (unique up to isomorphism) totally definite hermitian space over E with $V^\iota_{\mathbf{A}^\infty}\cong V_{\mathbf{A}^\infty}$; we fix such an identification. Then we may write $x=h^{-1}x'$ for $x'\in (V^\iota)^r$ and $h\in\mathrm{H}(\mathbf{A}^\infty)$. Let $\mathrm{H}(x')$ be the unitary group of the subspace $V(x')^\perp\subset V^\iota$, where $V(x'):=\mathrm{Span}_E(x'_1,\ldots,x'_r)$; let $K_{x'}:=hKh^{-1}\cap \mathrm{H}(x')(\mathbf{A}^\infty)$. We also denote by $\mathrm{H}(x)(\mathbf{A})$ the unitary group of $V(x)^\perp$, and set $K_x:=K\cap \mathrm{H}(x)(\mathbf{A})$.

The natural inclusion $j_{x'}\colon \mathrm{H}(x')\hookrightarrow \mathrm{H}^\iota$ of unitary groups induces a morphism of Shimura varieties

$$X_{\mathrm{H}(x'),K_{x'}} \xrightarrow{j_{x',K}} X_{hKh^{-1}} \xrightarrow{\cdot h} X_K.$$

Let $\mathscr{L}_K$ be the Hodge bundle on X_K (or its base-change to E_C). We then define a cycle (see [Ful98, § 2.5] for general background)

$$\mathrm{Z}^{tC}(x)_K := c_1(\mathscr{L}_K^\vee)^{\dim V(x)-r}_{|X_K^{tC}} \frown [C\nu(K_x):C]^{-1}\cdot [j_{x,K}(X^{tC\nu(K_x)}_{\mathrm{H}(x),K_x})]\in \mathrm{Ch}_{r-1}(X_{K,E_C})_{\mathbf{Q}},$$

where we have used the suggestive notation[7]

$$j_{x,K}(X^S_{\mathrm{H}(x),K_x}) := j_{x',K}(X^{\nu^{-1}(h)S}_{\mathrm{H}(x'),K_{x'}})h, \tag{2.2.3}$$

in which the right multiplication denotes the action of $\mathrm{H}(\mathbf{A}^\infty)$ on the tower $(X_{\mathrm{H},K})_K$.

The definition is independent of the auxiliary choices made.

– If x does not satisfy (2.2.2), we put $\mathrm{Z}^C(x)_K:=0$.

It is clear that $\mathrm{Tr}_{E_{C'}/E_C}\mathrm{Z}^{C'}(x)_K=\mathrm{Z}^C(x)_K$ whenever $C'\subset C\in\mathscr{C}$.

For a locally constant function $\chi:\Gamma\to\overline{\mathbf{Q}}_p$, we also define

$$\mathrm{Z}(x,\chi)_K := \sum_{t\in\Gamma_{/C}} \chi(t)\mathrm{Z}^{tC}(x)_K \qquad \in \mathrm{Ch}_{r-1}(X_{K,E_C})_{\overline{\mathbf{Q}}_p},$$

for any $C\in\mathscr{C}$ such that χ factors through $\Gamma_{/C}$. (Thus $\mathrm{Z}^C(x)_K=\mathrm{Z}(x,\mathbf{1}_C)_K$ for any $C\in\mathscr{C}$.)

Remark 2.2.1. — Suppose that $\chi:\Gamma\to\overline{\mathbf{Q}}_p^\times$ is a locally constant character. For every $x\in V^r_{\mathbf{A}^\infty}$, $\gamma\in\Gamma$, and $h\in\mathrm{H}(x)(\mathbf{A}^\infty)$, we have

$$\mathrm{Z}(x,\chi)^\gamma_K = \chi^{-1}(\gamma)\mathrm{Z}(x,\chi)_K. \tag{2.2.4}$$

and

$$\mathrm{Z}^C(x)_K h = \mathrm{Z}^{\nu(h)C}(h^{-1}x)_{h^{-1}Kh}, \qquad \mathrm{Z}(x,\chi)_K h = \chi_{\mathrm{H}}^{-1}(h)\mathrm{Z}(h^{-1}x,\chi)_{h^{-1}Kh}. \tag{2.2.5}$$

where we still denote simply as a right multiplication the pushforward action on cycles induced by the right action of $\mathrm{H}(\mathbf{A}^\infty)$ on $(X_{\mathrm{H},K})_K$.

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(7) This notation can likely be given a substantive meaning in the framework of [ST].

2.2.3. *Projection to the ρ-component.* — Let $C \in \mathscr{C}$, and let $K \subset \mathrm{H}(\mathbf{A}^\infty) \cap \nu^{-1}(C)$ be an open compact subgroup. Denote by $\mathrm{Fil}^{\bullet,C} \subset H^{2r}_{\text{ét}}(X_{K,E_C}, \mathbf{Q}_p(r))$ the filtration induced by the Hochschild–Serre spectral sequence $H^i(E_C, H^{2r-i}_{\text{ét}}(X_{K,E_C}, \mathbf{Q}_p(r))) \Rightarrow H^{2r}_{\text{ét}}(X_{K,E_C}, \mathbf{Q}_p(r))$. We have an absolute cycle class map

$$\mathrm{AJ}\colon \mathrm{Ch}_{r-1}(X_{K,E_C})_{\overline{\mathbf{Q}}_p} \to H^{2r}_{\text{ét}}(X_{K,E_C}, \mathbf{Q}_p(r))/\mathrm{Fil}^{2,C}.$$

Lemma 2.2.2. — *The Hecke-eigenprojection*

$$e_\rho\colon \bigoplus_{i\in\mathbf{Z}} H^i_{\text{ét}}(X_{K,\overline{E}}, \mathbf{Q}_p(r)) \to M_{\rho,K}$$

induces a Hecke-equivariant projection, still denoted

$$e_\rho\colon H^{2r}_{\text{ét}}(X_{K,E_C}, \mathbf{Q}_p(r))/\mathrm{Fil}^{2,C} \to H^1(E_C, M_{\rho,K}),$$

such that the composition

$$\begin{aligned}(-)_\rho\colon \mathrm{Ch}_{r-1}(X_{K,E_C})_{\mathbf{Q}} \xrightarrow{\mathrm{AJ}} H^{2r}_{\text{ét}}(X_{K,E_C}, \mathbf{Q}_p(r))/\mathrm{Fil}^{2,C} &\xrightarrow{e_\rho} H^1(E_C, M_{\rho,K})\\ \mathrm{Z} \mapsto \mathrm{Z}_\rho := e_\rho \mathrm{AJ}(\mathrm{Z})&\end{aligned}$$

takes values in $H^1_f(E_C, M_{\rho,K})$.

Proof. — As in [Dis, Lemma 4.2]. □

2.2.4. *Generating series.* — Let $\chi\colon \Gamma \to \overline{\mathbf{Q}}_p$ be a locally constant function, and let $C(\chi) \in \mathscr{C}$ be maximal such that χ factors through $\Gamma_{/C(\chi)}$. For $\phi \in \mathscr{S}(V^r_{\mathbf{A}^\infty})$ and any $K \subset \mathrm{H}(\mathbf{A}^\infty)$ fixing ϕ, let

$$\mathrm{Z}_T(\phi,\chi)_K := \sum_{x\in K\backslash V^r_{\mathbf{A}^\infty}\,:\,T(x)=T} \phi(x)\mathrm{Z}(x,\chi)_K \quad \in \mathrm{Ch}_{r-1}(X_K)_{\mathbf{Q}};$$

in the special case $\chi = \mathbf{1}_{tC}$ for $C \in \mathscr{C}$, we write

$$\mathrm{Z}^{tC}_T(\phi) := \mathrm{Z}_T(\phi, \mathbf{1}_{tC}), \qquad {}^{\mathrm{q}}\Theta^{tC}(\phi)_{\rho,K} = {}^{\mathrm{q}}\Theta(\phi, \mathbf{1}_{tC})_{\rho,K}.$$

We define

$${}^{\mathrm{q}}\Theta(\phi,\chi)_{\rho,K}(a) := \mathrm{vol}(K) \sum_{t\in\Gamma/C} \chi(t) \sum_{x\in K\backslash V^r_{\mathbf{A}^\infty}} \phi(xa)\mathrm{Z}^{\nu(m(a))tC}(x)_{K,\rho}\, q^{T(x)} \quad \in H^1_f(E_{C(\chi)}, M_{\rho,K}) \otimes_{\mathbf{Q}_p} \underline{\mathrm{SF}}_{\overline{\mathbf{Q}}_p},$$

where vol is as in [LL21, Definition 3.8].

Lemma 2.2.3. — *Let $K' \subset K \subset \mathrm{H}(\mathbf{A}^\infty)$ be compact open subgroups, let $\phi \in \mathscr{S}(V^r_{\mathbf{A}^\infty})^K$, and let $\mathrm{p} = \mathrm{p}_{K'/K}\colon X_{K'} \to X_K$ denote the projection map. Then for every locally constant $\chi\colon \Gamma \to \overline{\mathbf{Q}}_p$, we have*

$$\mathrm{vol}(K')\,\mathrm{p}_*\mathrm{Z}_T(\phi,\chi)_{K'} = \mathrm{vol}(K)\,\mathrm{Z}_T(\phi,\chi)_K.$$

Proof. — It suffices to show that $\mathrm{p}_*\mathrm{Z}^C_T(\phi)_{K'} = [K:K']\,\mathrm{Z}^C_T(\phi)_K$ for every $C \in \mathscr{C}$.

Suppose first that, in the notation of §2.2.2, we may identify x with an element of $(V^\iota)^r$. The finite surjective map $X^{C\nu(K'_x)}_{\mathrm{H}(x),K'_x} \to X^{C\nu(K_x)}_{\mathrm{H}(x),K_x}$ has degree $[K_x : K'_x]\cdot[C\nu(K_x) : C\nu(K'_x)]^{-1}$. Since

$$\mathrm{p}^*\mathscr{L}^\vee_K = \mathscr{L}^\vee_{K'},$$

by the definitions and the projection formula, we find

$$\mathrm{p}_*\mathrm{Z}^C(x)_{K'} = [K_x : K'_x]\cdot[C\nu(K_x) : C\nu(K'_x)]^{-1}[C\nu(K'_x) : C]^{-1}\,[j_{x,K}(X^{C\nu(K_x)}_{\mathrm{H}(x),K_x})] = [K_x : K'_x]\,\mathrm{Z}^C(x)_K.$$

It is easy to verify that this result remains valid without the assumption $x \in (V^\iota)^r$.

Then, setting

$$\phi_{|T}(x) := \mathbf{1}_{[T(x)=T]}\phi(x), \tag{2.2.6}$$

we find

$$\mathrm{p}_*\mathrm{Z}^C_T(\phi)_{K'} = \sum_{x\in K'\backslash V^r_{\mathbf{A}^\infty}} \phi_{|T}(kx)\mathrm{p}_*\mathrm{Z}^C(kx)_{K'} = \sum_{x\in K\backslash V^r_{\mathbf{A}^\infty}} \sum_{k\in K'\backslash K/(K'_x\backslash K_x)} \phi_{|T}(kx)[K_{kx}:K'_{kx}]\mathrm{Z}^C(kx)_K.$$

Now all the last three terms are independent of k, so that the inner sum equals

$$\sum_{x\in K\backslash V^r_{\mathbf{A}^\infty}} [K:K']\phi_{|T}(x)\mathrm{Z}^C(x)_K = [K:K']\mathrm{Z}^C_T(\phi)_K,$$

as desired. □

Corollary 2.2.4. — *The construction of* ${}^{\mathrm{q}}\Theta(\phi,\chi)_{\rho,K}$ *is compatible under pushforward in the tower* $(X_K)_K$.

2.2.5. *Modularity.* — For the history and evidence in favour of the following conjecture (which is [DL24, Conjecture 4.17]),[(8)] see [Dis, Remark 4.4] and references therein.

***Hypothesis 2.2.5* (Modularity).** — *For every* $\phi\in\mathscr{S}(V^r_{\mathbf{A}^\infty})$ *and every* $K\subset \mathrm{H}(\mathbf{A}^\infty)$ *fixing* ϕ*, there exists a unique*

$$\Theta(\phi)_{\rho,K}\in H^1_f(E,M_{\rho,K})\otimes_{\overline{\mathbf{Q}}_p}\mathscr{H}_{\overline{\mathbf{Q}}_p}$$

such that

$$\underline{\mathbf{q}}(\Theta(\phi)_{\rho,K}) = {}^{\mathrm{q}}\Theta(\phi)_{\rho,K}.$$

We can amplify the modularity to the other generating series.

Proposition 2.2.6. — *Assume Hypothesis 2.2.5. Then for every locally constant function* $\chi:\Gamma\to\overline{\mathbf{Q}}_p$*, every* $\phi\in\mathscr{S}(V^r_{\mathbf{A}^\infty})$*, and every* $K\subset\mathrm{H}(\mathbf{A}^\infty)$ *fixing* ϕ*, there exists a unique*

$$\Theta(\phi,\chi)_{\rho,K}\in H^1_f(E_\chi,M_{\rho,K})\otimes_{\overline{\mathbf{Q}}_p}\mathscr{H}_{\overline{\mathbf{Q}}_p}$$

such that

$$\underline{\mathbf{q}}(\Theta(\phi,\chi)_{\rho,K}) = {}^{\mathrm{q}}\Theta(\phi,\chi)_{\rho,K}.$$

As usual, we will write $\Theta^{tC}(\phi)_{\rho,K} := \Theta(\phi,\mathbf{1}_{tC})_{\rho,K}$.

Proof. — By Corollary 2.2.4, we may assume that K satisfies $\nu(K)\subset C$. We define a slightly different Siegel–Fourier expansion by

$${}^{\mathrm{q}}\widetilde{\Theta}(\phi,\chi)_{\rho,K}(a) := \mathrm{vol}(K)\sum_{x\in K\backslash V^r_{\mathbf{A}^\infty}} \omega(m(a))\phi(xa)\mathrm{Z}(x,\chi)_{K,\rho}\, q^{T(x)}$$

and we put ${}^{\mathrm{q}}\widetilde{\Theta}^{tC}(\phi)_{\rho,K} = {}^{\mathrm{q}}\widetilde{\Theta}(\phi,\mathbf{1}_{tC})_{\rho,K}$. It suffices to prove the proposition when χ is a character, in which case

$${}^{\mathrm{q}}\Theta(\phi,\chi)_{\rho,K}(a) = \chi(\nu_{\mathrm{G}}(m(a)))^{-1}\widetilde{\Theta}(\phi,\chi)_{\rho,K}(a),$$

so that if $\widetilde{\Theta}(\phi,\chi)_{\rho,K}$ is a (Selmer-group-valued) Siegel modular form with q-expansion ${}^{\mathrm{q}}\widetilde{\Theta}(\phi,\chi)_{\rho,K}$, then

$$\Theta(\phi,\chi)_{\rho,K}(g) := \chi^{-1}\circ\nu_{\mathrm{G}}\otimes\widetilde{\Theta}(\phi,\chi)_{\rho,K}$$

is a Siegel modular form with q-expansion ${}^{\mathrm{q}}\Theta(\phi,\chi)_{\rho,K}$. Therefore it is equivalent to prove the modularity of the series ${}^{\mathrm{q}}\widetilde{\Theta}(\phi,\chi)_{\rho,K}$ for all χ, and we may restrict to $\chi=\mathbf{1}_{tC}$ for $t\in\Gamma_{/C}$.

Now we have ${}^{\mathrm{q}}\widetilde{\Theta}(\phi,\mathbf{1}_{tC})_{K,\rho} = \mathrm{r}^{tC}_*\,{}^{\mathrm{q}}\Theta(\phi)_{\rho,K}$, were r^{tC}_* is induced by (2.2.1). Then

$$\widetilde{\Theta}^{tC}(\phi)_{\rho,K} := \mathrm{r}^{tC}_*\Theta(\phi)_{\rho,K} \quad \in H^1_f(E_C,M_{\rho,K})\otimes_{\overline{\mathbf{Q}}_p}\mathscr{H}_{\overline{\mathbf{Q}}_p}$$

(8) The formulation in *loc. cit.* is slightly different but easily seen to be equivalent.

satisfies $\underline{\mathbf{q}}(\widetilde{\Theta}^{tC}(\phi)_{\rho,K}) = {}^{\mathrm{q}}\widetilde{\Theta}^{tC}(\phi)_{\rho,K}$, as desired. □

2.3. The Euler system of Theta cycles. — From now on we assume that Hyptohesis 2.2.5 holds.

If $C \in \mathscr{C}$ and E' is a finite extension of E_C, for $z \in H^1_f(E', M^C_{\rho,K})$ and $f \in \sigma$ we denote

$$z.f := f_* z \in H^1_f(E', \rho).$$

2.3.1. *Theta cycles.* — For a relevant representation $\pi' \subset \mathscr{H}_{\overline{\mathbf{Q}}_p}$, denote by $\Phi \mapsto \Phi_{\pi'}$ the Hecke-eigenprojection $\mathscr{H}_{\overline{\mathbf{Q}}_p} \to \pi'$, and by $\langle , \rangle_{\pi'} \colon \pi'^{\vee} \otimes \pi' \to \overline{\mathbf{Q}}_p$ the canonical duality. We also abbreviate $\langle \varphi', \Phi \rangle_{\pi'} := \langle \varphi', \Phi_{\pi'} \rangle_{\pi'}$ for $\varphi' \in \pi'$, $\Phi \in \mathscr{H}_{\overline{\mathbf{Q}}_p}$, and use the same names for any base-change.

For every $\varphi \in \pi^\vee$, $f \in \sigma$, $C \in \mathscr{C}$, and every locally constant function $\chi : \Gamma_{/C} \to \overline{\mathbf{Q}}_p$, we define

$$\begin{aligned} \Theta(\varphi, \phi, \chi)_{\rho,K} &:= \langle \varphi, \Theta(\phi, \chi)_{\rho,K} \rangle_\pi \quad && \in H^1_f(E_C, M_{\rho,K}), \\ \Theta(\varphi, \phi, f, \chi) &:= \Theta(\varphi, \phi, \chi)_{\rho,K}.f && \in H^1_f(E_C, \rho), \end{aligned}$$

where $K \subset \mathrm{H}(\mathbf{A}^\infty)$ is any open compact subgroup fixing f and ϕ.

As usual, in the special case $\chi = \mathbf{1}_C$ for $C \in \mathscr{C}$, we will put $\Theta^C(-) := \Theta(-, \mathbf{1}_C)$.

If $\chi \in \widehat{\Gamma}$ (viewed as an automorphic character of $\mathrm{T}(\mathbf{A})$), let $\chi_{\mathrm{G}} := \chi \circ \nu_{\mathrm{G}}$, $\chi_{\mathrm{H}} := \chi \circ \nu_{\mathrm{H}}$, and denote by

$$\mathscr{S}_\chi(V^r_{\mathbf{A}^\infty})$$

the space $\mathscr{S}(V^r_{\mathbf{A}^\infty})$ with $\mathrm{G}(\mathbf{A}^\infty) \times \mathrm{H}(\mathbf{A}^\infty)$-action by $\omega_\chi := \omega \otimes \chi_{\mathrm{G}}^{-1} \otimes \chi_{\mathrm{H}}$. Let

$$\Lambda_{\rho,\chi} := \left(\pi \otimes \mathscr{S}_\chi(V^r_{\mathbf{A}^\infty}) \otimes \sigma\right)_{\mathrm{G}(\mathbf{A}^\infty) \times \mathrm{H}(\mathbf{A}^\infty)}, \tag{2.3.1}$$

a $\overline{\mathbf{Q}}_p$-line. For $C \in \mathscr{C}$, we also put

$$\Lambda_{\rho,C} := \bigoplus_{\chi \in \widehat{\Gamma}_{/C}} \Lambda_{\rho,\chi}.$$

***Lemma 2.3.1* (Equivariance).** — *Let $\chi \in \widehat{\Gamma}$. If Hypothesis 2.2.5 holds, the map*

$$\begin{aligned} \Theta(\,\cdot\,, \chi) \colon \pi \otimes \mathscr{S}(V^r_{\mathbf{A}^\infty}) \otimes \sigma &\to H^1_f(E, \rho(\chi)) \\ (\varphi, \phi, f) &\mapsto \Theta(\varphi, \phi, f, \chi) \end{aligned}$$

factors through $\Lambda_{\rho,\chi}$.

It follows from the lemma that for any $C \in \mathscr{C}$, the map Θ^C factors through $\Lambda_{\rho,C}$.

Proof. — It follows from (2.2.4) that the target is $H^1_f(E, \rho(\chi)) \subset H^1_f(E_{C(\chi)}, \rho)$. The equivariance for the action of $\mathrm{G}(\mathbf{A}^\infty)$ is clear. We then need to show that for every $\phi \in \mathscr{S}(V^r_{\mathbf{A}^\infty})$, $f \in \pi$, $h \in \mathrm{H}(\mathbf{A}^\infty)$,we have

$$\Theta(h\phi, \chi)_\rho.hf = \chi_{\mathrm{H}}^{-1}(h)\Theta(\phi, \chi)_\rho.f \tag{2.3.2}$$

Let $K \subset \mathrm{H}(\mathbf{A}^\infty)$ satisfy that f and ϕ are invariant under $K \cap h^{-1}Kh$, and $\nu(K) \subset C := C(\chi)$. Then, with the notation ϕ_T of (2.2.6), we have

$$\begin{aligned} Z_T(h\phi, \chi)_{\rho,K}.hf &= \sum_{x \in K \backslash V^r_{\mathbf{A}^\infty}} \phi_{|T}(h^{-1}x) Z(x, \chi)_K h.f \\ &= \sum_{x \in K \backslash V^r_{\mathbf{A}^\infty}} \phi_{|T}(h^{-1}x) \chi_{\mathrm{H}}^{-1}(h) Z(h^{-1}x, \chi)_{h^{-1}Kh}.f \quad = \chi_{\mathrm{H}}^{-1}(h) Z_T(\phi, \chi).f, \end{aligned}$$

where we have used (2.2.5) and a change of variables. This proves (2.3.2). □

2.3.2. *Choices of test vectors.* — Denote by:

- $\wp$ a fixed set of p-adic places of F such that for all $v|p$, v splits in E, the Galois representation ρ is Panchishkin-ordinary (Definition 3.3.1) at each place of E above v, and the associated representation π_v satisfies the technical condition (3.3.4) below;
- S a fixed set of finite places of F, containing all the places $v \notin \wp$ such that for some place $w|v$, the representation $\rho_{|G_{E_w}}$ is ramified or w is ramified over $\mathbf{Q}$;
- $\mathscr{M}_1$ the set of split places of E not in S;
- $\mathscr{M}$ the set of subsets of $\mathscr{M}_1$ (we will identify $\mathscr{M}$ and $\wp$ with a set of squarefree ideals in $\mathscr{O}_F$);
- $\mathscr{M}[\wp]$ the set of ideals of the form $m\prod_{v\in\wp} v^{s_v}$ for $m \in \mathscr{M}$ and $s = (s_v) \in \mathbf{Z}_{\geqslant 0}^{\wp}$.

For $v \notin S\wp$, let $\mathscr{V}_v \subset V_v$ be a self-dual hermitian lattice in V_v, let $K_v^\circ \subset H_v$ be the stabliser of $\mathscr{V}_v^r$, and let $U_v^\circ \subset G_v$ be the stabliser of $\sum_{i=1}^r \mathscr{O}_{E_v} e_i$. Fix decompositions $\pi^\vee = \otimes_v' \pi_v^\vee$, $\sigma = \otimes_v' \sigma_v$, where the restricted tensor products are with respect to some spherical vectors $\varphi_v^\circ \in \pi_v^{\vee,\circ}$, $f_v^\circ \in K_v^\circ$ for all $v \notin S$.

We make the following choices of test vectors in $\pi_v \otimes \mathscr{S}(V_v^r) \otimes \sigma_v$ at all finite places v of F:

- for $v \notin S\wp$, define
$$\phi_v^\circ := \mathbf{1}_{\mathscr{V}_v^r} \in \mathscr{S}(V_v^r)^{K_v^\circ \times U_v^\circ},$$
$$\lambda_v^\circ := \varphi_v^\circ \otimes \phi_v^\circ \otimes f_v^\circ.$$
- for $v \in S$, we let $\lambda_v = \varphi_v \otimes \phi_v \otimes f_v \in \pi_v \otimes \mathscr{S}(V_v^r) \otimes \sigma_v$ be any element whose image in $\Lambda_{\rho,v}$ is nonzero;
- for $v \in \mathscr{M}_1$, we will define another Schwartz function
$$\phi_v^\bullet := (3.2.1) \in \mathscr{S}(V_v^r)^{K_v^\circ \times U_v^\circ}$$
below (where the subscripts v will be omitted from the notation), and we put
$$\lambda_v^\bullet := \varphi_v^\circ \otimes \phi_v^\bullet \otimes f_v^\circ;$$
- for $v \in \wp$, we will define vectors $\varphi^{\mathrm{a}} \in \pi^\vee$, $f^{\mathrm{a}} \in \sigma$ and a sequence of Schwartz functions
$$\phi_v^{(s)} \in \mathscr{S}(V_v^r), \qquad s \geqslant 0,$$
in Definition 3.3.10 below (where the subscripts v will be omitted from the notation). We put
$$\lambda_v^{(s)} := \varphi^{\mathrm{a}} \otimes \phi_v^{(s)} \otimes f^{\mathrm{a}}.$$

For $m = \prod_{v|m, v\in\mathscr{M}_1} v \prod_{v\in\wp} v^{s_v} \in \mathscr{M}[\wp]$, we put
$$\lambda^{(m)} := (\otimes_{v\notin Sm\wp} \lambda_v^\circ) \otimes (\otimes_{v|m, v\in\mathscr{M}_1} \lambda_v^\bullet) \otimes (\otimes_{v\in\wp} \lambda_v^{(s_v)}) \otimes \otimes_{v\in S} \lambda_v \qquad \in \pi^\vee \otimes \mathscr{S}(V_{\mathbf{A}^\infty}^r) \otimes \sigma;$$
we also put $\lambda^{(0)} := \lambda^{(1)}$, and define $\varphi^{(m)} \in \pi^\vee$, $f_v^{(m)} \in \sigma$ in the obvious way so that $\lambda^{(m)} = \varphi^{(m)} \otimes \phi^{(m)} \otimes f^{(m)}$. Note that $\varphi^{(m)}$ and $f^{(m)}$ are in fact independent of $m \in \mathscr{M}[\wp] \cup \{0\}$.

2.3.3. *The Euler system.* — For $m \in \mathscr{M}[\wp]$, we set
$$C(m) := (1 + m\widehat{\mathscr{O}}_E) \cap \mathrm{T}(\mathbf{A}^\infty), \qquad E[m] := E_{C(m)}.$$
For $m = 0$, we put $C(0) = \mathrm{T}(\mathbf{A}^\infty)$, $E[0] := E$.

Definition 2.3.2. — The *Euler system of Theta cycles* is the system of classes $(\Theta_m)_{m\in\mathscr{M}[\wp]\cup\{0\}}$ defined by
$$\Theta_m := \Theta^{C(m)}(\lambda^{(m)}) \qquad \in H_f^1(E[m], \rho).$$

The following theorem says precisely that $(\Theta_m)_m$ is an Euler system in the sense of Jetchev-Nekovář-Skinner. For a place w of E at which ρ is unramified, let $\mathrm{Fr}_w \in G_{E_w}$ be a geometric Frobenius at w, and let $P_w(t) := \det(1 - t\mathrm{Fr}_w | \rho^*(1))$.

Theorem 2.3.3. — *The system of classes* $(\Theta_m)_{m\in\mathscr{M}[\wp]\cup\{0\}}$ *of Definition 2.3.2 satisfies* $\mathrm{Tr}_{E[1]/E}\Theta_1 = \Theta_0$ *and the following conditions.*

1. Integrality. *There exists a* G_E*-stable* $\overline{\mathbf{Z}}_p$*-lattice* $\rho_0 \subset \rho$ *such that for every* $m \in \mathscr{M}[\wp]\cup\{0\}$,
$$\Theta_m \in H^1_f(E[m], \rho_0).$$
2. Horizontal norm relations. *For every* $m \in \mathscr{M}$ *and every* $v \in \mathscr{M}_1$ *not dividing* m,
$$\mathrm{Tr}_{E[mv]/E[m]}\Theta_{mv} = P_w(\mathrm{Fr}_w)\Theta_m,$$
where w *is any one of the places of* E *above* v.
3. Vertical norm relations. *For every* $m \in \mathscr{M}[\wp]$ *and every* $v \in \wp$,
$$\mathrm{Tr}_{E[mv]/E[m]}\Theta_{mv} = \Theta_m.$$

Remark 2.3.4. — By construction, the $\overline{\mathbf{Q}}_p$-vector space $\Lambda_{\rho,S} = \otimes_v \Lambda_{\rho,v}$ is 1-dimensional; thus the 'base class' Θ_0, which only depends on the image of $\lambda_S = \otimes_v \lambda_v$ in $\Lambda_{\rho,S}$, is independent of choices up to a scalar (after the initial choice of the descent π). The following proposition verifies the resulting necessary condition for the nonvanishing of (the base class of) our Euler system.

We say that ρ is *exceptional* at a place $v \in \wp$ if for some (equivalently,[9] every) place $w|v$ of E and embedding $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$, the Deligne–Langlands γ-factor
$$\gamma(\mathrm{WD}_\iota(\rho_w^+), \psi_{E,w}, s)$$
of the complex Weil–Deligne representation attached to ρ_w^+ by [Fon94] does not have a pole at $s=0$. A consideration of weighs shows that if ρ is crystalline at all $w|v$, then it is not exceptional.

Proposition 2.3.5. — *The image of* $\lambda^{(0)}$ *in* Λ_ρ *is nonzero if and only if* ρ *is not exceptional at any place* $v \in \wp$.

This is clearly a local statement, which will be proved in § 3.3.6.

2.4. Reduction of the Euler-system properties to local statements. — We reduce Theorem 2.3.3 to several local results, to be proved in the next section; for clarity, these results are marked with a '→'.

2.4.1. *Integral structures.* — Let $K \subset \mathrm{H}(\mathbf{A}^\infty)$, $U \subset \mathrm{G}(\mathbf{A}^\infty)$ be compact open subgroups fixing $\varphi^{(m)}$ and $f^{(m)}$. We consider the following integral structures on our representations.

- We let $\rho_{\overline{\mathbf{Z}}_p}$ be a $\overline{\mathbf{Z}}_p$-lattice in ρ, stable under G_E (this may require a choice that we now fix);
- Let $M_{\rho,\overline{\mathbf{Z}}_p,K} \subset M_{\rho,K}$ be a $\overline{\mathbf{Z}}_p$-lattice such that for each C, the image of $\mathrm{Ch}_{r-1}(X_{K,E_C})_{\overline{\mathbf{Z}}_p}$ of the cycle class map $(-)_\rho$ from Lemma 2.2.2 is contained in $H^1(E_C, M_{\rho,\overline{\mathbf{Z}}_p,K})$. (As explained in [Nek95, § II.1.10], we may take $M_{\rho,\overline{\mathbf{Z}}_p,K} = p^{-a}e_\rho H^{2r-1}_{\text{ét}}(X_{K,\overline{E}}, \overline{\mathbf{Z}}_p(r))$, where p^a is the order of the torsion subgroup of $H^{2r-1}_{\text{ét}}(X_{K,\overline{E}}, \overline{\mathbf{Z}}_p(r))$.) Let $M_{\sigma,\overline{\mathbf{Z}}_p,K} := M_{\sigma,K} \cap M_{\rho,\overline{\mathbf{Z}}_p,K}$. We define
$$\sigma^K_{\overline{\mathbf{Z}}_p} := \mathrm{Hom}_{\overline{\mathbf{Q}}_p[K\backslash \mathrm{H}_V(\mathbf{A}^\infty)/K]}(M_{\sigma,\overline{\mathbf{Z}}_p,K}, \rho_{\overline{\mathbf{Z}}_p}),$$
a $\overline{\mathbf{Z}}_p$-lattice.
- Let $\mathscr{H}_{\overline{\mathbf{Z}}_p}$ be the preimage of $\underline{\mathrm{SF}}_{\overline{\mathbf{Z}}_p}$ under the map $\underline{\mathbf{q}}$ of (2.1.1). We define
$$\pi^{U,\vee}_{\overline{\mathbf{Z}}_p} := \pi^{U,\vee} \cap \mathrm{Hom}(\mathscr{H}^U_{\overline{\mathbf{Z}}_p}, \overline{\mathbf{Z}}_p),$$

[9]This follows from multiplicativity and functional equation of γ-factors, the selfduality of ρ, and the fact that, by weight considerations, for every w the factor $\gamma(\mathrm{WD}_\iota(\rho_w), \psi_{E,w}, s)$ has neither a zero nor a pole at $s=0$.

where $\pi^{U,\vee}$ is viewed as a subspace of $\mathrm{Hom}(\mathscr{H}_{\overline{\mathbf{Q}}_p},\overline{\mathbf{Q}}_p)$ via the composition of the natural duality and the projection $\mathscr{H}_{\overline{\mathbf{Q}}_p}\to\pi$.

- For $x\in V^r_{\mathbf{A}^\infty}$ and $C\in\mathscr{C}$, let $V(x)$ and $\mathrm{H}(x)(\mathbf{A}^\infty)$ be as in § 2.2.2, and let $K_x:=K\cap\mathrm{H}(x)$, $K_x^C:=K_x\cap\nu^{-1}_{\mathrm{H}(x)}(C)$. For each $C\in\mathscr{C}$, we define
$$\mathscr{S}(V^r_{\mathbf{A}^\infty},\overline{\mathbf{Z}}_{p,C})^K\subset\mathscr{S}(V^r_{\mathbf{A}^\infty},\overline{\mathbf{Q}}_p)^K$$
to be the $\overline{\mathbf{Z}}_p$-module of functions satisfying that for every $x\in\mathrm{Spt}(\phi)$,
$$\mathrm{vol}(K)\cdot\phi(x)\cdot[K_x:K_x^C]^{-1}\in\overline{\mathbf{Z}}_p.$$

Similar integrality properties for Schwartz functions are considered by Shah in [Sha, §3.5].

Remark 2.4.1. — The $\overline{\mathbf{Z}}_p$-submodule $\mathscr{H}^U_{\overline{\mathbf{Z}}_p}$ is a $\overline{\mathbf{Z}}_p$-lattice in the subspace $\mathscr{H}^{U,\circ}_{\overline{\mathbf{Q}}_p}\subset\mathscr{H}^U_{\overline{\mathbf{Q}}_p}$ consisting of forms with (uniformly) bounded q-expansions. (We conjecture that $\mathscr{H}^{U,\circ}_{\overline{\mathbf{Q}}_p}=\mathscr{H}^U_{\overline{\mathbf{Q}}_p}$; at least when U is hyperspecial at p-adic places, this should be provable by considering q-expansion maps on integral models of PEL Shimura varieties related to G, cf. [Lan12, Remark 5.2.14].) This implies that $\pi^{U,\vee}_{\overline{\mathbf{Z}}_p}\subset\pi^{U,\vee}$ contains a $\overline{\mathbf{Z}}_p$-lattice.

Lemma 2.4.2. — *For every $C\in\mathscr{C}$, we have*
$$\varphi\in\pi^{U,\vee}_{\overline{\mathbf{Z}}_p},\quad\phi\in\mathscr{S}(V^r_{\mathbf{A}^\infty},\overline{\mathbf{Z}}_{p,C}))^{U\times K},\quad f\in\sigma^K_{\overline{\mathbf{Z}}_p}\qquad\Longrightarrow\qquad\Theta^C(\varphi,\phi,f)\in H^1_f(E_C,\rho_{\overline{\mathbf{Z}}_p}).$$

Proof. — By the definitions, it suffices to prove that
$$\Theta^C(\phi)_K\in\mathrm{Ch}_{r-1}(X_{K,E_C})_{\overline{\mathbf{Z}}_p}\otimes_{\overline{\mathbf{Z}}_p}\mathscr{H}^U_{\overline{\mathbf{Z}}_p},$$
that is, that for all $x\in\mathrm{Spt}(\phi)$ with $T(x)\in\mathrm{Herm}_r(F)$ and for all $t\in T(\mathbf{A}^\infty)$, we have
$$\mathrm{vol}(K)\cdot\phi(x)\cdot[K_x:K_x^C]^{-1}\cdot\left(c_1(\mathscr{L}_K^\vee)^{\dim V(x)-r}_{|X_K^{tC}}\frown[j_x(X^{tC}_{\mathrm{H}(x),K_x})]\right)\in\mathrm{Ch}_{r-1}(X_{K,E_C})_{\overline{\mathbf{Z}}_p}.$$
This is immediate from the definition of $\mathscr{S}(V^r_{\mathbf{A}^\infty},\overline{\mathbf{Z}}_{p,C}))^{U\times K}$. □

2.4.2. *Integrality.* — We reduce the integrality of the Euler system to a result on the local integrality of our Schwartz functions.

For $x_v\in V_v^r$, let $V(x_v)=\mathrm{Span}(x_{v,1},\ldots,x_{v,r})^\perp$, let $H(x_v)=U(V(x_v))$. For open commpact subgroups $U_v\subset G_v$, $K_v\subset H_v$ and $C_v\subset T_v$, let $K_{x_v}:=K_v\cap H(x_v)$, $K_{x_v}^{C_v}:=K_{x_v}\cap\nu^{-1}_{H(x_v)}(C_v)$. We define
$$\mathscr{S}(V_v^r),\overline{\mathbf{Z}}_{p,C_v})^{U_v\times K_v}\subset\mathscr{S}(V_v^r)^{U_v\times K_v}$$
to be the $\overline{\mathbf{Z}}_p$-module of functions ϕ_v satisfying that for every $x_v\in\mathrm{Spt}(\phi_v)$,
$$\mathrm{vol}(K_v)\cdot\phi_v(x_v)\cdot[K_{x_v}:K_{x_v}^{C_v}]^{-1}\in\overline{\mathbf{Z}}_p.$$

For a finite place v of F, we denote by ϖ_u a fixed uniformiser. If v is a split finite place of F, we denote $C(\varpi_v^e):=1+\varpi_v^e\mathscr{O}_{E,v}\cap T_v$. The following will be proved as Propositions 3.2.3 and 3.3.13 below.

Proposition$^\to$ 2.4.3. — *We have:*

1. *for $v\in\mathscr{M}_1$,*
$$\phi_v^\bullet\in\mathscr{S}(V_v^r),\overline{\mathbf{Z}}_{p,C(\varpi_v)})^{U_v^\circ\times K_v^\circ};$$
2. *for $v\in\wp$, there exist open compact subgroups $U_v\subset G_v$, $K_v\subset H_v$ fixing respectively φ_v^{a} and f_v^{a}, such that for every $s\geqslant 0$,*
$$\phi_v^{(s)}\in\mathscr{S}(V_v^r,\overline{\mathbf{Z}}_{p,C(\varpi_v^s)})^{U_v\times K_v}.$$

Proof of Theorem 2.3.3.1, assuming Proposition 2.4.3. — Proposition 2.4.3 implies that there are compact open subgroups $U \subset \mathrm{G}(\mathbf{A}^\infty)$ and $K \subset \mathrm{H}(\mathbf{A}^\infty)$, and an integer e_2 such that for every $m \in \mathscr{M}[\wp]$,

$$\phi^{(m)} \in p^{-e_2}\mathscr{S}(V^r_{\mathbf{A}^\infty}, \overline{\mathbf{Z}}_{p,C(m)})^{K\times U}.$$

Now Lemma 2.4.2 shows that if $e_1, e_3 \in \mathbf{Z}$ are such that $\varphi^{(m)} \in p^{-e_1}\pi^{\vee,U}_{\overline{\mathbf{Z}}_p}$, $f^{(m)} \in p^{-e_3}\sigma^K_{\overline{\mathbf{Z}}_p}$ (recall that $\varphi^{(m)}$ and $f^{(m)}$ are independent of m, so this is possible by Remark 2.4.1), then for the $\overline{\mathbf{Z}}_p$-lattice

$$\rho_0 := p^{-e_1-e_2-e_3}\rho_{\overline{\mathbf{Z}}_p} \subset \rho$$

we have $\Theta_m \in H^1_f(E[m], \rho_0)$, as desired. □

2.4.3. *Horizontal norm relations.* — We first reduce the norm relation of Theorem 2.3.3.2 to the following proposition.

Proposition→ 2.4.4. — *For all $m \in \mathscr{M}$ and $v \in \mathscr{M}_1$ with $v \nmid m$, all characters $\chi : \Gamma_{/C(m)} \to \overline{\mathbf{Q}}_p^\times$, and for every $\lambda_v \in \pi^{\vee,v} \otimes \mathscr{S}(V^r_{\mathbf{A}^{v\infty}}) \otimes \sigma^v$, we have*

$$\Theta(\lambda^v\lambda^\bullet_v, \chi) = P_w(\mathrm{Fr}_w)\Theta(\lambda^v\lambda^\circ_v, \chi) \tag{2.4.1}$$

in $H^1_f(E[m], \rho) \otimes_{\overline{\mathbf{Q}}_p} \mathscr{H}_{\overline{\mathbf{Q}}_p}$.

Proof of Theorem 2.3.3.2, assuming Proposition 2.4.4. — The identity (2.4.1) remains valid for any function χ on $\Gamma_{/C(m)}$. Then it suffices to apply it to $\chi = \mathbf{1}_{C(m)}$ and $\lambda^v = \lambda^{(m),v}$. □

We can further reduce Proposition 2.4.4 to the following abstract local analogue, to be proved in § 3.2.2. Analogously to (2.3.1), denote by $\mathscr{S}_{\chi_v}(V^r_v)$ the space $\mathscr{S}(V^r_v)$ equipped with the $(G_v \times H_v)$-action by $\omega_{v,\chi_v} := \omega_v \otimes \chi^{-1}_{\mathrm{G},v} \otimes \chi_{\mathrm{H},v}$, and let

$$\Lambda_{\rho_v,\chi_v} := \left(\pi^\vee_v \otimes \mathscr{S}_{\chi_v}(V^r_v) \otimes \sigma_v\right)_{G_v\times H_v}. \tag{2.4.2}$$

Proposition 2.4.5. — *For every $v \in \mathscr{M}_1$ and every unramified character χ of $E^\times_v$, we have*

$$[\lambda^\bullet_v] = L(\rho^*(1)_w, \chi^{-1}_w, 0)^{-1} \cdot [\lambda^\circ_v] \tag{2.4.3}$$

in $\Lambda_{\rho,\chi,v}$.

Proof of Proposition 2.4.4, assuming Proposition 2.4.5. — Characters $\chi : \Gamma_{/C(m)} \to \overline{\mathbf{Q}}_p^\times$ are unramified at $v \nmid m$, and by (2.2.4), Fr_w acts by $\chi^{-1}(\mathrm{Fr}_w)$ on the generating series $\Theta(\phi,\chi)_\rho$. Thus the Galois element $P_w(\mathrm{Fr}_w)$ acts by the scalar $P_w(\chi^{-1}(\mathrm{Fr}_w)) = L(\rho^*(1)_w, \chi^{-1}_w, 0)^{-1}$, and the desired identity (2.4.1) simplifies to

$$\Theta(\lambda^v\lambda^\bullet_v, \chi) = L(\rho^*(1)_w, \chi^{-1}_w, 0)^{-1} \cdot \Theta(\lambda^v\lambda^\circ_v, \chi).$$

This identity is implied by Proposition 2.4.5 since, by Lemma 2.3.1, the map $\lambda_v \mapsto \Theta(\lambda^v\lambda_v, \chi)$ factors through $\Lambda_{\rho,\chi,v}$. □

2.4.4. *Vertical norm relations.* — We reduce Theorem 2.3.3.3 to the following.

Proposition→ 2.4.6. — *Let $v \in \wp$. For every compact open subgroup $C_v \subset E^\times_v$, the image of $\lambda^{(s)}_v$ in $\Lambda^{C_v}_{\rho_v} = \bigoplus_{\chi_v \in \widehat{E^\times_v/C}} \Lambda_{\rho_v,\chi_v,v}$ is independent of $s \geqslant 0$.*

Proof of Theorem 2.3.3.3, assuming Proposition 2.4.6. — Let $s = v(m)$. We have

$$\mathrm{Tr}_{E[mv]/E}\Theta_{mv} = \mathrm{Tr}_{E[mv]/E[m]}\Theta^{C(mv)}(\lambda^{(m),v}\lambda^{(s+1)}_v) = \Theta^{C(m)}(\lambda^{(m),v}\lambda^{(s)}_v) = \Theta_m.$$

□

3. Local study

The goal of this section is to prove Propositions 2.4.3, 2.4.5, and 2.4.6, after giving the definition of the test vectors.

3.1. Preliminaries

3.1.1. *Notation.* — Let v be a finite place of F split in E. We work in a local setting over F_v and drop all subscripts v (thus writing F, E, V, W, … for F_v, E_v, V_v, W_v, …). We denote by $d \in \mathscr{O}_F$ a generator of the different ideal of F.

We fix an ordering of the two primes w_1, w_2 above v and write $E = E_{w_1} \times E_{w_2} = F \times F$; we put $\mathrm{i} := (1,-1) \in E$. We fix a uniformiser ϖ of F, and we denote by k the residue filed of F, by q its cardinality.

Write $V = V_v = V_1 \oplus V_2$ where the isotropic subspaces $V_i := V \otimes_F E_{w_i}$. Write $W = E^{2r} = W^+ \oplus W^-$ where $W^+ = \mathrm{Span}(e_1, \dots, e_r)$, $W^- = \mathrm{Span}(e_{r+1}, \dots, e_{2r})$. For $? = \emptyset, +, -$, write $W^? = W_1^? \oplus W_2^?$. Let $\mathscr{W} := \bigoplus_{i=1}^n \mathscr{O}_E e_i$, and if '?' is any decoration, let $\mathscr{W}_? = W_? \cap \mathscr{W}$.

We denote

$$H = \mathrm{H}(F) \cong \mathrm{Aut}_F(V_1), \qquad G = \mathrm{G}(F) \cong \mathrm{Aut}_F(W_1), \qquad T = \mathrm{T}(F) \cong F^\times,$$

where H acts on V_1 on the left, and G acts on W_1 on the right.

Fix an isometry between V and $W^\vee$, where the latter is endowed with the hermitian form dual to the form $(y, y')_W := \mathrm{i}\langle y, y' \rangle_W$ on W. The chosen isometry between V and (the hermitian space attached to) $W^\vee$ induces isomorphisms[10]

$$H = \mathrm{Aut}_F(V_1) \to \mathrm{Isom}_F(W_1^\vee, V_1) \leftarrow \mathrm{Aut}_F(W_1^\vee) = G. \tag{3.1.1}$$

By the isomorphism $W_i \cong V_i^\vee$ (for $i = 1, 2$), we have $\mathscr{O}_F$-lattices $\mathscr{V}_i = \mathscr{W}_i^\vee$, and direct summands $V_i^\pm = (W_i^\pm)^\vee \subset V_i$; we let $\mathscr{V}_i^\pm := \mathscr{V}_i \cap V_i^\pm$. We will often write

$$x_i^\pm = \begin{pmatrix} x_{i,+}^\pm \\ x_{i,-}^\pm \end{pmatrix} \in V_i \otimes W_i^\pm = \mathrm{Hom}(V_i^\pm, V_i) = \mathrm{Hom}(V_i^\pm, V_i^+) \oplus \mathrm{Hom}(V_i^\pm, V_i^-).$$

3.1.2. *Subgroups of G and H.* — We denote still by H the algebraic group over $\mathscr{O}_F$ with $\mathrm{H}(R) = \mathrm{Aut}_R(\mathscr{V}_1 \otimes_{\mathscr{O}_F} R)$ for any $\mathscr{O}_F$-algebra R; we similarly extend G to a group over $\mathscr{O}_F$ (isomorphic to H). We write $P = \mathrm{P}(F) \subset G = \mathrm{GL}(W_1) = \mathrm{GL}_n(F)$ for the Siegel parabolic, with Levi $M \cong \mathrm{GL}(W_1^+) \times \mathrm{GL}(W_1^-) =: G^+ \times G^-$ and unipotent radical $N = \mathrm{N}(F)$.

We put

$$K^\circ := \mathrm{G}(\mathscr{O}_F),$$

and define (deeper) pro-p parahoric subgroups of $\mathrm{G}(\mathscr{O}_F)$ of level $s \geqslant 1$ by

$$I_s = \mathrm{G}(\mathscr{O}_F) \times_{\mathrm{G}(\mathscr{O}_F/\varpi^s \mathscr{O}_F)} \mathrm{N}(\mathscr{O}_F/\varpi^s \mathscr{O}_F).$$

Via the identification (3.1.1), we may also view the above as subgroups of H; when the context is ambiguous (and the distinction is needed), we will add a superscript 'G' or 'H' to the notation for those subgroups in order to distinguish the ambient group in question.

Finally, we will need the following definition.

Definition 3.1.1. — Let H' be a general linear group over F and let $s \geqslant 0$ be an integer. We say that a subgroup $K \subset H'$ has *Galois-level at least s* if $\det(K) \subset 1 + \varpi^s \mathscr{O}_F$.

[10]Note that G acts on the left on $W_1^\vee$.

For the rest of this section, unless noted otherwise: all tensor products of finite-dimensional F-vector spaces are taken over F; Schwartz spaces consist of $\overline{\mathbf{Q}}_p$-valued functions; all tensor products of $\overline{\mathbf{Q}}_p$-vector spaces are taken over $\overline{\mathbf{Q}}_p$.

3.1.3. *Weil action and linear action on Schwartz spaces.* — For any smooth character of $F^\times$, denote by

$$\mathscr{S}'_\chi(V_1 \otimes W_1)$$

the Schwartz space of $V_1 \otimes W_1$ endowed with the action of $G \times H$ given by

$$(g,h).\phi'(y) = \chi^{-1}(\det g)\chi(\det h)|\det h|^{-r}|\det g|^r \phi'(h^{-1}yg). \tag{3.1.2}$$

We still denote by $\mathscr{S}_\chi(V \otimes_E W^+)$ the Schwartz space of

$$V \otimes_E W^+ \cong (V_1 \otimes W_1^+) \oplus (V_2 \otimes W_2^+)$$

endowed with the twisted Weil action $\omega_\chi = \omega \otimes \chi_G^{-1} \otimes \chi_H$. (When we are interested in the Schwartz space only and not the specific action, we will omit the subscript χ.)

Define a linear map

$$\begin{aligned}\mathscr{F} : \mathscr{S}'_\chi(V_1 \otimes W_1) = \mathscr{S}'_\chi((V_1 \otimes W_1^+) \oplus (V_1 \otimes W_1^-))\\ \to \mathscr{S}_\chi((V_1 \otimes W_1^+) \oplus (V_2 \otimes W_2^+)) = \mathscr{S}_\chi(V \otimes_E W^+)\end{aligned} \tag{3.1.3}$$

by the partial Fourier transform

$$\mathscr{F}\phi'(x_1^+, x_2^+) = \int_{V_1 \otimes W_1^-} \phi'(x_1^+, x_1^-)\psi(\langle x_1^-, x_2^+ \rangle)\, \mathrm{d}x_1^-$$

where $\langle\, ,\, \rangle$ is the natural duality between $V_1 \otimes W_1^-$ and $V_2 \otimes W_2^+$ given by the restriction of $(\, ,\,)_V \otimes (\, ,\,)_W$, and $\mathrm{d}x_1^-$ is the self-dual Haar measure, which assigns volume $|d|^{r^2}$ to $\mathscr{V}_1 \otimes_{\mathscr{O}_F} \mathscr{W}_1^+$.

Lemma 3.1.2. — *The map $\mathscr{F} : \mathscr{S}'_\chi(V_1 \otimes W_1) \to \mathscr{S}_\chi(V \otimes_E W^+)$ is an isomorphism of $(G \times H)$-modules.*

Proof. — It is easy to verify by explicit computation that $\mathscr{F}$ is $(H \times G)$-equivariant (for a brief discussion in a more general context, see [GQT14, § 2.9]). It is also clear that the dual partial Fourier transform gives an explicit inverse. □

3.1.4. *Godement–Jacquet zeta integrals as models for the Howe correspondence.* — We fix an embedding $\overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ extending ι°, via which we may base-change all L-values, functionals, and representations – without changing the notation.

By the uniqueness of the Howe correspondent σ in § 2.1.5 together with Lemma 3.1.3 below, we have $\sigma \cong \pi$ under the isomorphism (3.1.1). Fixing a nontrivial H-equivariant pairing $(\, ,\,)_\pi : \pi^\vee \otimes \pi \to \mathbf{C}$, we have the Godement–Jacquet zeta integral

$$\begin{aligned}\zeta(\cdot,\chi) \colon \pi^\vee \otimes \mathscr{S}'_\chi(V_1 \otimes W_1) \otimes \sigma \to \mathbf{C}\\ (\varphi, \phi', f) \mapsto \int_G (g\varphi, f)_\pi \cdot \phi'(g)\chi^{-1}(\det g)|\det g|^r \, \mathrm{d}g\end{aligned} \tag{3.1.4}$$

where $\mathrm{d}g$ is the Haar measure on G assigning volume $|d|^{n^2/2}$ to $K^\circ := \mathrm{Aut}_{\mathscr{O}_F}(\mathscr{V}_1) \subset G \cong H$. Let

$$\theta(\cdot,\chi) := \zeta(\cdot,\chi) \circ \mathscr{F}^{-1} \in \Lambda^\vee_{\rho,\chi} \tag{3.1.5}$$

(where $\Lambda_{\rho,\chi} = (2.4.2)$).

Lemma 3.1.3. — *The functional $\theta(\cdot,\chi)$ is a generator of $\Lambda^\vee_{\rho,\chi}$.*

Proof. — By Lemma 3.1.2, this is equivalent to the assertion that $\zeta(\cdot,\chi)$ is a nonzero element of $\mathrm{Hom}_{G\times H}(\pi^\vee\otimes\mathscr{S}'_\chi(V_1\otimes W_1)\otimes\sigma,\mathbf{C})$. The belonging is easily verified. For the nonvanishing, it suffices to apply $\zeta(\cdot,\chi)$ to a triple (φ,ϕ',f) such that $(\varphi,f)_\pi\neq 0$ and ϕ' has small support near the identity. □

3.2. Test vectors, norm relations and integrality at places in $\mathscr{M}_1$. — Suppose that $v\in\mathscr{M}_1$.

3.2.1. *Definition of the Schwartz function* $\phi^\bullet$. — If P is a logical proposition, denote by

$$\mathbf{1}[P]\in\{1=\text{true},0=\text{false}\}$$

its truth value (thus for the characteristic function of a set A we have $\mathbf{1}_A(x)=\mathbf{1}[x\in A]$). Let

$$\phi'^\bullet:=\mathbf{1}_{\mathrm{Aut}_{\mathscr{O}_F}(\mathscr{V}_1)}=\sum_{h\in K^\circ/I_1}h\phi_0'^{(v)}\qquad\in\mathscr{S}'(V_1\otimes W_1),$$

where

$$\phi_0'^\bullet(x_1):=\mathbf{1}\left[x_1=\begin{pmatrix}x_{1,+}^+ & x_{1,+}^-\\ x_{1,-}^+ & x_{1,-}^-\end{pmatrix}\in\begin{pmatrix}\mathrm{id}_{V_1^+}+\varpi\mathrm{End}(\mathscr{V}_1^+) & \mathrm{Hom}(\mathscr{V}_1^-,\mathscr{V}_1^+)\\ \varpi\mathrm{Hom}(\mathscr{V}_1^+,\mathscr{V}_1^-) & \mathrm{id}_{V_1^-}+\varpi\mathrm{End}(\mathscr{V}_1^-)\end{pmatrix}\right].$$

The stabiliser of $\phi_0'^\bullet$ under the action (3.1.2) (for $\chi=\mathbf{1}$) of $K^\circ\subset H=\mathrm{Aut}_F(V_1)$ is I_1.

W define

$$\phi^\bullet:=\mathscr{F}\phi'^\bullet=\sum_{h\in K^\circ/I_1}h\phi_0^\bullet,\qquad \phi_0^\bullet:=\mathscr{F}\phi_0'^\bullet. \tag{3.2.1}$$

The choice of $\phi^\bullet$ is motivated by Proposition 3.2.1 below.

3.2.2. *Local horizontal norm relations.* — We prove Proposition 2.4.5 by a computation of zeta integrals.

Proposition 3.2.1. — *For* $?\in\{\circ,\bullet\}$, *let* $\lambda^?:=\varphi^\circ\otimes\phi^?\otimes f^\circ\in\pi\otimes\mathscr{S}(V^r)\otimes\sigma$. *For every unramified character* χ *of* $F^\times$, *we have*

$$[\lambda^\bullet]=L(1/2,\pi(\chi)^\vee)^{-1}\cdot[\lambda^\circ]$$

in $\Lambda_{\rho,\chi}$.

Proof. — By [GJ72, Lemma 6.10] and, respectively, the definition, we have

$$\zeta(\varphi^\circ,\phi'^\circ,f^\circ,\chi)=L(1/2,\pi(\chi)^\vee)\cdot(\varphi^\circ,f^\circ)_\pi,$$
$$\zeta(\varphi^\circ,\phi'^\bullet,f^\circ,\chi)=(\varphi,f)_\pi.$$

By Lemma 3.1.3, this implies the desired result. □

Proof of Proposition 2.4.5. — It is equivalent to Proposition 3.2.1, once noted that $L(\rho^*(1)_w,\chi_w^{-1},s)=L(s+1/2,\pi(\chi)_v^\vee)$. □

3.2.3. *A decomposition.* — We begin a study of the function $\phi^\bullet$, with the final goal of establishing its integrality properties.

We have

$$\begin{aligned}\mathscr{F}\phi_0'^\bullet(x_1^+,x_2^+)&=\mathbf{1}\left[\begin{pmatrix}x_{1,+}^+ & x_{2,+}^+\\ x_{1,-}^+ & x_{2,-}^+\end{pmatrix}\in\begin{pmatrix}\mathrm{id}_{V_1^+}+\varpi\mathrm{End}(\mathscr{V}_1^+) & \varpi^{-1}\mathrm{End}(\mathscr{V}_2^+)\\ \varpi\mathrm{Hom}(\mathscr{V}_1^+,\mathscr{V}_1^-) & \mathrm{Hom}(\mathscr{V}_2^+,\mathscr{V}_2^-)\end{pmatrix}\right]\cdot\psi(\mathrm{Tr}(x_{2,+}^+))\\ &=\sum_{y\in Y}\phi_{0,y}^\bullet(x_1^+,x_2^+),\end{aligned}$$

where $Y:=\varpi^{-1}\mathrm{End}(\mathscr{V}_2^+)/\mathrm{End}(\mathscr{V}_2^+)$ and

$$\phi_{0,y}^\bullet(x_1^+,x_2^+):=\psi(\mathrm{Tr}(y))\cdot\mathbf{1}\left[\begin{pmatrix}x_{1,+}^+ & x_{2,+}^+\\ x_{1,-}^+ & x_{2,-}^+\end{pmatrix}\in\begin{pmatrix}\mathrm{id}_{V_1^+}+\varpi\mathrm{End}(\mathscr{V}_1^+) & y+\mathrm{End}(\mathscr{V}_2^+)\\ \varpi\mathrm{Hom}(\mathscr{V}_1^+,\mathscr{V}_1^-) & \mathrm{Hom}(\mathscr{V}_2^+,\mathscr{V}_2^-)\end{pmatrix}\right].$$

3.2.4. *Rationality.* — It is clear that the functions $\phi^{\bullet}_{0,y}$ take values in $\mathbf{Q}(\mu_{\ell^\infty})$, where ℓ is the rational prime underlying v. The following lemma is not strictly necessary, but we include it for completeness.

Lemma 3.2.2. — *For $u\in \mathbf{Z}_\ell^\times$, let $\sigma_u\in \mathrm{Gal}(\mathbf{Q}(\mu_{\ell^\infty}/\mathbf{Q}))$ be its image under the reciprocity map.*

1. *For all $u\in\mathbf{Z}_p^\times$, we have $\phi^{\bullet}_{0,y}(x)^{\sigma_u}=\phi^{\bullet}_{0,u^{-1}y}\left(\begin{pmatrix}1_r & \\ & u1_r\end{pmatrix}x\right)$.*
2. *The function $\phi^\bullet$ takes values in $\mathbf{Z}$.*

Proof. — Part 1 follows from the relation $\psi(t)^{\sigma_u}=\psi(u^{-1}t)$ (for all $t\in F$) and the definitions, using the description of the action of $\begin{pmatrix}1_r & \\ & u1_r\end{pmatrix}\in \mathrm{GL}_n(F)\cong H$ on $V\otimes_E W^+$ given in (3.2.3), (3.2.4) below. From part 1 and the definitions, since $\begin{pmatrix}1_r & \\ & u1_r\end{pmatrix}\in K^\circ$ it follows that $\mathrm{Gal}(\mathbf{Q}(\mu_\ell^\infty/\mathbf{Q}))$ fixes $\phi^\bullet$; it is also clear that the values of ϕ are integers. $\square$

3.2.5. *Levels and integrality.* — We now consider V as endowed with the dual basis $b_1,\ldots,b_n$ to the standard basis of $W=E^n$; thus $V_1^+=\mathrm{Span}(b_1,\ldots,b_r)$, $V_1^-=\mathrm{Span}(b_{r+1},\ldots,b_n)$ and the hermitian form on V has matrix $\mathrm{i}\cdot\begin{pmatrix}0_r & -1_r\\ 1_r & 0_r\end{pmatrix}$ in this basis. We may then identify $V_1^\pm=F^n$ and $H=\mathrm{GL}(V_1)=\mathrm{GL}(W_1^\vee)=\mathrm{GL}_n(F)$.

We will prove the follwoing more precise form of Proposition 2.4.3.

Proposition 3.2.3. — *For $x\in V^r$, set $K_x^{(v)}:=K_x\cap \nu^{-1}(C(\varpi_v))\subset K_x$.*

1. *There exists a decomposition into $\mathbf{Z}$-valued Schwartz functions with disjoint supports*
$$\phi^\bullet=\phi^{\bullet,\mathbb{O}}+\phi^{\bullet,\times}$$
such that:
   - *$\phi^{\bullet,\mathbb{O}}$ takes values in $(q-1)\mathbf{Z}$;*
   - *if $x\in\mathrm{Spt}(\phi^{\bullet,\times})$, then $K_x^{(v)}:=K\cap H$ has Galois-level at least 1.*
2. *For all $x\in\mathrm{Spt}(\phi^\bullet)$, we have $\phi^\bullet(x)[K_x:K_x^{(v)}]^{-1}\in\mathbf{Z}$.*

We start by studying the stabilisers of the functions $\phi^\bullet_{0,y}$.

Lemma 3.2.4. — *Let $y\in Y\cong\varpi^{-1}\mathrm{M}_r(\mathscr{O}_F)/\mathrm{M}_r(\mathscr{O}_F)$, and let*
$$K'_y:=\{d\in\mathrm{GL}_r(\mathscr{O}_F)\mid (d^{\mathrm{t}}-1_r)y\subset \mathrm{M}_r(\mathscr{O}_F)\}.$$
The stabiliser K_y of $\phi^\bullet_{0,y}$ under the action of $K^\circ\subset H$ is the subgroup

$$\begin{pmatrix}1_r+\varpi\mathrm{M}_r(\mathscr{O}_F) & \mathrm{M}_r(\mathscr{O}_F)\\ \varpi\mathrm{M}_r(\mathscr{O}_F) & K'_y\end{pmatrix}\supset I_1. \tag{3.2.2}$$

Proof. — It is clear that $I_1\subset K_y$, so computing K_y is equivalent to computing its image $\overline{K}_y\subset \mathrm{H}(k)$.

Let $h=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in K^\circ$, where $a,b,c,d\in\mathrm{M}_r(\mathscr{O}_F)$. We have

$$h\begin{pmatrix}x_{1,+} & x_{2,+}\\ x_{1,-} & x_{2,-}\end{pmatrix}=\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x_{1,+}\\ x_{1,-}\end{pmatrix},\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\begin{pmatrix}x_{2,+}\\ x_{2,-}\end{pmatrix}\right) \tag{3.2.3}$$

where the unitarity of h means that $\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}$ is characterised by

$$\begin{cases}a^{\mathrm{t}}d'-c^{\mathrm{t}}b'=1_r\\ d^{\mathrm{t}}a'-b^{\mathrm{t}}c'=1_r\end{cases}\qquad \begin{cases}b^{\mathrm{t}}d'-d^{\mathrm{t}}b'=0_r\\ c^{\mathrm{t}}a'+a^{\mathrm{t}}c'=0_r.\end{cases} \tag{3.2.4}$$

Denote by '$\equiv$' the relation of congruence modulo ϖ on free $\mathscr{O}_F$-modules. If $h \in K_y$, then for $\begin{pmatrix} x_{1,+} \\ x_{1,-} \end{pmatrix} = \begin{pmatrix} 1_r \\ 0_r \end{pmatrix}$ we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_{1,+} \\ x_{1,-} \end{pmatrix} \equiv \begin{pmatrix} 1_r \\ 0_r \end{pmatrix}$, so that

$$a \equiv 1_r, \qquad c \equiv 0_r,$$

and (3.2.4) implies $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \equiv \begin{pmatrix} d^{\mathrm{t},-1} & d^{\mathrm{t},-1}b^{\mathrm{t}} \\ 0_r & 1_r \end{pmatrix}$. From this, (3.2.3), and the definition of $\phi^{\bullet}_{0,y}$, we see that $h \in K_y$ if and only if further

$$d^{\mathrm{t},-1}y + d^{\mathrm{t},-1}b^{t}\mathrm{M}_r(\mathscr{O}_F) \subset y + \mathrm{M}_r(\mathscr{O}_F),$$

that is $(d^{\mathrm{t}} - 1_r)y \subset \mathrm{M}_r(\mathscr{O}_F)$, as desired. □

For $y \in Y$, write $d(y) = d$ if the image of ϖy in $\mathrm{GL}_r(k)$ has rank d, and let

$$Y^{\times} := \{y \in Y \mid d(y) = r\} \subset Y.$$

From Lemma 3.2.4, we deduce the following.

Lemma 3.2.5. — *Suppose $y \in Y - Y^{\times}$, and let $K_y :=$ (3.2.2). The integer $|K_y/I_1|$ is a multiple of $q-1$.*

Proof. — Note that if $d(y) = d$, then the reduction $\overline{K}'_y \subset \mathrm{GL}_r(k)$ of K'_y is $\mathrm{GL}_r(k)$-conjugate to $\overline{K}'_d := \begin{pmatrix} 1_d & \mathrm{M}_{d,r-d}(k) \\ 0_{r-d,r} & \mathrm{GL}_{r-d}(k) \end{pmatrix}$. Therefore $|K_y/I_1| = |\overline{K}'_d|$, and when $d < r$ the determinant maps the last group onto $k^{\times}$. □

Remark 3.2.6. — For $x \in \mathrm{Spt}(\phi^{\bullet})$, it is easy to see that there is exactly one integer $0 \le d \le r$ such that for some $y \in Y$ with $d(y) = d$ and some $h \in K^{\circ}$, we have $h \in \mathrm{Spt}(h\phi^{\bullet}_{0,y})$. We denote this integer by $d(x)$.

Let us identify $V \otimes_E W^{+} = V^{r}$ and denote a typical element by x (rather than x^{+}). For $x \in V^{r}$, denote $V(x) = \mathrm{Span}(x)^{\perp} \subset V$ (an r-dimensional hermitian subspace), and let $H(x) = U(V(x)) \subset U(V) = H$.

We complement the result of Lemma 3.2.5 by a lower bound for the Galois-level (Definition 3.1.1) along part of the support of $\phi^{\bullet}_{0,y}$ for $y \in \mathscr{Y}^{\times}$.

Lemma 3.2.7. — *Let $y \in Y^{\times}$ and let $x \in \mathrm{Spt}(\phi^{\bullet}_{0,y}) \subset V^{r}$. The group $K_x := K^{\circ} \cap H(x)$ has Galois-level at least* 1.

Proof. — The same calculations as in the proof of Lemma 3.2.4 (with the same notation '$\equiv$') show that if $h \in K_x \subset H(x)$, so that $hx = x$ (in particular $hx\begin{pmatrix} 1_r & \\ & \varpi 1_r \end{pmatrix} \equiv x\begin{pmatrix} 1_r & \\ & \varpi 1_r \end{pmatrix}$), then $h \in K_y = I_1$. Thus $\det h \equiv 1$. □

Proof of Proposition 3.2.3. — Part 2 follows from part 1: if $x \in \mathrm{Spt}(\phi^{\bullet,\mathbb{O}})$, it suffices to observe that $[K_x : K_x^{(v)}]$ is obviously a divisor of $q - 1 = [C(1) : C(\varpi)]$; whereas if $x \in \mathrm{Spt}(\phi^{\bullet,\times})$, we have $[K_x : K_x^{(v)}] = 1$ and $\phi(x) \in \mathscr{O}$.

We now prove part 1. We have

$$\begin{aligned}\phi^{\bullet} &= \sum_{h \in K^{\circ}/I_1} \sum_{y \in Y} h\phi^{\bullet}_{0,y} = \sum_{y \in Y} \sum_{h \in K^{\circ}/K_y} |K_y/I_1| \cdot h\phi^{\bullet}_{0,y} \\ &= \phi^{\bullet,\mathbb{O}} + \phi^{\bullet,\times}\end{aligned}$$

with
$$\phi^{\bullet,\oplus} := \sum_{y\in Y-Y^\times}\sum_{h\in K^\circ/K_y} |K_y/I_1|\cdot h\phi^\bullet_{0,y}.$$
$$\phi^{\bullet,\times} := \sum_{y\in Y^\times}\sum_{h\in K^\circ/I_1} h\phi^\bullet_{0,y}.$$
By Remark 3.2.6, the supports of $\phi^{\bullet,\oplus}$ and $\phi^{\bullet,\times}$ are disjoint. Both functions take values in $\mathbf{Q}$ by the same argument as in Lemma 3.2.2.2, and both are clearly integral.

By Lemma 3.2.5, the function $\phi^{\bullet,\oplus}$ takes values in $(q-1)\mathbf{Z}$. By Lemma 3.2.7, if $x\in \mathrm{Spt}(\phi^{\bullet,\times}) = \bigcup_{h\in K^\circ, y\in Y^\times}\mathrm{Spt}(h\phi^\bullet_{0,y})$, then $K_x^{(v)}$ has Galois-level at least 1. □

3.3. Test vectors, norm relations and integrality at places in $\wp$. — We suppose now that $v\in\wp$.

3.3.1. *P-ordinary representations.* — We recall two notion of ordinariness and show that they correspond under Langlands duality.

Definition 3.3.1. — A de Rham representation $\rho\colon G_F\to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is said to be *Panchishkin-ordinary* (after [Nek93, § 6.7]) if there exists a short exact sequence
$$0\to\rho^+\to\rho\to\rho^-\to 0$$
of de Rham representations of G_F with coefficients in $\overline{\mathbf{Q}}_p$, such that
$$\mathrm{F}^0\mathbf{D}_{\mathrm{dR}}(\rho^+) = \mathbf{D}_{\mathrm{dR}}(\rho^-)/\mathrm{F}^0\mathbf{D}_{\mathrm{dR}}(V^-) = 0.$$

We denote by $\mathrm{Fr}\in G_F$ a lift of the geometric Frobenius corresponding to the chosen uniformiser ϖ. We denote by $\mathrm{WD}(\rho)$ the Weil–Deligne representation over $\overline{\mathbf{Q}}_p$ attached to a de Rham representation ρ by [Fon94] (see also [TY07, § 1]).

Remark 3.3.2. — Suppose that ρ is Panchishkin-ordinary, and put $\mathrm{r}^\pm := \mathrm{WD}(\rho^\pm)^{\text{Fr-ss}}$, where the superscript denotes Frobenius-semisemplification. By construction, the multiset of slopes ($=p$-adic valuations of eigenvalues) of Fr on $\mathrm{r}^\pm$ coincides with the multiset of slopes of the $[F_0:\mathbf{Q}_p]^{\text{th}}$ power of the crystalline Frobenius on $\mathbf{D}_{\mathrm{pst}}(\rho^\pm)$. In particular, we observe all eigenvalues of of Fr on r^+ (respectively r^-) have strictly negative (respectively non-negative) p-adic valuation, and this condition uniquely determines $\mathrm{r}^\pm$ and $\rho^\pm$ (up to isomorphism).

Remark 3.3.3. — Suppose that ρ is Panchishkin-ordinary, and assume that the $\jmath$-Hodge–Tate weight of $\det\rho^\pm$ is independent of $\jmath\colon F\hookrightarrow \mathbf{C}_p$ and equal to $\mathrm{w}^\pm$. Let $\chi_{\mathrm{cyc}}\colon G_F\to\overline{\mathbf{Q}}_p^\times$ be the cyclotomic character, and let

$$\alpha := \chi_{\mathrm{cyc}}^{\mathrm{w}^+}\cdot\det\rho^+(\mathrm{Fr}). \tag{3.3.1}$$

Since the Newton and Hodge polygons of $\mathbf{D}_{\mathrm{pst}}(\rho^+)$ have the same endpoints, we have that $\alpha\in\overline{\mathbf{Z}}_p^\times$.

Denote by Ind_P^G the unitarily normalised induction, and denote by $\xi_{\pi^?}$ the central character of a representation $\pi^?$ of a general linear griup. The following definition is adapted from [Hid98].

Definition 3.3.4. — Let π be a smooth irreducible generic representation of $G=\mathrm{GL}_n(F)$ with coefficients in $\overline{\mathbf{Q}}_p$. Let $\mathrm{w}^+\in\mathbf{Z}_{\leq -1}$, $\mathrm{w}^-\in\mathbf{Z}_{\geqslant 0}$. We say that π is *P-ordinary for the Hodge–Tate weights* $(\mathrm{w}^+,\mathrm{w}^-)$ if there exists a G-equivariant surjection

$$\mathrm{p}_\pi\colon \mathrm{Ind}_P^G(\pi^-\boxtimes\pi^+)\to\pi \tag{3.3.2}$$

for some irreducible admissible representations $\pi^\pm$ of $G_r := \mathrm{GL}_r(F)$ such that

$$\xi_{\pi^+}|\cdot|^{r/2+w^+}(\varpi)\in\overline{\mathbf{Z}}_p^\times. \tag{3.3.3}$$

Lemma 3.3.5. — *Let $\rho\colon G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a de Rham representation. Let π be the smooth irreducible representation of $\mathrm{GL}_n(F)$ over $\overline{\mathbf{Q}}_p$ that corresponds to $\mathrm{WD}(\rho)$ under the local Langlands correspondence, (re)normalised so that $L(\rho, s) = L(s + 1/2, \pi)$.*

Suppose that ρ is Panchishkin-ordinary and the $\jmath$-Hodge–Tate weight of $\det \rho^{\pm}$ is independent of $\jmath : F \hookrightarrow \mathbf{C}_p$ and equal to $\mathrm{w}^{\pm}$.

Then π is P-ordinary for the Hodge–Tate weights $(\mathrm{w}^+, \mathrm{w}^-)$, with $\pi^{\pm}$ in (3.3.2) the representation corresponding to $\mathrm{WD}(\rho^{\pm})$ under the above local Langlands correspondence, and

$$\alpha = |\varpi|^{r/2+\mathrm{w}^+} \xi_{\pi^+}(\varpi).$$

Proof. — We freely use the theory of Bernstein–Zelevinsky and the properties of the local Langlands correspondence as summarised for instance in [Dis20, § 2], with the notation used there. We denote by

$$\pi_{-1/2}\colon \mathrm{r}' \mapsto \pi_{\mathrm{u}}(\mathrm{r}(-1/2)), \qquad \pi_{-1/2,\mathrm{ss}}\colon \mathrm{r}' \mapsto \pi_{\mathrm{u,ss}}(\mathrm{r}'_{|W_F}(-1/2))$$

the twists of the unitarily normalised local Langlands correspondence and, respectively, semisimple local Langlands correspondence of *loc. cit.*

Let $\pi^{\pm} = \pi_{-1/2}(\mathrm{r}^{\pm})$, and write $\pi^? = \pi_{\mathrm{u}}(\mathbf{s}^?)$ for some multisegments $\mathbf{s}^?$. Since π is generic, the segments in $\mathbf{s}$ can be ordered so as to satisfy the 'does not precede' condition above [Dis20, (2.2.1)]. By construction, the same is true of $\mathbf{s}^{\pm}$, which implies that $\pi^{\pm}$ is generic; and, together with Remark 3.3.2, no segment in $\mathbf{s}^+$ precedes a segment in $\mathbf{s}^-$. Now (ii) implies that the unique irreducible quotient of $\mathrm{Ind}_P^G(\pi^- \boxtimes \pi^+)$ is generic; and by construction, its supercuspidal support is the representation $\pi_{-1/2,\mathrm{ss}}(\mathrm{r})$. But there is a unique up to isomorphism generic irreducible representation with a given supercuspidal support. Since π is also generic and irreducible and has supercuspidal support $\pi_{-1/2,\mathrm{ss}}(\mathrm{r})$, we conclude that the surjection (3.3.2) exists. The formula for α is clear. $\square$

3.3.2. *The elements φ^{a} and f^{a}.* — We specialise back to our running assumptions, so that ρ is pure of weight -1 and Panchishkin-ordinary with $\jmath$-Hodge–Tate weights $\{-r, \ldots, r-1\}$ (for every $\jmath : F \hookrightarrow \mathbf{C}_p$); and π is the associated P-ordinary representation of G. In the notation of the previous paragraphs, we have $\mathrm{w}^+ = -\binom{r}{2}$, $\mathrm{w}^- = \binom{r-1}{2}$. Then $\pi^{\vee}$ is also P-ordinary for the weight $(\mathrm{w}^+, \mathrm{w}^-)$ with respect to the representations $\pi^{\vee,\pm} := \pi^{\mp,\vee}$. We note that $\alpha = q^{r^2/2}\xi_{\pi^+}(\varpi)$ and put

$$\alpha^{\vee} := q^{r^2/2}\xi_{\pi^{\vee,+}}(\varpi) = \xi_{\pi}^{-1}(\varpi)\alpha \in \overline{\mathbf{Z}}_p^{\times}.$$

We fix perfect parings $(\ ,\ )_{\pi^{\pm}} \colon \pi^{\vee,\mp} \otimes \pi^{\pm} \to \overline{\mathbf{Q}}_p$.

For any $s \in \mathbf{Z}$, we define elements of G by

$$t := \begin{pmatrix} \varpi 1_r & \\ & 1_r \end{pmatrix}, \qquad w_s := \begin{pmatrix} & 1_r \\ -\varpi^s 1_r & \end{pmatrix} = w t^s$$

and we put $t^w = w^{-1} t w = \left(\begin{smallmatrix} 1_r & \\ & \varpi 1_r \end{smallmatrix}\right)$. We also put

$$U_t = \sum_{b \in \mathrm{M}_r(k)} \begin{pmatrix} 1_r & b \\ & 1_r \end{pmatrix} t \qquad \in \mathbf{Z}[G];$$

then for all $s' \geqslant s \geqslant 1$, the double coset operator

$$I_{s'} t I_s$$

acts by U_t on any smooth G-module.

We introduce the following condition, which we assume from now on:

(3.3.4) $\qquad$ either π is unramified or $\pi^{\pm}$ are both supercuspidal.

Remark 3.3.6. — The reason for imposing this technical condition is that it(s second part) appears in the current version of [Mar.a]; it is expected to be removed in a future update.

Proposition 3.3.7. — *There exist*

$$\varphi^{\pm}\in\pi^{\vee,\pm},\quad f^{\pm}\in\pi^{\pm},\quad \varphi^{\rm ord}\in\pi,\quad f^{\rm ord}\in\pi$$

satisfying the following conditions:

1. $(\varphi^{-},f^{+})_{\pi^{+}}=(\varphi^{+},f^{-})_{\pi^{-}}=1$;
2. *there is a constant $c(\pi)$ such that the vectors $\varphi^{\rm ord}$, $f^{\rm ord}$ are invariant under $I_{c(\pi)}$ and satisfy*
$$U_t\varphi^{\rm ord}=\alpha^{\vee}\varphi^{\rm ord},\qquad U_tf^{\rm ord}=\alpha f^{\rm ord};$$
3. *for each $c\geqslant c(\pi)$, setting*
$$\varphi_c^{\rm a}:=q^{cr^2}\alpha^{\vee,-c}\pi^{\vee}(w_c)\varphi^{\rm ord},\qquad f_c^{\rm a}:=q^{cr^2}\alpha^{-c}\pi(w_c)f^{\rm ord},$$
we have, for every $y_{\pm}\in{\rm GL}_r(F)$,
$$(\pi^{\vee}(\left(\begin{smallmatrix}y_+&\\&y_-\end{smallmatrix}\right)w^{-1})\varphi_c^{\rm a},f_c^{\rm a})_{\pi}=|\det y_+|^{r/2}|\det y_-|^{-r/2}\cdot(\pi^{\vee,+}(y_+)\varphi^{+},f^{-})_{\pi^{-}}(\pi^{\vee,-}(y_-)\varphi^{-},f^{+})_{\pi^{+}}.$$
(The superscripts 'a' stand for 'anti-ordinary'.)

Proof. — Consider first the case where $\pi^{\pm}$ are supercuspidal. We deduce the proposition form the results of Marcil in [Mar.a, Mar.b]. Let $c(\pi)\geqslant 1$ be the minimal integer (denoted by r in [Mar.a, Mar.b]) for which the construction that we are about to cite can be performed. Let

$$f^{+},\,f^{-};\quad \varphi^{-},\,\varphi^{+};\quad f^{\rm M}_{c(\pi)}\in\pi^{I_{c(\pi)}},\,\varphi^{\rm M}_{c(\pi)}\in(\pi^{\vee})^{I^{\rm t}_{c(\pi)}}$$

be the vectors denoted respectively

$$\phi_a,\,\phi_b;\quad \mu\tilde\phi_a,\mu\tilde\phi_b;\quad \varphi,\,\tilde\phi$$

in [Mar.b, pp. 14-15] (we omit all the subscripts 'w' when transcribing notation from [Mar.a, Mar.b]), where we adjust the scalar $\mu\in\mathbf{Q}^{\times}$ so that $(\varphi^{\mp},f^{\pm})_{\pi^{\pm}}=1$. As noted in [Mar.b, Remarks 2.8, 2.11], the existence of such vectors, which may depend on some choices that we fix, is guaranteed under our assumptions by [Mar.a, Theorem 4.3, Lemma 4.6].

Moreover, denote by $\delta_P\colon\left(\begin{smallmatrix}g_+&\\&g_-\end{smallmatrix}\right)\mapsto|\det g_+|^{r}|\det g_-|^{-r}$ the modulus character of P; then by those results we have

$$I_{c(\pi)}t^{-1}I_{c(\pi)}f^{\rm M}_{c(\pi)}=\delta_P^{-1/2}(t)\xi_{\pi^{\vee,+}}(\varpi)f^{\rm M}_{c(\pi)},$$

that is

$$I_{c(\pi)}t^{w}I_{c(\pi)}f^{\rm M}_{c(\pi)}=\delta_P^{-1/2}(t)\xi_{\pi^{\vee,-,\vee}}(\varpi)f^{\rm M}_{c(\pi)}=\alpha f^{\rm M}_{c(\pi)};\tag{3.3.5}$$

and similarly

$$I^{\rm t}_{c(\pi)}tI^{\rm t}_{c(\pi)}\varphi^{\rm M}_{c(\pi)}=\alpha^{\vee}\varphi^{\rm M}_{c(\pi)}.\tag{3.3.6}$$

Finally, by the proof of [Mar.b, Proposition 4.3]:

$$(\pi^{\vee}(\left(\begin{smallmatrix}g_+&\\&g_-\end{smallmatrix}\right))\varphi_c^{\rm M},f_c^{\rm M})_{\pi}=\mu'_c\cdot|\det g_+|^{r/2}|\det g_-|^{-r/2}\cdot(\pi^{\vee,+}(g_+)\varphi^{+},f^{-})_{\pi^{-}}(\pi^{\vee,-}(g_-)\varphi^{-},f^{+})_{\pi^{+}},\tag{3.3.7}$$

for some constant μ'_c.

Let

$$f^{\rm a}_{c(\pi)}:={\mu'_{c(\pi)}}^{-1}\cdot f^{\rm M}_{c(\pi)},\qquad \varphi^{\rm a}_{c(\pi)}:=\pi^{\vee}(w)\varphi^{\rm M}_{c(\pi)},$$
$$f^{\rm ord}:=w^{-1}_{c(\pi)}f^{\rm a}_{c(\pi)},\qquad \varphi^{\rm ord}:=w^{-1}_{c(\pi)}\varphi^{\rm a}_{c(\pi)};$$

then by the definitions and (3.3.5), (3.3.6) we have part 2; and by (3.3.7) we have part 3.

The case where π is unramified can be similarly deduced from [DL24, Proposition 3.25], by taking $c(\pi)=1$ and $\varphi_1^{\mathrm{a}}=w_1\varphi^{\mathrm{ord}}$, $f_1^{\mathrm{a}}=w_1 f^{\mathrm{ord}}$ to be the vectors denoted respectively by $\varphi_v^{\vee}$ and φ_v in *loc. cit.* □

3.3.3. *Schwartz functions.* — Let us first introduce some notation. For a compact open subgroup $J\subset \mathrm{GL}_r(F)$ and any Haar measure $\mathrm{d}y$ on J, put

$$\delta_J(y):=\mathrm{vol}(J,\mathrm{d}y)^{-1}\cdot \mathbf{1}_J(y)$$
$$\psi_J(y'):=\mathrm{vol}(J,\mathrm{d}y)^{-1}\cdot\int_J \psi(\mathrm{Tr}(y^{\mathrm{t}}y'))\,\mathrm{d}y.$$

If $\Phi=(\Phi_{ij})$ is a matrix of $\overline{\mathbf{Q}}_p$-valued functions of the variables X_{ij} (viewed as the entries of a matrix X), we write

$$\Phi(X):=\prod_{i,j}\Phi_{ij}(X_{ij}).$$

Let $J_+\subset \mathrm{GL}_r(\mathscr{O}_F)$, respectively $J_-\subset\mathrm{GL}_r(\mathscr{O}_F)$, be an open subgroup fixing φ^+ and f^-, respectively φ^- and f^+. We fix the generator $d=\varpi^{v(d)}\in\mathscr{O}_F$ of the different ideal to be a power of ϖ. For every pair of integers $s,s'\geqslant 0$, let[11]

(3.3.8)
$$\phi^{(s,s')}\left(\begin{pmatrix} x & x'\\ y & y'\end{pmatrix}\right):=\mathbf{1}[x^{\mathrm{t}}y'-y^{\mathrm{t}}x'\in d^{-1}\mathrm{M}_r(\mathscr{O})]\cdot\begin{pmatrix}\mathbf{1}_{\varpi^{-s}d^{-1}\mathrm{M}_r(\mathscr{O})} & \mathbf{1}_{\varpi^{-s'}\mathrm{M}_r(\mathscr{O})}\\ \delta_{\varpi^{-s}d^{-1}J_-} & \delta_{\varpi^{-s'}J_+}\end{pmatrix}\left(\begin{pmatrix} x & x'\\ y & y'\end{pmatrix}\right)$$

Remark 3.3.8. — This is a variant of the function $\phi_{2,v}^{[(s,s')_v]}$ defined before [DL24, Lemma 4.29]; more precisely, let $m_d:=\begin{pmatrix} d1_r & \\ & 1_r\end{pmatrix}$; then, when $J_\pm=\mathrm{GL}_r(\mathscr{O}_F)$, we have $\phi^{(s,s')}=m_d\phi_{v,2}^{[(s,s')_v]}$.

For the next two lemmas, recall we have fixed an isomorphism between the isomorphic groups G and H, but that they act rather differently on $\mathscr{S}_\chi(V^r)$; thus we add a superscript 'G' in the notation for $U_t\in\mathbf{Z}[G]$, and denote by $U_t^H\in\mathbf{Z}[H]$ the corresponding operator for H; likewise for the subgroups $I_c^?$ of G and H.

Lemma 3.3.9. — *There exists an integer $c\geqslant 1$ such that for all $s,s'\geqslant 0$, $\phi^{(s,s')}$ is fixed under $I_c^G\times I_c^H$, and*

$$\omega_\chi(U_t^G)\phi^{(s,s')}=\chi(\varpi)^{-r}\phi^{(s+1,s')},$$
$$\omega_\chi(U_t^H)\phi^{(s,s')}=\chi(\varpi)^{r}\phi^{(s,s'+1)}.$$

Proof. — It suffices to check that $\phi^{(0,0)}$ is fixed under $\mathrm{N}(\mathscr{O}_F)^G\times\mathrm{N}(\mathscr{O}_F)^H\subset G\times H$, which is straightforward, and the two formulas.

The first formula is verified as in [DL24, Lemma 4.29 (1)]; to compare, see Remark 3.3.8, and note that our U_t^G equals the operator $m_d\mathrm{U}_v^{1_{w_1}}m_d^{-1}$ used in *loc. cit.*

For the second formula, by the definitions and (3.2.4), we have

$$U_t^H\phi\left(\begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}x'\\ y'\end{pmatrix}\right)=\chi(\varpi)^r\sum_{b\in\mathrm{M}_r(\mathscr{O}_F/\varpi)}\phi\left(\begin{pmatrix}\varpi^{-1}1_r & \varpi^{-1}b\\ & 1_r\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}1_r & b^{\mathrm{t}}\\ & \varpi 1_r\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix}\right).$$

For $\phi=\phi^{(s,s')}$, it is easy to check that each term in the sum vanishes unless $\begin{pmatrix} x & x'\\ y & y'\end{pmatrix}$ belongs to the support of $\phi^{(s,s'+1)}$, whereas if this condition is satisfied, then the sum contains only one nonzero term,

[11] For convenience, the choices in our naming of the coordinates here are different from those of § 3.2.

with value $\mathrm{vol}(J_+)^{-1}\mathrm{vol}(J_-)^{-1}$, which is the one indexed by the class

$$b = -xy^{-1} \equiv (x'y'^{-1})^{\mathrm{t}} \in \mathrm{M}_r(\mathcal{O}_F/\varpi).$$

□

Definition 3.3.10. — Let $c(\pi)$ and φ_c^{a}, f_c^{a} be as in Proposition 3.3.7, and let $c \geqslant c(\pi)$ be the minimal integer satisfying the conditions of Lemma 3.3.9. We define for all $s \geqslant 0$:

$$\varphi^{\mathrm{a}} := \varphi_c^{\mathrm{a}} \quad \in \pi, \qquad \phi^{(s)} := \alpha^{-s-v(d)}\alpha^{\vee,-s}\cdot\phi^{(s,s)} \quad \in \mathcal{S}(V^r), \qquad f^{\mathrm{a}} := f_c^{\mathrm{a}} \quad \in \sigma,$$
$$\lambda^{(s)} := \varphi^{\mathrm{a}} \otimes \phi^{(s)} \otimes f^{\mathrm{a}}.$$

3.3.4. *Norm relations.* — We prove the local form of the vertical norm relations.

Lemma 3.3.11. — *For every smooth admissible $G\times H$-module $\mathcal{S}$, every $1 \le d \le c$, and every $\phi \in \mathcal{S}^{I_c^G \times I_c^H}$, we have*

$$[\varphi^{\mathrm{a}} \otimes U_t^G \phi] = \alpha\cdot[\varphi^{\mathrm{a}}\otimes\phi] \quad \text{in } (\pi^\vee\otimes\mathcal{S})_G,$$
$$[U_t^H \phi \otimes f^{\mathrm{a}}] = \alpha^\vee\cdot[\phi\otimes f^{\mathrm{a}}] \quad \text{in } (\mathcal{S}\otimes\sigma)_H.$$

Proof. — For the first equality, dropping all superscripts G, we have

$$\begin{aligned}[\varphi^{\mathrm{a}}\otimes I_d t I_d\phi] &= [\pi^\vee(I_d t^{-1} I_d w_c)\varphi\otimes\phi^{\mathrm{ord}}] = [\pi^\vee(I_c t^{-1} I_c w_c)\varphi^{\mathrm{ord}}\otimes\phi]\\ &= [\pi^\vee(w_c I_c t^{w,-1} I_c)\varphi^{\mathrm{ord}}\otimes\phi] = \xi_{\pi^\vee}(\varpi)^{-1}\alpha^\vee\cdot[\pi^\vee(w_c)\varphi^{\mathrm{ord}}\otimes\phi] = \alpha\cdot[\varphi^{\mathrm{a}}\otimes\phi].\end{aligned}$$

The proof of the second equality is virtually identical. □

Proposition 3.3.12 (= Proposition 2.4.6). — *For every open compact subgroup $C \subset F^\times$, the image of $\lambda^{(s)}$ in Λ_ρ^C is independent of $s \geqslant 0$.*

Proof. — This follows from Definition 3.3.10, Lemma 3.3.9, and Lemma 3.3.11 applied to $\mathcal{S} = \mathcal{S}_\chi(V^r)$ for each character χ of $F^\times/C$. □

3.3.5. *Integrality*

Proposition 3.3.13. — *For every $s \geqslant 0$, we have*

$$\phi^{(s)} \in \mathcal{S}(V^r, \overline{\mathbf{Z}}_{p,C(\varpi^s)})^{I_c\times I_c}.$$

Proof. — It suffices to prove that for each $x \in \mathrm{Spt}(\phi^{(s)})$, the group K_x has Galois-level at least s (Definition 3.1.1). This is proved in [DL24, Lemma 4.36] (for a slightly different Schwartz function, but the same proof goes through). □

Proof of Proposition 2.4.3. — It follows from Propositions 3.2.3.2 and 3.3.13. □

3.3.6. *Local non-vanishing.* — We conclude by studying the image of $\lambda^{(s)}$ in Λ_ρ.

Proposition 3.3.14. — *For every $s \geqslant 0$, we have*

$$\theta(\lambda^{(s)}) = q^{v(d)r^2/2}\gamma(1/2, \pi^{\vee,+}, \psi)^{-1}.$$

Proof of Proposition 2.3.5, assuming Proposition 3.3.14. — We restore the notation used in the global context. Denote by $[\lambda_v^{(0)}]$ the image of $\lambda_v^{(0)}$ in $\Lambda_{\rho,v}$. It is clear that $[\lambda_v^{(0)}] \neq 0$ at all $v \in S$. At $v \notin S_\wp$, the nontriviality of $[\lambda_v^{(0)}]$ follows from [Dis, Proposition 3.6.4]. At $v \in \wp$, the desired assertion follows from Proposition 3.3.14 since $\gamma(1/2+s, \pi_v^{\vee,+}, \psi_w) = \gamma(\mathrm{WD}(\rho_{w_2}^+), \psi_w, s)$. □

In order to prove Proposition 3.3.14, we need a lemma. Let $\phi' := \mathscr{F}\phi^{(0)}$, and consider the map

$$g : \mathrm{GL}_r(F) \times \mathrm{M}_r(F) \times \mathrm{M}_r(F) \times \mathrm{GL}_r(F) \longrightarrow G = \mathrm{GL}_n(F)$$

$$g(y_+, x_+, x_-, y_-) := \begin{pmatrix} 1 & x_+ \\ & 1 \end{pmatrix} \begin{pmatrix} y_+ & \\ & y_- \end{pmatrix} \begin{pmatrix} 1 & \\ x_- & 1 \end{pmatrix} w^{-1}. \tag{3.3.9}$$

Lemma 3.3.15. — *We have*

$$\phi'(g(y_+, x_+, x_-, y_-)) = |d|^{r^2/2} \cdot \begin{pmatrix} \psi_{J_+} & \mathbf{1}_{\mathrm{M}_r(\mathscr{O})} \\ \mathbf{1}_{\mathrm{M}_r(\mathscr{O})} & \delta_{d^{-1}J_-} \end{pmatrix} \left(\begin{pmatrix} y_+ & x_+ \\ x_- & y_- \end{pmatrix} \right).$$

Proof. — With the change of variables $x'_+ = y_-^{\mathrm{t},-1} x_+^{\mathrm{t}} y'_- + x''_+$, we have

$$\begin{aligned}
\phi'\left(\begin{pmatrix} x_+ & y_+ \\ y_- & x_- \end{pmatrix} \right) &= \int_{\mathrm{M}_r(F)} \int_{\mathrm{M}_r(F)} \phi\left(\begin{pmatrix} x_+ & x'_+ \\ y_- & y'_- \end{pmatrix} \right) \psi(x_+'^{\mathrm{t}} x_-)\, \psi(-y_-'^{\mathrm{t}} y_+)\, \mathrm{d}x'_+\, \mathrm{d}y'_- \\
&= \mathbf{1}_{d^{-1}\mathrm{M}_r(\mathscr{O})}(x_+) \delta_{d^{-1}J_-}(y_-) \int_{\mathrm{M}_r(F)} \int_{\mathrm{M}_r(\mathscr{O})} \psi((y_-'^{\mathrm{t}} x_+ y_-^{-1} + x_+''^{\mathrm{t}}) x_-)\, \delta_{J_+}(y'_-)\, \psi(-y_-'^{\mathrm{t}} y_+)\, \mathrm{d}x''_+\, \mathrm{d}y'_- \\
&= \mathbf{1}_{d^{-1}\mathrm{M}_r(\mathscr{O})}(x_+) \delta_{d^{-1}J_-}(y_-) \int_{\mathrm{M}_r(\mathscr{O})} \psi(x_+''^{\mathrm{t}} x_-)\, \mathrm{d}x''_+ \int_{\mathrm{M}_r(F)} \delta_{J_+}(y'_-) \psi(y_-'^{\mathrm{t}}(x_+ y_+^{-1} x_- - y_+))\, \mathrm{d}y'_- \\
&= |d|^{r^2/2} \mathbf{1}_{d^{-1}\mathrm{M}_r(\mathscr{O})}(x_+) \delta_{d^{-1}J_-}(y_-) \mathbf{1}_{d^{-1}\mathrm{M}_r(\mathscr{O})}(x_-) \psi_{J_+}(x_+ y_-^{-1} x_- - y_+) \\
&= |d|^{r^2/2} \begin{pmatrix} \mathbf{1}_{d^{-1}\mathrm{M}_r(\mathscr{O})} & \psi_{J_+} \\ \delta_{d^{-1}J_-} & \mathbf{1}_{d^{-1}M_r(\mathscr{O})} \end{pmatrix} \left(\begin{pmatrix} x_+ & x_+ y_-^{-1} x_- - y_+ \\ y_- & x_- \end{pmatrix} \right).
\end{aligned}$$

Then the desired formula follows from evaluating at

$$g(y_+, x_+, x_-, y_-) = \begin{pmatrix} x_+ y_- & -y_+ - x_+ y_- x_- \\ y_- & -y_- x_- \end{pmatrix}.$$

$\square$

Proof of Proposition 3.3.14. — By the definitions, we have

$$\theta(\lambda^{(s)}) = \theta(\lambda^{(0)}) = \int_G (g \varphi_c^{\mathrm{a}}, f_c^{\mathrm{a}})_\pi \cdot \phi'(g) |\det g|^r\, dg.$$

We integrate over the full-measure subset that is the image of the map $g = (3.3.9)$, for which

$$\mathrm{d}g(y_+, x_+, x_-, y_-) = |\det y_+|^{-r} |\det y_-|^r\, \mathrm{d}x_-\, \mathrm{d}y_+\, \mathrm{d}y_-\, \mathrm{d}x_+.$$

where the Haar measures $\mathrm{d}y_\pm$ on $G_r = \mathrm{GL}_r(F)$ and $\mathrm{d}x_\pm$ on $\mathrm{M}_r(F)$ are normalised by assigning volume $|d|^{r^2/2}$ respectively to $\mathrm{GL}_r(\mathscr{O}_F)$ and $\mathrm{M}_r(\mathscr{O}_F)$. Then we obtain

$$\begin{aligned}
\theta(\lambda^{(0)}) &= \int_G \left(\pi^\vee\left(g(y_+, x_+, x_-, y_-)\right) \varphi^{\mathrm{a}}, f^{\mathrm{a}}\right)_\pi \cdot \phi'\left(g(y_+, x_+, x_-, y_-)\right) |\det y_+ \det y_-|^r\, \mathrm{d}g(y_+, x_+, x_-, y_-) \\
&= \alpha^{-v(d)} |d|^{r^2/2} \int_{\mathrm{M}_r(\mathscr{O}_F)} \int_{\mathrm{GL}_r(F)} \int_{\mathrm{M}_r(\mathscr{O}_F)} \int_{\mathrm{GL}_r(F)} \left(\pi^\vee\left(\left(\begin{smallmatrix} 1 & x_+ \\ & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} y_+ & \\ & y_- \end{smallmatrix}\right) w^{-1} \left(\begin{smallmatrix} 1 & -x_- \\ & 1 \end{smallmatrix}\right) \right) \varphi^{\mathrm{a}}, f^{\mathrm{a}}\right)_\pi \\
&\qquad \cdot \psi_{J_+}(y_+) \delta_{d^{-1}J_-}(y_-) |\det y_-|^{2r}\, \mathrm{d}x_-\, \mathrm{d}y_+\, \mathrm{d}y_-\, \mathrm{d}x_+.
\end{aligned}$$

Since φ^{a} and f^{a} are both invariant under $\mathrm{N}(\mathscr{O})$, the integrations in $\mathrm{d}x_\pm$ give 1, and we get

$$\theta(\lambda^{(0)}) = \alpha^{-v(d)} |d|^{r^2/2} \int_{\mathrm{GL}_r(F)} \int_{\mathrm{GL}_r(F)} \left(\pi^\vee\left(\left(\begin{smallmatrix} y_+ & \\ & y_- \end{smallmatrix}\right) w^{-1} \right) \varphi^{\mathrm{a}}, f^{\mathrm{a}}\right)_\pi \cdot \psi_{J_+}(y_+) \delta_{d^{-1}J_-}(y_-) |\det y_-|^{2r}\, \mathrm{d}y_-\, \mathrm{d}y_+.$$

By the formula for the matrix coefficient in Proposition 3.3.7, we deduce

$$\theta(\lambda^{(0)}) = \mathrm{Z}_+ \mathrm{Z}_-,$$

where

$$Z_+ := |d|^{-r^2/2} \int_{\mathrm{GL}_r(F)} (\pi^{\vee,+}(y_+)\varphi^+, f^-)_{\pi^-} \cdot \psi_{J_+}(y_+) |\det y_+|^{r/2} \, \mathrm{d}y_+,$$

$$Z_- := \alpha^{-v(d)} |d|^{r^2} \int_{\mathrm{GL}_r(F)} (\pi^{\vee,-}(y_-)\varphi^-, f^+)_{\pi^+} \delta_{d^{-1}J_-}(y_-) |\det y_-|^{3r/2} \, \mathrm{d}y_-.$$

Since $\alpha = |\varpi|^{r^2/2} \xi_{\pi^+}(\varpi)$, we have

$$Z_- = |d|^{r^2} \alpha^{-v(d)} \xi_{\pi^{\vee,-}}(d^{-1}) |d|^{3r^2/2} = 1;$$

whereas by the Godement–Jacquet functional equation ([Jac79, Proposition 1.2 (3)], which has a typo corrected in (1.3.7) *ibid.*),

$$|d|^{r^2/2} \cdot Z_+ = \gamma(1/2, \pi^{\vee,+}, \psi)^{-1} \int_{\mathrm{GL}_r(F)} (\varphi^+, \pi^-(y_+) f^-)_{\pi^-} \delta_{J_+}(y_+) |\det y_+|^{r/2} \, \mathrm{d}y_+ = \gamma(1/2, \pi^{\vee,+}, \psi)^{-1}.$$

This completes the proof. □

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