GAN–GROSS–PRASAD CYCLES
AND DERIVATIVES OF p-ADIC L-FUNCTIONS
PRELIMINARY VERSION

DANIEL DISEGNI AND WEI ZHANG

ABSTRACT. We study the p-adic analogue of the arithmetic Gan–Gross–Prasad (GGP) conjectures for unitary groups. Let Π be a hermitian cuspidal automorphic representation of $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$ over a CM field, which is algebraic of minimal regular weight at infinity. We first show the rationality of twists of the ratio of L-values of Π appearing in the GGP conjectures. Then, when Π is p-ordinary at a prime p, we construct a cyclotomic p-adic L-function $\mathscr{L}_p(\mathrm{M}_\Pi)$ interpolating those twists. Finally, under various local assumptions, we prove a precise formula relating the first derivative of $\mathscr{L}_p(\mathrm{M}_\Pi)$ to the p-adic heights of Selmer classes arising from arithmetic diagonal cycles on unitary Shimura varieties. We deduce applications to the p-adic Beilinson–Bloch–Kato conjecture for the motive attached to Π. All proofs are based on some relative-trace formulas in p-adic coefficients.

CONTENTS

1. Introduction

The pioneering formulas of Gross–Zagier and Perrin-Riou [GZ86, PR87] revealed a remarkable relation between Heegner points and derivatives of complex and p-adic L-functions. They had immediate applications to the (classical and p-adic) Birch and Swinnerton-Dyer conjectures, soon strengthened by the Selmer-group bounds proved by Kolyvagin [Kol88].

A "furtive caress"[1] between those formulas and one on central L-values, by Waldspurger [Wal85b], did not escape Gross; and in [Gro04], he turned it into a representation-theoretic marriage, which would blossom in [YZZ12] (and later p-adically in [Dis17]).

The seeds for a new generation were sown in a paper by Gan, Gross, and Prasad [GGP12]. Their influential work conjectured a pair of non-vanishing criteria in the context of embeddings of unitary groups: one for automorphic periods, in terms of Rankin–Selberg L-values (generalizing [Wal85b]); and one for algebraic cycles in Shimura varieties, in terms of (complex) L-derivatives (the *arithmetic* GGP conjecture, generalizing [GZ86]).

The conjecture on automorphic periods was refined to an exact formula by Ichino–Ikeda and N. Harris [II10, Har14], and recently proved in this form in [BPLZZ21, BPCZ22]. On the other hand, despite considerable progress (see [Zha] for a review), the arithmetic GGP conjecture remains open outside of cases where it can be reduced to Heegner points [YZZ12, Xue19].[2]

The purpose of this work is to formulate and, under some local assumptions, prove a p-adic variant of the arithmetic GGP conjecture. The result in fact takes the form of a precise formula, in the spirit of [PR87, Dis17, II10, Har14]. It has immediate applications to the p-adic Beilinson–Bloch–Kato conjecture for the relevant motives, which can be further strengthened by the Selmer bounds recently established in [LTX+22].

(Indeed, one advantage of working in p-adic rather than archimedean coefficients is that we obtain a nonvanishing criterion in Selmer groups, rather than Chow groups: while the class map from the latter to the former should be injective, this is not known beyond curves.)

---

[1]Words borrowed from [Wei40].

[2]The analogous conjecture for orthogonal groups is also known for 1-cycles in threefolds [YZZ].

In the rest of this introduction, we state our main results, discuss their history and context, and give some ideas on the proofs.

In § 1.1, we describe our p-adic L-function (Theorem B), preceded by a rationality result for twisted Rankin–Selberg L-values (Theorem A) that should be of independent interest.

In § 1.2 we state our applications on the the p-adic Beilinson–Bloch–Kato conjecture (Theorem C; the order of presentation is dictated by ease of exposition rather than logic). In § 1.3 we define the Gan–Gross–Prasad cycles and state our formula for their p-adic heights (Theorem D).

In § 1.4, we give a sketch of our methods: inspired by the strategy proposed by Jacquet–Rallis for the Ichino–Ikeda conjecture [JR11], and by one of us [Zha12] for the arithmetic GGP conjecture (in archimedean coefficients), we construct a p-adic relative-trace formula from which we extract the p-adic L-function; then, we compare it to another relative-trace formula encoding the p-adic heights of GGP cycles.

1.1. **The p-adic L-function.**

Let F_0 be a number field, and denote by $\mathbf{A}$ the adèles of F_0, by $D_{F_0} = \prod_{v \nmid \infty} D_{F_0,v}$ be the discriminant of F_0 (here $D_{F_0,v}$ is the norm of the different ideal of $F_{0,v}$). Let F be quadratic extension of F_0, let $\mathrm{c} \in \mathrm{Gal}(F/F_0)$ be the nontrivial element, and let $\eta \colon F_0^\times \backslash \mathbf{A}^\times \to \{\pm 1\}$ be the associated quadratic character. Define a reductive group over F_0 by

$$\mathrm{G}' \coloneqq (\mathrm{Res}_{F/F_0} \mathrm{GL}_n \times \mathrm{Res}_{F/F_0} \mathrm{GL}_{n+1})/\mathrm{GL}_1 \times \mathrm{GL}_1,$$

where $\mathrm{GL}_1 \times \mathrm{GL}_1$ is embedded in the center $\mathrm{Res}_{F/F_0}(\mathrm{GL}_1 \times \mathrm{GL}_1)$ of G'. Let $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ be an (irreducible) automorphic representation of $\mathrm{G}'(\mathbf{A})$. Define[3] a Rankin–Selberg and an Asai L-function[4] for Π and a character χ of $F^\times \backslash \mathbf{A}^\times$ by

$$L(s, \Pi \otimes \chi) \coloneqq L(s, \Pi_n \times (\Pi_{n+1} \otimes \chi \circ \mathrm{Nm}_{F/F_0}))$$

$$L(s, \Pi, \mathrm{As}^\star) \coloneqq L(s, \Pi_n, \mathrm{As}^{(-1)^n}) L(s, \Pi_{n+1}, \mathrm{As}^{(-1)^{n+1}}).$$

We say that a cuspidal automorphic representation Π_ν is *hermitian* if $\Pi \circ \mathrm{c} \cong \Pi^\vee$ and $L(s, \Pi_\nu, \mathrm{As}^{(-1)^\nu})$ is regular at $s = 1$. We say that $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ is hermitian if Π_n, Π_{n+1} are. For such a representation Π, we define

$$\mathscr{L}(s, \Pi_v, \chi_v) \coloneqq D_{F_0,v}^{n+1} \prod_{i=1}^{n+1} L(i, \eta_v^i) \cdot \frac{L(s, \Pi_v \otimes \chi_v)}{L(1, \Pi_v, \mathrm{As}^\star)}, \tag{1.1.1}$$

and

$$\mathscr{L}(s, \Pi, \chi) \coloneqq \prod_{v \nmid \infty} \mathscr{L}(s, \Pi_v, \chi_v).$$

Here, the abelian factor may be interpreted in terms of L-values of motives of unitary groups (§ 2.2.1).

1.1.1. *Rationality of $\mathscr{L}$.*

Assume from now on that F_0 is totally real and F is CM. Let $\arg(z) \coloneqq z/|z|$ (a character of $\mathbf{C}^\times$), let $\Pi_{\nu,\mathbf{R}}^\circ$ be the representation of $\mathrm{GL}_\nu(\mathbf{C})/\mathrm{GL}_1(\mathbf{R})$ induced by the

---

[3]Throughout the introduction (but *differently* from the rest of the paper) L-functions do not include archimedean factors.

[4]See [GGP12, § 7] for the definition of $L(s, \Pi_n, \mathrm{As}^\pm)$.

character $\arg^{\nu-1} \otimes \arg^{\nu-3} \otimes \ldots \otimes \arg^{1-\nu}$ of the torus $(\mathbf{C}^{\times})^{\nu}$, and define the representation

$$\Pi_{\infty}^{\circ} = \bigotimes_{v|\infty} \Pi_{\mathbf{R}}^{\circ} := \bigotimes_{v|\infty} \Pi_{n,\mathbf{R}}^{\circ} \otimes \Pi_{n+1,\mathbf{R}}^{\circ}$$

of $\mathrm{G}'(F_{0,\infty})$. Let us also denote by $\mathbf{1}_{\infty}$ the trivial representation of $\mathrm{G}'(F_{0,\infty})$ over $\mathbf{Q}$.

Let $\Pi = \Pi^{\infty} \otimes \mathbf{1}_{\infty}$ be a representation of $\mathrm{G}'(\mathbf{A})$ on a characteristic-zero field L (admitting embeddings into $\mathbf{C}$). We say that Π is a *trivial-weight (algebraic) cuspidal automorphic representation* if for every $\iota \colon L \hookrightarrow \mathbf{C}$, the representation $\Pi^{\iota} := \iota\Pi^{\infty} \otimes \Pi_{\infty}^{\circ}$ is an (irreducible) cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$. (It is known that every cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over $\mathbf{C}$ such that $\Pi_{\infty} \cong \Pi_{\infty}^{\circ}$ arises in this manner for some number field L.) We say that Π is hermitian if Π^{ι} is for some (equivalently, every) ι.

We first prove the following strong rationality property for the values of $\mathscr{L}$. For an ideal $m \subset \mathscr{O}_{F_0}$, let $Y(m)_{/\mathbf{Q}}$ be the finite étale scheme of characters of $F_0^{\times} \backslash \mathbf{A}^{\times} / F_{0,\infty}^{\times}(\widehat{\mathscr{O}}_{F_0}^{\times} \cap 1 + m\widehat{\mathscr{O}}_{F_0})$. Let $Y := \varinjlim_{m} Y(m)$, the ind-finite scheme over $\mathbf{Q}$ of locally constant characters of $F_0^{\times} \backslash \mathbf{A}^{\times} / F_{0,\infty}^{\times}$.

Theorem A. *Let Π be a trivial-weight hermitian cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ defined over a characteristic-zero field L. Then there is an element*

$$\mathscr{L}(\mathrm{M}_{\Pi}, \cdot) \in \mathscr{O}(Y_L). \tag{1.1.2}$$

such that

$$\mathscr{L}(\mathrm{M}_{\Pi}, \chi) = \frac{\mathscr{L}(1/2, \Pi^{\iota}, \chi)}{\varepsilon(\tfrac{1}{2}, \chi^2)^{\binom{n+1}{2}}}$$

for all $\chi \in Y_L(\mathbf{C})$ with underlying embedding $\iota \colon L \hookrightarrow \mathbf{C}$.

For the notation 'M_{Π}', see Remark 1.2.2.

Remark 1.1.1. For $n = 1$, Theorem A is a variant of a classical result of Shimura [Shi78]. A conditional proof of the rationality of $\mathscr{L}(1/2, \Pi, \mathbf{1})$ for a more general class of Π was recently obtained by Grobner and Lin [GL21, Theorem C]. (In fact, their rationality result is also a consequence of the Ichino–Ikeda conjecture, but the method of [GL21] is different.) See also [Rag16] for a related result.

Remark 1.1.2. Strictly speaking, all of our main theorems rely on an explicit formula for a certain Rankin–Selberg local integral (Hypothesis 5.2.6), which is expected to be proven in forthcoming work by Li–Liu–Sun [LLS].

1.1.2. *p-adic interpolation.* Fix from now on a rational prime p. For $v|p$ a place of F_0, let $N_v^{\circ} \subset G_v' := \mathrm{G}'(F_{0,v})$ be the subgroup of integral unipotent upper-triangular matrices, and let $T_v^+ \subset G_v'$ be the monoid of diagonal matrices such that $tN_v^{\circ}t^{-1} \subset N_v^{\circ}$. Let Π be a trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p$. We say that Π is *v-ordinary* if $\Pi_v^{N_v^{\circ}}$ contains a nonzero vector (necessarily unique up to scalar multiple) on which all the operators $U_{t,v} := \sum_{x \in N_v^{\circ}/tN_v^{\circ}t^{-1}} [xt]$, for $t \in T_v^+$, act by units in $\mathscr{O}_L$. We say that Π is *ordinary* if it is v-ordinary for all $v|p$.

For any finite extension E/F, denote $\Gamma_E := E^{\times} \backslash \mathbf{A}_E^{\infty\times} / \prod_{w\nmid p} \mathscr{O}_{E,w}^{\times}$, and let

$$\mathscr{Y} := \operatorname{Spec} \mathbf{Z}_p[\![\Gamma_{F_0}]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

We have a natural map $Y(p^\infty) := \varinjlim_r Y(p^r) \hookrightarrow \mathscr{Y}$.

If L'/L is a field extension and S/L is an (ind-) scheme, we denote $S_{L'} := S \times_{\mathrm{Spec}\,L} \mathrm{Spec}\,L'$.

Theorem B. *Let Π be an ordinary, hermitian, trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p$. Assume that for each place $v|p$ of F_0 that does not split in F, the representation Π_v is unramified.*

There exists a unique function

$$\mathscr{L}_p(\mathrm{M}_\Pi) \in \mathscr{O}(\mathscr{Y}_L)$$

whose restriction to $Y(p^\infty)_L$ satisfies

$$\mathscr{L}_p(\mathrm{M}_\Pi)(\chi) = e_p(\mathrm{M}_{\Pi\otimes\chi})\,\mathscr{L}(\mathrm{M}_\Pi, \chi) \tag{1.1.3}$$

where $\mathscr{L}(\mathrm{M}_\Pi)$ is as in (1.1.2), and $e_p(\mathrm{M}_{\Pi\otimes\chi}) = \prod_{v|p} e(\Pi_v, \chi_v)$ is the product of the explicit local terms (5.2.9).

Remark 1.1.3. We conjecture that the theorem remains true without the non-ramification condition at nonsplit p-adic places.

Remark 1.1.4. We say that Π is *non-exceptional* if $e_p(\mathrm{M}_\Pi) \neq 0$. According to Hypothesis 5.2.6, the factor $e_p(\mathrm{M}_{\Pi\otimes\chi})$ is as conjectured by Coates and Perrin-Riou [Coa91]; this implies that if Π_v is an irreducible principal series for all $v|p$, then Π is non-exceptional (see Remark 5.2.9).

Remark 1.1.5. Januszewski [Jan16] has proven a variant of Theorem B in a more general context, by the method of modular symbols. Our method is similar locally but very different globally, see § 1.4.2 below.

1.2. On the p-adic Beilinson–Bloch–Kato conjectures for Rankin–Selberg motives.

Before moving to discuss our main result, we give its main arithmetic application, which can be stated without much further background.

Let

$$\Pi = \Pi_n \boxtimes \Pi_{n+1}$$

be a hermitian trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p$. Denote by G_F the absolute Galois group of F, by $\overline{\mathbf{Q}}_p$ an algebraic closure of L and let $\rho_{\Pi_\nu, \overline{\mathbf{Q}}_p}: G_F \to \mathrm{GL}_\nu(\overline{\mathbf{Q}}_p)$ be the semisimple representation attached to Π_ν by the global Langlands correspondence (as described in [Car12, Theorem 1.1]). Assuming that $\varepsilon(\Pi) := \varepsilon(1/2, \Pi_n^\iota \times \Pi_{n+1}^\iota) = -1$ for any (equivalently, all) $\iota: L \hookrightarrow \mathbf{C}$, we construct a continuous representation

$$\rho_\Pi: G_F \longrightarrow \mathrm{GL}_{n(n+1)}(L) \tag{1.2.1}$$

whose base-change $\rho_\Pi \otimes_L \overline{\mathbf{Q}}_p$ is isomorphic, up to semisimplication, to $\rho_{\Pi_n, \overline{\mathbf{Q}}_p} \otimes \rho_{\Pi_{n+1}, \overline{\mathbf{Q}}_p}(n)$ (Remark 10.1.3). It satisfies $\rho_\Pi^c \cong \rho_\Pi^*(1)$, where $\rho^c(g) := \rho(c^{-1}gc)$ for any lift $c \in G_F$ of c.

The Beilinson–Bloch–Kato (BBK) conjecture relates the dimension of the Bloch–Kato Selmer group

$$H^1_f(F, \rho_\Pi)$$

to the order of vanishing of $\mathscr{L}(s, \Pi^\iota)$ at $s = 1/2$, for any $\iota\colon L \hookrightarrow \mathbf{C}$. Assuming that Π is ordinary, we can consider a variant in terms of

$$\operatorname{ord}_{\chi=\mathbf{1}} \mathscr{L}_p(\mathrm{M}_\Pi) \coloneqq \sup\left\{r \mid \mathscr{L}_p(\mathrm{M}_\Pi) \in \mathfrak{m}_{\mathbf{1}}^r \subset \mathscr{O}(\mathscr{Y}_L)\right\},$$

where $\mathfrak{m}_{\mathbf{1}}$ is the ideal of functions vanishing at $\chi = \mathbf{1}$. We prove the following.

Theorem C. *Let Π be an ordinary, hermitian, trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p$. Assume that $\varepsilon(\Pi) = -1$, and that the following further conditions are satisfied:*

- *F/F_0 is unramified; in particular, $F_0 \neq \mathbf{Q}$;*

- *all places $v|2$ are split in F/F_0;*

- *$p > 2n$ if $n > 1$;*

- *for every place $v|p$ of F_0, we have that v splits in F and Π_v is unramified;*

- *for every place v of F_0 that splits in F, at least one of $\Pi_{n,v}$ and $\Pi_{n+1,v}$ is unramified;*

- *for every place v of F_0 that is inert in F, each of $\Pi_{n,v}$ and $\Pi_{n+1,v}$ is either unramified or almost unramified (namely, the base change of an almost unramified representation of unitary group), and if $\Pi_{n,v}$ is almost unramified then $\Pi_{n+1,v}$ is also almost unramified.*

Then

$$\operatorname{ord}_{\chi=\mathbf{1}} \mathscr{L}_p(\mathrm{M}_\Pi) = 1 \implies \dim_L H^1_f(F, \rho_\Pi) \geq 1. \tag{1.2.2}$$

If moreover p is an admissible prime for Π in the sense of [LTX$^+$22, *Definition 8.1.1*]*, then*

$$\operatorname{ord}_{\chi=\mathbf{1}} \mathscr{L}_p(\mathrm{M}_\Pi) = 1 \implies \dim_L H^1_f(F, \rho_\Pi) = 1. \tag{1.2.3}$$

This result is a consequence of a non-vanishing criterion for certain explicit elements of $H^1_f(F, \rho_\Pi)$ arising as classes of algebraic cycles, which we describe in the rest of this section. The stronger (1.2.3) follows from combining that criterion with the Selmer bound of [LTX$^+$22] (whose admissibility condition is expected to be mild, see *ibid.* Remark 1.1.5). In particular, in this case we have that $H^1_f(F, \rho_\Pi)$ is generated by the class of an algebraic cycle – a result analogous to the finiteness of the p^∞-torsion of the Tate–Shafarevich group of an elliptic curve.

Remark 1.2.1. The history of theorems of type (1.2.2) consists of several works for similar 2-dimensional Galois representations over CM fields (starting with [PR87] and continuing with [Nek95, Kob13, Shn16, Dis17, Dis, Dis22]), together with a very recent result by Y. Liu ad one of us for a family of higher-dimensional representations [DL, Theorem 1.7]. Theorems of type (1.2.3) were previously only known in 2-dimensional cases, based on generalizations of [Kol88].

Remark 1.2.2. Our notation (and the definitions going back to (1.1.1)) suggest that one may think of $\mathscr{L}_{(p)}(\mathrm{M}_\Pi)$ as attached to the virtual motive M_Π over F_0 whose p-adic realization is (up to abelian factors)

$$\mathrm{M}_{\Pi,p} \coloneqq (\operatorname{Ind}_{G_F}^{G_{F_0}} \rho_\Pi) \ominus \operatorname{As}^\star(\rho_\Pi).$$

Here, $\mathrm{As}^\star(\rho_\Pi) = \mathrm{As}^\star(\rho_{\Pi_n}) \oplus \mathrm{As}^\star(\rho_{\Pi_{n+1}})$ with the factors defined by

$$\mathrm{As}^\pm(\rho_{\Pi_\nu}) \colon G_{F_0} \longrightarrow \mathrm{GL}(L^\nu \otimes_L L^\nu)$$

$$G_F \ni g \longmapsto \rho_{\Pi_\nu}(g) \otimes \rho_{\Pi_\nu}^{\mathrm{c}}(g),$$

$$c \longmapsto (x \otimes y \longmapsto \pm y \otimes x)$$

and the sign $\star = (-1)^\nu$ on the ν-factor.

Then the p-adic BBK conjecture would rather relate $\mathrm{ord}_{\chi=1} \mathscr{L}_p(\mathrm{M}_\Pi)$ with

$$\dim_L H_f^1(F_0, \mathrm{Ind}_{G_F}^{G_{F_0}} \rho_\Pi) - \dim_L H_f^1(F_0, \mathrm{As}^\star(\Pi)).$$

The first term equals $\dim_L H_f^1(F, \rho_\Pi)$. Under our assumption that Π is hermitian, $\mathrm{As}^\star(\Pi_\nu)$ coincides with the adjoint representation defined in the opening paragraphs of [NT] (cf. [GGP12, Proposition 7.4]). By the results obtained there and in [Tho], under some irreducibility assumptions on ρ_{Π_ν}, we have $H_f^1(F_0, \mathrm{As}^\star(\rho_\Pi)) = 0$.

Remark 1.2.3. In addition to Hypothesis 5.2.6, Theorem C and Theorem D below rely on a decomposition of the tempered part of the cohomology of unitary Shimura varieties (Hypothesis 10.1.2), which is expected to be proven in a sequel to [KSZ]. (At a more basic level, we also freely use the results of [Mok15, KMSW] on automorphic representations of unitary groups.) Finally, we also need to admit a non-vanishing result for certain local spherical characters, Hypothesis 11.2.1, which is expected to be proven in [Dan].

In the next subsection we describe, after some preliminaries, the construction of the Selmer classes of interest and our formula relating those to the derivative of $\mathscr{L}_p(\mathrm{M}_\Pi)$ (Theorem D).

1.3. **The p-adic arithmetic Gan–Gross–Prasad conjecture.** The cycles of interest arise from Shimura varieties attached to certain unitary group. We start by describing the representation-theoretic background.

1.3.1. *Incoherent unitary groups and their representations.* For a place v of F_0, denote by $\mathscr{V}_v$ the set of isomorphism classes of pairs $V_v = (V_{n,v}, V_{n+1,v})$ of (non-degenerate) $F_v/F_{0,v}$-hermitian spaces over F_v, where $V_{n,v}$ has rank n and $V_{n+1,v} = V_{n,v} \oplus F_v e$ with e a vector of norm 1. Let $\mathscr{V}^\circ$ be the set of collections $V = (V_v)_v$ with $V_v \in \mathscr{V}_v$ such that $V_{n,v}$ is positive-definite for all archimedean places, and for all but finitely many places v, the Hasse–Witt invariant

$$\epsilon(V_v) := \eta_v((-1)^{\binom{n}{2}} \det V_{n,v}) \tag{1.3.1}$$

equals $+1$.

We say that $V \in \mathscr{V}^\circ$ is *coherent* if there exists a (unique up to isomorphism) pair of F/F_0-hermitian spaces, still denoted $V = (V_n, V_{n+1})$, whose v-localization is V_v. This holds if and only if $\epsilon(V) := \prod_v \epsilon(V_v)$ equals $+1$. When $\epsilon(V)$ equals -1, we refer to V as an *incoherent* pair of F/F_0-hermitian spaces. For $V \in \mathscr{V}^\circ$, we denote by

$$\mathrm{H}_v^{V_v} := \mathrm{U}(V_{n,v}) \subset \mathrm{G}_v^{V_v} := \mathrm{U}(V_{n,v}) \times \mathrm{U}(V_{n+1,v}), \tag{1.3.2}$$

(where the embedding is diagonal), by $H_v^{V_v} \subset G_v^{V_v}$ their $F_{0,v}$-points. When V is coherent, these are localizations of unitary groups $\mathrm{H}^V := \mathrm{U}(V_n) \hookrightarrow \mathrm{G}^V := \mathrm{U}(V_n) \times \mathrm{U}(V_{n+1})$ over F_0. When V is

incoherent, we still use the notation

$$\mathrm{H}^V \subset \mathrm{G}^V$$

for the collections (1.3.2), which we refer to as *incoherent* unitary groups over F_0, and we denote $\mathrm{G}^V(\mathbf{A}^S) = \prod'_{v \notin S} G_v^{V_v}$.

In § 2.2, for each $V_v \in \mathscr{V}_v$, we fix measures dh_v on $H_v = H_v^{V_v}$ such that (i) if v is finite, dh_v is $\mathbf{Q}$-valued; (ii) if v is archimedean and V_v is positive definite, $\mathrm{vol}(H_v, dh_v) \in \mathbf{Q}^\times$; if $V \in \mathscr{V}^\circ$ is coherent, $\prod_v dh_v$ is the Tamagawa measure on $\mathrm{H}^V(\mathbf{A})$.

Suppose that $V \in \mathscr{V}^\circ$ is incoherent. If v is a place of F_0 non-split in F, we let $V(v) \in \mathscr{V}$ be the coherent collection with $V(v)_w = V_w$ if $w \neq v$, and $V(v)_v \in \mathscr{V}_v$ is the unique element different from V_v if v is non-archimedean, and the element such that $V(v)_{n,v}$ has signature $(n-1, 1)$ if v is archimedean. We let $\mathrm{G}^{(v)} = \mathrm{G}^{V(v)}$.

Let $\underline{\pi}_{\mathbf{R}}^\circ$ be the set of (isomorphism classes of) tempered representations of the real group $U(n-1, 1) \times U(n, 1)$ whose base-change to $\mathrm{GL}_n(\mathbf{C})/\mathbf{R}^\times \times \mathrm{GL}_{n+1}(\mathbf{C})/\mathbf{R}^\times$ is Π_∞°. For a characteristic-zero field L and an incoherent $\mathrm{G} = \mathrm{G}^V$, a *cuspidal automorphic representation of* $\mathrm{G}(\mathbf{A})$ *over* L *trivial at infinity* is a representation $\pi = \pi^\infty \otimes \mathbf{1}_\infty$ of $\mathrm{G}(\mathbf{A})$ over L, such that for every $\iota \colon L \hookrightarrow \mathbf{C}$, every $v | \infty$, and some (equivalently, every) $\pi_v^\circ \in \underline{\pi}_{\mathbf{R}}^\circ$, the complex representation of $\mathrm{G}^{(v)}(\mathbf{A})$

$$\pi^{\iota, v} := \iota\pi^v \otimes \pi_v^\circ$$

is irreducible, cuspidal and automorphic. If each π^ι is tempered and admits a *cuspidal* automorphic base-change to $\mathrm{G}'(\mathbf{A})$, we say that π is *stable*; the base-change of π^ι is necessarily of the form Π^ι for a trivial-weight representation Π over L that we call the base-change of π and denote $\mathrm{BC}(\pi)$.

1.3.2. *Arithmetic diagonal cycles.* When V is incoherent, we may attach to $\mathrm{G} = \mathrm{G}^V$ a tower of Shimura varieties $(X_K)_{K \subset \mathrm{G}(\mathbf{A}^\infty)}$ over F of dimension $2n-1$, and to $\mathrm{H} = \mathrm{H}^V$ a tower of Shimura varieties $(Y_{K'})_{K' \subset \mathrm{H}(\mathbf{A}^\infty)}$ over F of dimension n. They are proper provided that $F_0 \neq \mathbf{Q}$, a condition that we henceforth assume.

The embedding $\jmath \colon \mathrm{H}(\mathbf{A}) \to \mathrm{G}(\mathbf{A})$ induces a morphism of Shimura varieties still denoted by $\jmath$. Consider the (well-defined) normalized fundamental class $[Y]^\circ := \lim_{K'} \mathrm{vol}(K')[Y_{K'}] \in \varprojlim_{K'} \mathrm{Ch}^0(Y_{K'})_{\mathbf{Q}}$ and the *arithmetic diagonal cycle* $\jmath_*([Y]^\circ) \in \varprojlim_K \mathrm{Ch}^n(X_K)_{\mathbf{Q}}$ (where $\mathrm{Ch}^i(Z)_{\mathbf{Q}}$ denotes the Chow group of codimension-i cycles on Z with rational coefficients). The p-adic absolute cycle class of $\jmath_*([Y]^\circ)$ can be projected to an element

$$Z \in H^1_f(F, M_{\mathrm{temp}})$$

where $M^{\mathrm{temp}} = \varprojlim_K H^{2n-1}_{\text{ét}}(X_{K, \overline{F}}, \mathbf{Q}_p(n))_{\mathrm{temp}}$, and the superscript 'temp' refers to the tempered part of cohomology (see § 10.1.3).

1.3.3. *Gan–Gross–Prasad cycles.* Let π be a stable,[5] cuspidal automorphic representation of $\mathrm{G}(\mathbf{A})$ trivial at infinity, over some finite extension L of $\mathbf{Q}_p$; let $\Pi = \mathrm{BC}(\pi)$. According to

---

[5]If π is only assumed to be tempered but not stable, we can still define Z_π with values in the Selmer group of a certain Galois representation (see § 10.2.3).

Hypothesis 10.1.2, there is an injective map

$$\pi \longrightarrow \mathrm{Hom}_{\mathbf{Q}_p[G_F]}(M^{\mathrm{temp}}, \rho_\Pi),$$

well-defined uniquely up to scalar multiples. We identify π with the image of this map, and define the *Gan–Gross–Prasad functional*

$$Z_\pi \colon \pi \longrightarrow H^1_f(F, \rho_\Pi)$$
$$\phi \longmapsto Z_\pi(\phi) := \phi_* Z. \qquad (1.3.3)$$

We call elements in its image *Gan–Gross–Prasad cycles.*

1.3.4. *The p-adic arithmetic Gan–Gross–Prasad conjecture.* By construction, we have

$$Z_\pi \in \mathrm{Hom}_{\mathrm{H}^V(\mathbf{A})}(\pi, L) \otimes_L H^1_f(F, \rho_\Pi).$$

The space $\mathrm{Hom}_{\mathrm{H}^V(\mathbf{A})}(\pi, L)$ is known to be of dimension 0 or 1; in the latter case, π is said to be *distinguished.* By the local Gan–Gross–Prasad conjecture proved in [BP16, BP20], for a given representation Π over L as in Theorem A, there exists a unique (up to isomorphism) pair (V, π) where $V \in \mathscr{V}^\circ$ and π is a representation of $\mathrm{G}^V(\mathbf{A})$ as above that is distinguished. Moreover, π can be defined over L, and V is incoherent if and only if $\varepsilon(\Pi) = -1$ (see § 2.5.3).

The following is a *p-adic* analogue of the arithmetic Gan–Gross–Prasad conjecture [GGP12, Conjecture 27.1] for unitary groups.

Conjecture 1.3.1. *Let Π be a representation as in Theorem B. Assume that $\varepsilon(\Pi) = -1$ and that Π is not exceptional. The following conditions are equivalent:*

(1) $\mathrm{ord}_{\chi=\mathbf{1}} \mathscr{L}_p(\mathrm{M}_\Pi) = 1$.

(2) for the unique distinguished π with $\mathrm{BC}(\pi) = \Pi$, we have

$$Z_\pi \neq 0.$$

Remark 1.3.2. According to the *p-adic* BBK conjecture, both conditions are also equivalent to

(3) $\dim_L H^1_f(F, \rho_\Pi) = 1$.

The implication *(2)* $\implies$ *(3)* is [LTX+22, Theorem 1.1.9] (when p is admissible for Π).

As a refinement of Conjecture 1.3.1, we prove (under some conditions) a formula that 'measures' the product $Z_\pi \otimes Z_{\pi^\vee}$ in terms of the derivative of $\mathscr{L}_p(\mathrm{M}_\Pi)$; in order to state it, we need to define some pairings.

1.3.5. *Dualities.* Continue with the setup of § 1.3.3. Fix a non-degenerate pairing

$$\langle \, , \, \rangle_\Pi \colon \rho_\Pi \otimes_L \rho_{\Pi^\vee} \longrightarrow L(1),$$

and let $\langle \, , \, \rangle \colon M_{\mathrm{temp}} \otimes M_{\mathrm{temp}} \to L(1)$ be the pairing induced by Poincaré duality. Then we define a pairing

$$(\, , \,)_\pi \colon \pi \otimes \pi^\vee \longrightarrow L \qquad (1.3.4)$$

by $(\phi, \phi')_\pi := \phi \circ u(\phi'^*(1))$, where $\phi'^*(1) \colon \rho_{\Pi^\vee}^*(1) \to M_{\mathrm{temp}}^*(1)$ is the transpose, and $u \colon M_{\mathrm{temp}}^*(1) \to M_{\mathrm{temp}}$ is the isomorphism induced by $\langle \, , \, \rangle$.

1.3.6. *Invariant functionals.* If π is distinguished, there is a canonical generator

$$\alpha \in \mathrm{Hom}_{\mathrm{H}^V(\mathbf{A})}(\pi, L) \otimes_L \mathrm{Hom}_{\mathrm{H}^V(\mathbf{A})}(\pi^\vee, L)$$

defined as follows. Pick a factorization $(\,,\,)_\pi = \prod_v (\,,\,)_{\pi_v}$, where each factor is a pairing on $\pi_v \otimes \pi_v^\vee$. Then α is defined on factorizable elements $\phi = \otimes_{v \nmid \infty} \phi_v$, $\phi' = \otimes_{v \nmid \infty} \phi'_v$ by the product of absolutely convergent integrals

$$\iota\alpha(\phi, \phi') := \mathrm{vol}(H_\infty^V, dh_\infty) \cdot \prod_{v \nmid \infty} \mathscr{L}(1/2, \iota\Pi_v)^{-1} \int_{H_v} \iota(\pi(h)\phi, \phi')_\pi \, dh_v, \qquad (1.3.5)$$

where $\iota \colon L \hookrightarrow \mathbf{C}$ is any embedding, $\mathrm{vol}(H_\infty^V, dh_\infty) = \prod_{v \mid \infty} \mathrm{vol}(H_v^{V_v}, dh_v) \in \mathbf{Q}^\times$, and almost all factors are equal to 1.

1.3.7. *p-adic heights and main result.* Assume that Π is ordinary. Then ρ_Π is Panchishkin-ordinary in the sense of [Nek93] (recalled in § A.2.1). By Nekovář's theory (see [Nek93] or § A), the pairing $\langle\,,\,\rangle_\Pi$ and the natural projection $\lambda \colon \Gamma_F \to \Gamma_{F_0}$ induce a height pairing

$$h_\pi \colon H_f^1(F, \rho_\Pi) \otimes_L H_f^1(F, \rho_{\Pi^\vee}) \longrightarrow \Gamma_{F_0} \hat{\otimes} L.$$

For $\mathscr{L} \in \mathscr{O}(\mathscr{Y})_L$, set

$$\partial\mathscr{L} := [\mathscr{L} - \mathscr{L}(1)] \in T_\mathbf{1}^* \mathscr{Y}_L = \mathfrak{m}_\mathbf{1}/\mathfrak{m}_\mathbf{1}^2 \otimes_{\mathbf{Q}_p} L = \Gamma_{F_0} \hat{\otimes} L.$$

The following is a p-adic analogue of the refined arithmetic Gan–Gross–Prasad conjecture (cf. [Xue19, Conjecture 5.1]), in the spirit of the Ichino–Ikeda refinement of the usual Gan–Gross–Prasad conjecture. The case $n = 1$ is essentially equivalent to the p-adic Gross–Zagier formula as in [Dis17].

Conjecture 1.3.3. *Let $V \in \mathscr{V}^\circ$ be an incoherent pair, and let π be a distinguished, stable, ordinary, cuspidal automorphic representation of $\mathrm{G}^V(\mathbf{A})$, trivial at infinity, over a finite extension L of $\mathbf{Q}_p$. Let $\Pi := \mathrm{BC}(\pi)$ and assume that it is ordinary and non-exceptional. Then for all $\phi \in \pi$, $\phi' \in \pi^\vee$, we have*

$$h_\pi(Z_\pi(\phi), Z_{\pi^\vee}(\phi')) = e_p(\mathrm{M}_\Pi)^{-1} \cdot \frac{1}{4} \partial\mathscr{L}_p(\mathrm{M}_\Pi) \cdot \alpha(\phi, \phi')$$

in $\Gamma_{F_0} \hat{\otimes} L$.

Remark 1.3.4. This conjecture implies the direction $(1) \implies (2)$ in Conjecture 1.3.1; the converse implication is reduced to the conjectural non-degeneracy of h_π.

We have the following theorem, confirming the above refined conjecture in certain cases.

Theorem D. *Conjecture 1.3.3 holds if we assume further that:*
− *F/F_0 is unramified; in particular, $F_0 \neq \mathbf{Q}$;*
− *all places $v \mid 2$ are split in F/F_0;*
− *$p > 2n$ if $n > 1$;*
− *for every place $v \mid p$ of F_0, we have that v splits in F and π_v is unramified;*
− *for every finite place v of F_0 that splits in F/F_0, at least one of $\pi_{n,v}$ and $\pi_{n+1,v}$ is unramified;*

− *for every finite place v of F_0 that is inert in F/F_0, $\pi_{n,v}$ and $\pi_{n+1,v}$ are either unramified or almost unramified, and if $\pi_{n,v}$ is almost unramified then $\pi_{n+1,v}$ is also almost unramified.*

Remark 1.3.5. Besides the p-adic Gross–Zagier results mentioned in Remark 1.2.1, the only other p-adic height formula in the literature is the recent [DL, Theorem 1.11], which is however conditional on a certain modularity conjecture (this is arguably the main obstacle towards achieving a result of type (1.2.3) in that setting). While our setup and global approach to the proof are different, a theorem on p-local heights in [DL] is essential for us.

1.4. p-adic relative trace formulas and the proofs.

Our approach to Theorem D is based on the comparison of a pair of relative-trace formulas with p-adic coefficients, analogously to the approach proposed by one of us [Zha12] over archimedean coefficients. We give a brief overview; unexplained terminology will be defined in the main body of the paper.

1.4.1. *Rationality.* Let us first explain the proof of Theorem A. For each $\chi \in Y(\mathbf{C})$, we have a Jacquet–Rallis relative-trace distribution

$$I(-, \chi) \colon \mathscr{H}(\mathrm{G}'(\mathbf{A}), \mathbf{C}) \longrightarrow \mathbf{C}$$

on the Hecke algebra for G'. For a 'nice' $f' \in \mathscr{H}(\mathrm{G}(\mathbf{A}), \mathbf{C})$, it admits a spectral and a geometric expansion

$$\sum_{\Pi} \frac{1}{4} \mathscr{L}(1/2, \Pi, \chi) \prod_{v} I_{\Pi_v}(f'_v, \chi_v) = I(f', \chi) = \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I_\gamma(f', \chi), \qquad (1.4.1)$$

where: Π ranges over isomorphism classes of cuspidal representations of $\mathrm{G}'(\mathbf{A})$; the I_{Π_v} are local spherical characters; $\mathrm{B}'_{\mathrm{rs}/F_0}$ is a variety parametrizing the 'regular semisimple' orbits in a certain double quotient of G'; and the I_γ are products of local orbital integrals.

The only possible sources of irrationality in the right-hand side of (1.4.1) are essentially the archimedean orbital integrals. However, there is a particularly well-behaved class of $f'_\infty \in \mathscr{H}(\mathrm{G}'(F_{0,\infty}))$ (and corresponding $f' \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$), the so-called (rational) Gaussians, whose orbital integrals are controlled. Building on [BPLZZ21], we are able to show that for Π as in Theorem A, there exist L-rational Gaussians f' annihilating every automorphic representation of $\mathrm{G}'(\mathbf{A})$ but Π. Moreover, we need to show that one can pick f' to be nice: more precisely, supported at the regular semisimple elements of $\mathrm{G}'(F_{0,v})$ for some place v. This turns out to be a rather hard local problem, which is solved by a pair of explicit local computations for a certain f'_v of Iwahori level: one on the spectral side, which is Hypothesis 5.2.6; and one on the geometric side, which is Proposition 5.3.2, whose proof occupies the entire § 7. Then the rationality of $\mathscr{L}(1/2, \Pi, \chi)$ can be deduced from (1.4.1).

1.4.2. *p-adic analytic distribution.* We have a p-adic variant of $I(-, \chi)$, that we describe at first in a slightly idealized form. For any 'convenient' subgroup $K'_p \subset \mathrm{G}'(F_{0,p})$, we construct a distribution

$$\mathscr{I} = \mathscr{I}_{K'_p} \colon \mathscr{H}(\mathrm{G}'(\mathbf{A}^p))^\circ_{K'_p, \mathrm{rs}, \mathrm{qc}} \longrightarrow \mathscr{O}(\mathscr{Y})$$

on a certain space of regularly supported, $\mathbf{Q}_p$-rational Gaussian elements of the Hecke algebra away from p. It admits a spectral and a geometric expansion

$$\sum_\Pi \frac{1}{4} \mathscr{L}_p(\mathrm{M}_\Pi, \chi) \prod_{v \nmid p} \mathscr{I}_{\Pi_v}(f'_v, \chi_v) = \mathscr{I}(f'^p, \chi) = \int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} \mathscr{I}_\gamma(f'^p, \chi) \, dI^{\mathrm{ord}}_{\gamma, K'_p, p}, \qquad (1.4.2)$$

where Π ranges over representations as in Theorem B with nontrivial K'_p-invariants; the $\mathscr{I}_{\Pi_v}$, $\mathscr{I}_\gamma$ are $\mathscr{O}(\mathscr{Y})$-valued spherical characters and orbital integrals, respectively; and finally, $dI^{\mathrm{ord}}_{-, K'_p, p}$ is a certain generalized Radon measure on $\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)$. In fact, we construct $\mathscr{I}$ from its geometric expansion, and prove Theorem B by extracting $\mathscr{L}_p(\mathrm{M}_\Pi)$ from $\mathscr{I}$.

We note that Urban [Urb11, § 6] has constructed a p-adic Arthur–Selberg trace formula; it would be interesting to compare or combine our two approaches.

1.4.3. *The derivative.* For suitable f'^p, we then have a similar expansion for the derivative of $\mathscr{I}$. We will be especially interested in those f'^p that 'purely match' an $f^p \in \mathscr{H}(\mathrm{G}^V(\mathbf{A}^p))$ for some incoherent V, in the sense that $I_\gamma(f'_v, \mathbf{1}) = J_\gamma(f_v)$ for orbital integrals

$$J_\delta(f_v) = \int_{\mathrm{H}^{V_v}(F_{0,v})^2} f_v(h^{-1}\gamma h') \, dh \, dh';$$

here δ is the (regular semisimple) orbit in $\mathrm{H}^{V_v}(F_{0,v}) \backslash \mathrm{G}^{V_v}(F_{0,v}) / \mathrm{H}^{V_v}(F_{0,v})$ that 'matches' γ in the sense that certain invariants coincide.

For such f'^p, we have $\mathscr{I}(f'^p, \mathbf{1}) = 0$ and the $\Gamma_{F_0} \hat{\otimes} \mathbf{Q}_p$-valued expansions

$$\sum_\Pi \frac{1}{4} \partial \mathscr{L}_p(\mathrm{M}_\Pi) \prod_{v \nmid p} \mathscr{I}_{\Pi_v}(f'_v, \chi_v) = \partial \mathscr{I}(f'^p) = \int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} \partial \mathscr{I}_\gamma(f') \, dI^{\mathrm{ord}}_{\gamma, K'_p, p} \qquad (1.4.3)$$

for the derivative. Moreover

$$\partial \mathscr{I}_\gamma(f'^p) = \sum_{v \nmid p\infty \text{ nonsplit in } F} I_\gamma(f'^{vp}) \, \partial \mathscr{I}_\gamma(f'_v)$$

with $I_\gamma(f'^{vp}) = \mathscr{I}_\gamma(f'^{vp}, \mathbf{1})$. The v-component of the sum can be nonzero only if γ matches an orbit δ of $\mathrm{H}^{V(v)}(\mathbf{A}^p) \backslash \mathrm{G}^{V(v)}(\mathbf{A}^p) / \mathrm{H}^{V(v)}(\mathbf{A}^p)$ for the coherent pair $V(v) \in \mathscr{V}^\circ$ that is locally isomorphic to V at all places except v.

In practice, unless K'_p is suitably symmetric, we are only able to prove the geometric expansion in (1.4.2) after specialization at a $\chi \in Y(p^\infty)$, and with a generalized Radon measure $I^{\mathrm{ord}}_{-, K'_p, p}(\chi_p)$ depending on χ_p; nevertheless we can show that (1.4.3) still holds with $I^{\mathrm{ord}}_{-, K'_p, p} := I^{\mathrm{ord}}_{-, K'_p, p}(\mathbf{1})$.

1.4.4. *Arithmetic distribution.* Let $V \in \mathscr{V}^\circ$ be incoherent, $\mathrm{G} = \mathrm{G}^V$. For a convenient subgroup $K_p \subset \mathrm{G}(F_{0,p})$, we define another $\Gamma_{F_0} \hat{\otimes} \mathbf{Q}_p$-valued distribution on a suitable subset of $\mathscr{H}(\mathrm{G}(\mathbf{A}^p))$ by

$$\mathscr{J}_{K_p}(f^p) = h(Z^{\mathrm{ord}}_{K_p} T(f^p), Z^{\mathrm{ord}}_{K_p}),$$

where $Z^{\mathrm{ord}}_{K_p}$ is an ordinary modification of the arithmetic diagonal cycle in level K_p, and h is a limit of height pairings on the Selmer group of the tempered, ordinary part of $H^{2n-1}(X_{K^p K_p, \overline{F}_0}, \mathbf{Q}_p(n))$.

When the cycles have disjoint support on the generic fiber, the p-adic height pairing admits an expansion $h = \sum_{v \nmid \infty} h_v$ into local height pairings. The disjointness can be achieved by choosing

a test function f with regular support at some place v_0; we pick v_0 to be split and not above p, and f_{v_0} to match the regularly supported f'_{v_0} alluded to in § 1.4.1 – in particular, f_{v_0} has Iwahori level.

By results in [DL, LL21], the local height pairing at a place v away from p is related to the arithmetic intersection pairing on a regular v-integral model, at least after applying suitable Hecke correspondence to the cycles, and under some vanishing condition for the absolute cohomology of the model. After a base change, we may use the models constructed in the previous work of Rapoport, Smithling and the second author [RSZ20, RSZ21]; here, a technical difficulty is to verify the vanishing of cohomology in the case of non-trivial level structure, as required both in order to treat the ramification of Π in Theorem D, and for the place v_0 of regular support. Once this is settled:

− for *split* places away from p, we can show that the local arithmetic intersection numbers at *split* places away from p vanish, by refining an argument of [Zha12, RSZ20];

− for *inert* places v (thus away from p), by results in [Zha12] and [RSZ20], the local arithmetic intersection numbers admit geometric expansions over the orbits δ for $V(v)$, whose terms are products of local orbital integrals $J_\delta(f_{v'})$ ($v' \neq v$) and arithmetic intersection numbers $\mathscr{I}_\delta(f_v)$ in a certain v-adic Rapoport–Zink space.

On the other hand, the contribution of *p-adic* places vanishes: this is proved by a variant of an argument of Perrin-Riou, which in our higher-dimensional case relies on a foundational result of Y. Liu and one of us in [DL].

We then obtain a spectral and a geometric expansion

$$\sum_{\pi} \mathscr{I}_\pi(f^p) = \mathscr{I}_{K_p}(f^p) = \int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} \sum_{v \nmid p\infty \ \mathrm{nonsplit}} \mathbf{1}_{V(v)}(\gamma) J^{vp}_{\underline{\delta}(\gamma)}(f^{vp}) \mathscr{I}_{\underline{\delta}(\gamma),v}(f_v) \, dI^{\mathrm{ord}}_{\gamma,p,K'_p},$$

where: π ranges over equivalence classes of automorphic representations as in Theorem D; the geometric expansion is pulled back to $\mathrm{B}'_{\mathrm{rs}}$ via a 'matching of orbits' map $\underline{\delta}$, and $\mathbf{1}_{V(v)}$ is the indicator function of those orbits matching one on $\mathrm{G}^{V(v)}$; and finally, $dI^{\mathrm{ord}}_{\gamma,p,K'_p}$ is as in (1.4.3).

N.B.: we will also need to treat regular, non-semisimple orbits; this part is missing from the current preliminary version, cf. Remarks 4.2.4, 6.3.2.

1.4.5. *Comparison.* Theorem D is deduced from the spectral sides of an equality

$$\mathscr{I}_{K_p}(f^p) = \partial\mathscr{I}_{K'_p}(f'^p) \tag{1.4.4}$$

for suitable matching f^p, f'^p. The only substantial input in the deduction is the comparison in [BP21a, BP21] of the functionals I_{Π_v} with corresponding ones, J_{π_v}, that are related to the local components of $\alpha = (1.3.5)$.

We prove (1.4.4) by comparing the geometric expansions. By the definitions of local matching of Hecke elements (which can be globally assembled thanks to the Fundamental Lemma [Yun11, BP21b]), orbital integrals on either side are the same, thus we are reduced to identities

$$\mathscr{I}_{\underline{\delta}(\gamma),v}(f_v) = \partial\mathscr{I}_\gamma(f'_v) \tag{1.4.5}$$

for inert places v. For the spherical f'_v, f_v, the identity (1.4.5) is the Arithmetic Fundamental Lemma proposed by one of us [Zha12] and then proved in [Zha21, MZ]; for certain f'_v, f_v of maximal parahoric level, (1.4.5) was very recently proved by Z. Zhang [ZZh].

1.4.6. *Organization of the paper.* After some preliminaries in § 2, this paper is divided into two parts and en epilogue. In Part 1, we construct the analytic distribution $\mathscr{I}$ and prove the associated RTF, as well as Theorems A and B. In Part 2 (and the related appendix), we construct the distribution $\mathscr{J}$ and prove the associated RTF. In the epilogue, we compare the two RTFs to prove Theorems D and C. More details on the contents of the two parts are provided at the beginning of each.

Acknowledgements. We would like to thank Yifeng Liu for many helpful discussions, and especially for providing us with the material of § 4.4. We are also grateful to Binyong Sun for correspondence about Hypothesis 5.2.6.

2. NOTATION AND PRELIMINARIES

2.1. **Basic notation.** We set up some notation to be used throughout the paper unless otherwise noted.

2.1.1. *Fields.* We denote by $F \supset F_0$ a quadratic extension of number fields, as in the introduction, and by $c \in \mathrm{Gal}(F/F_0)$ the conjugation. We denote by $\mathbf{A}$ the adèles of F_0. From § 4 on, we will assume that F_0 is totally real and F is CM.

We denote with a bar the nontrivial automorphism of F/F_0, and by

$$\eta \colon F_0^\times \backslash \mathbf{A}^\times \longrightarrow \{\pm 1\}$$

the quadratic character associated with F/F_0. We fix an auxiliary element $\tau \in F$ such that $\tau^c = -\tau$, and an extension $\eta' \colon F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ of η.

If F' is a number field and S is a finite set of places of F', we denote by $F'_S = \prod_{v \in S} F'_v$, and by $\mathbf{A}_{F'}^S = \prod'_{v \notin S} F'_v$. If $F'' \subset F'$ is a subfield and ℓ is a place of of F'', for notational purposes we identify ℓ with the set of places of F' above ℓ.

2.1.2. *L-functions.* In the rest of the paper (*unlike* in the introduction), all global ζ- and L-functions are complete including the archimedean factors (this also includes the ratio of L-functions $\mathscr{L}(1/2, \Pi, \chi)$). If $L^S(s)$ is a global L-function, we denote by

$$L^{S,*}(s_0)$$

its leading term at $s = s_0$.

2.1.3. *Groups.* We now recall the groups under consideration in this paper, then discuss local and global base-change from unitary groups to general linear groups. We denote by $\mathbf{G}_m = \mathrm{Spec}\,\mathbf{Q}[T^{\pm 1}]$ the multiplicative group over $\mathbf{Q}$. If G is a (usually, group-) scheme over a global field F_0 and v is a place of F_0, we denote $G_v \coloneqq \mathrm{G}(F_{0,v})$ with its v-adic topology. We also denote

$$[G] = \mathrm{G}(F_0)\backslash \mathrm{G}(\mathbf{A}).$$

For $* = \emptyset, 0$ (where in this type of context, '$\emptyset$' will always mean 'no subscript') and $\nu \in \mathbf{N}$, let $G'_{\nu,*} := \operatorname{Res}_{F_*/F_0} \operatorname{GL}_\nu$. We consider

$$G' := G'_n/G_{1,0} \times G'_{n+1}/G'_{1,0}, \tag{2.1.1}$$

where $G'_{1,0}$ is the F_0-split center of G, and its subgroups

$$j_1 \colon H'_1 := G'_n \hookrightarrow G',$$

where $j_1(h) := [(\operatorname{diag}(h,1), h)]$, and

$$j_2 \colon H'_2 := G'_{n,0}/G'_{1,0} \times G'_{n+1,0}/G'_{1,0} \hookrightarrow G',$$

where j_2 is induced by $F_0 \hookrightarrow F$.

For unitary groups, we use the notation H^V, G^V introduced in § 1.3.1. We denote by $\mathscr{V}$ the set of isomorphism classes of pairs $V = (V_n, V_{n+1} = V_n \oplus Fe)$ of F/F_0-hermitian spaces with $(e,e) = 1$. When F_0 is totally real and F is CM, we denote by $V_\infty^\circ = (V_v^\circ)_{v|\infty}$ the pair such that $V_{n,v}$ is positive-definite, and by $\mathscr{V}^\circ \subset \mathscr{V}$ the set of (coherent or incoherent) pairs (V_v) such that $V_v = V_v^\circ$ for all $v|\infty$. We partition

$$\mathscr{V}^\circ = \mathscr{V}^{\circ,+} \sqcup \mathscr{V}^{\circ,-},$$

where $V \in \mathscr{V}^{\circ,\epsilon}$ if and only if $\epsilon(V) = \epsilon$.

2.2. **Measures.** Let F_0 be a number field, and let $D = |D_{F_0}|$ be the absolute value of its discriminant. Fixing a nontrivial character $\psi \colon F_0 \backslash \mathbf{A} \to \mathbf{C}^\times$, we denote by $dx = \prod_v dx_v$ the self-dual measure on $\mathbf{A}$ with respect to ψ; it satisfies $\operatorname{vol}(F_0 \backslash \mathbf{A}, dx) = 1$. For a finite place v, let d_v be a generator of the different ideal of $F_{0,v}$ and let $D_v := |d_v|^{-1}$. Assume for definiteness that $\operatorname{Ker}(\psi_v) = d_v^{-1} \mathscr{O}_{F_{0,v}}$ for all finite places v; then we have $\operatorname{vol}(\mathscr{O}_{F_{0,v}}, dx_v) = D_v^{-1/2}$. We have $D = \prod_{v\nmid\infty} D_v$, and for a finite set of places S of F_0 we define $D^S := \prod_{v\nmid S\infty} D_v$.

2.2.1. *Tamagawa measures.* If G is a reductive group over a local or global field E, we denote by M_G the Artin–Tate motive attached to (the quasi-split inner form of) G by Gross [Gro97]. If E is a local field, let

$$\Delta_G = L(M_G^\vee(1)), \qquad \Delta_G^\circ := D_v^{\dim G/2} \Delta_G.$$

Then the abelian term in (1.1.1) (including the factor $D_{F_{0,v}}^{n+1}$) equals $\Delta_{G^{V_v}}^\circ / \Delta_{H^{V_v}}^\circ$ where H^{V_v}, G^{V_v} are is in (1.3.2) (for any $V_v \in \mathscr{V}_v$.)

Assume from now on that E is the global field F_0. For a finite set S of places of F_0, let

$$\Delta_G^S := L^{S,*}(M_G^\vee(1), 0), \qquad \Delta_G^{S,\circ} := (D^S)^{\dim G/2} L^{S,*}(M_G^\vee(1), 0).$$

Let ω be any non-zero top-degree invariant differential form on G. We denote by

$$d_\omega g_v := |\omega|_v$$

its modulus with respect to dx_v ([Oes84, §4]), a Haar measure on $G(F_{0,v})$. We define

$$d^\natural g_v := \Delta_{G,v}^\circ \, d_\omega g_v$$

Then for all finite places v and any open compact subgroup $K_v \subset G_v$, we have $\mathrm{vol}(K_v, d^\natural g_v) \in \mathbf{Q}^\times$. Moreover if G_v is unramified and K_v is hyperspecial, we have $\mathrm{vol}(K_v, d^\natural g_v) = 1$. The Tamagawa measure on G is

$$dg := \Delta_{\mathrm{G}}^{\circ,-1} \prod_v d^\natural g_v. \qquad (2.2.1)$$

2.2.2. *Variants.* We define a variant

$$dg_v = \begin{cases} d^\natural g_v & \text{if } v \nmid \infty \\ \Delta_{\mathrm{G}}^{\circ,-1} d^\natural g_v = \Delta_{\mathrm{G}}^{\infty,\circ,-1} d_\omega g_v & \text{if } v = \infty \end{cases} \qquad (2.2.2)$$

so that $dg = \prod_v dg_v$. The 'rationale' for this choice is the following.

Lemma 2.2.1. *Suppose G_∞ is compact. Then $\mathrm{vol}(G_\infty, dg_\infty)$ is rational.*

Proof. We say that two measures μ, μ' are commensurable if $\mu = c\mu'$ for some $c \in \mathbf{Q}^\times$. Let $\mu := \prod_v \mu_v$ be the measure on $G(\mathbf{A})$ considered in § 9 of [Gro97], to which all citations in this proof will refer. The measure μ is nonzero by Propositions 9.4, 9.5. For almost all finite v, $\mu_v = dg_v$; for all finite v, μ_v gives rational volume to compact open subgroups (equation (5.2)), hence it is commensurable with dg_v; and μ is commensurable with dg (Theorem 9.9). It follows that dg_∞ is commensurable with μ_∞, which (again by equation (5.2)), gives rational volume to G_∞. $\qquad \square$

We also consider a different measure, for comparison with some of the literature (notably [Zha14b, § 2]). Let Z be the center of G, let $\mathrm{G}^{\mathrm{ad}} := \mathrm{G}/\mathrm{Z}$, and for $? = \circ, \emptyset$ write

$$\zeta_{\mathrm{G},v}^?(1) := \Delta_{\mathrm{Z},v}^?, \qquad \zeta_{\mathrm{G}}^{S,?,*}(1) := \Delta_{\mathrm{Z}}^{S,?},$$

sot that $\zeta_{\mathrm{G},v}^?(1)\Delta_{\mathrm{G}^{\mathrm{ad}},v}^? = \Delta_{\mathrm{G},v}^?$. Then we set

$$d^* g_v := \zeta_{\mathrm{G},v}(1)\, d^\tau g_v, \qquad d^* g = \prod_v d^* g_v$$

so that

$$dg = \zeta_{\mathrm{G},v}^*(1)^{-1} \prod_v d^* g_v$$

and for finite v, $dg_v = D_v^{\dim \mathrm{Z}/2} \Delta_{\mathrm{G}^{\mathrm{ad}},v}^\circ d^* g_v$.

2.2.3. *Local and incoherent measures.* The global measures do not depend on ω, but the local ones do. We fix the following explicit choices:

− if $\mathrm{G} = \mathrm{GL}_\nu$, we take

$$\omega := \det(g)^{-\nu} \wedge_{i,j} dg_{ij}.$$

− if G is a (product of) unitary groups over a local or a global field, we fix ω as in [Zha14b, § 2].

If G is a (product of) incoherent unitary groups, we then get a measure on $G(\mathbf{A})$ by (2.2.1), with a factorisation $dg = \prod_v dg$ as in (2.2.2).

2.3. **Hecke algebras.** Let G be a reductive group over a number field F_0, let v be a place of F_0, and let L be a characteristic-zero field, with $L = \mathbf{C}$ if v is archimedean or G_v is not compact.

We denote by $\mathcal{S}(G_v, L)$ the space of Schwartz functions on G_v valued in L: when $F_{0,v}$ is non-archimedean, this is the same as the smooth compactly supported L-valued functions, whereas when $F_{0,v}$ is archimedean this is defined in [Cas89, AG08]. We denote by $\mathscr{H}(G_v, L)$ the space of *Schwartz measures* on G_v: those are measures of the form $\dot{f}\,dg$ where $\dot{f} \in \mathcal{S}(G_v, L)$ and dg is the Haar measure fixed above. The field L will be omitted when it is unimportant or understood from context. For an open compact $K_v \subset G_v$, we denote

$$e_{K_v} := \frac{1}{\mathrm{vol}(K_v, dg_v)} \mathbf{1}_{K_v}\, dg_v$$

for any Haar measure dg_v; it is an idempotent in $\mathscr{H}(G_v)$.

When G'_v is the group (2.1.1), we define the *standard hyperspecial subgroup* $K_v^\circ \subset G'_v$ to be the image of $\mathrm{GL}_n(\mathscr{O}_{F_v}) \times \mathrm{GL}_{n+1}(\mathscr{O}_{F,v})$; when $G_v^{V_v}$ is the product of unramified unitary groups from (1.3.2), a *relative hyperspecial subgroup* $K_v^\circ \subset G_v^{V_v}$ is one of the form $\mathrm{U}(\Lambda_v) \times \mathrm{U}(\Lambda_v \oplus \mathscr{O}_{F,v}e)$ for some self-dual lattice $\Lambda_v \subset V_{n,v}$. For S a finite set of places of F_0, and G denoting either G' or G^V for some $V \in \mathscr{V}^\circ \cup_{\mathscr{V}^\circ,+} \mathscr{V}$, we consider the Hecke algebra

$$\mathscr{H}(\mathrm{G}(\mathbf{A}^S)) := \bigotimes_{v \notin S}' \mathscr{H}(G_v),$$

where the restricted tensor product is with respect to

$$f_v^\circ := e_{K_v^\circ} \tag{2.3.1}$$

for some relative hyperspecial $K_v^\circ \subset G_v$. If $K = \prod_v K_v \subset \mathrm{G}(\mathbf{A}^S)$ is an open compact subgroup, we denote $e_K := \prod_v e_{K_v}$. We say that an element $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}^\infty))$ is *supported in the set S* if we can write $f = f_S \otimes \prod_{v \notin S\infty} f_v^\circ$ for some $f_S \in \mathscr{H}(G_S)$.

For $f \in \mathscr{H}(G_v)$, we denote $f^\vee(x) := f(x^{-1})$. We denote by $*$ the convolution operation $f * f'(x) := \int_{G_v} f(xg)f'(g^{-1})$.

2.3.1. *Convention.* We stipulate that groups and Hecke algebras act on locally symmetric spaces, Shimura varieties, and their homology and algebraic cycles on the right; on automorphic forms on the left.

2.4. **Local base-change and distinction.** Let v be a place of F_0. If v is nonarchimedean, $\mathrm{G}^?$ is a reductive group over $F_{0,v}$, and L is a field admitting embeddings into $\mathbf{C}$, we say that an absolutely irreducible (that is, $\pi \otimes_L \overline{L}$ is irreducible) smooth admissible representation π of $G_v^?$ over L is *tempered* if $\pi \otimes_{L,\iota} \mathbf{C}$ is tempered for every $\iota \colon L \hookrightarrow \mathbf{C}$.

Let $\mathscr{V}_v$ be the set of isomorphism classes of pairs $V_v = (V_{n,v}, V_{n+1,v} = V_{n,v} \oplus F_{0,v}e)$ of hermitian spaces over $F_v/F_{0,v}$. Let $\mathrm{Temp}(G_v^?)(L)$ be the set of $\mathrm{Gal}(\overline{L}/L)$-orbits of isomorphism classes of irreducible tempered representations of $G_v^?$ over L.

2.4.1. *Local base-change.* Let v be a place of F_0. Thanks to [Mok15, KMSW], we have a local base-change map BC from complex irreducible admissible representations of $G_v^{V_v}$ to complex irreducible admissible representations of G'_v, whose definition is recalled in [BP21a, § 2.10]. It has the following properties:

(1) it restricts to a map

$$\mathrm{BC}\colon \mathrm{Temp}(G_v^{V_v})(\mathbf{C}) \longrightarrow \mathrm{Temp}(G_v')(\mathbf{C}); \tag{2.4.1}$$

(2) being defined by a map of L-groups, it is rational in the sense that it yields a map

$$\mathrm{BC}\colon \mathrm{Temp}(G_v^{V_v})(L) \longrightarrow \mathrm{Temp}(G_v')(L)$$

for any characteristic-zero field L;

(3) when v splits in F, we simply have $\mathrm{BC}(\pi) \coloneqq \pi \boxtimes \pi^\vee$ if we identify $G_v^{V_v} \cong G_{n,0,v}' \times G_{n+1,0,v}'$ for the unique $V_v \in \mathscr{V}_v$;

(4) when $G_v^{V_v} = U(n) \times U(n+1)$ over $\mathbf{R}$, the preimage of $\Pi_{\mathbf{R}}^\circ$ under (2.4.1) consists of the trivial representation only;

(5) when $G_v^{V_v} = U(n-1,1) \times U(n,1)$ over $\mathbf{R}$, the preimage

$$\pi_{\mathbf{R}}^\circ \coloneqq \mathrm{BC}^{-1}(\Pi_{\mathbf{R}}^\circ) \tag{2.4.2}$$

consists of the $n(n+1)$ discrete series representations having the Harish-Chandra parameter $\{\frac{1-\nu}{2}, \frac{3-\nu}{2}, \ldots, \frac{\nu-1}{2}\}$ on the $U(\nu-1,1)$-component. (See [LTX$^+$22, Proposition C.3.1].)

If v is non-archimedean and π_v, respectively Π_v, is a representation of $G_v^{V_v}$, respectively G_v', over a field L admitting embeddings into $\mathbf{C}$, we will write $\mathrm{BC}(\pi_v) = \Pi_v$ if $\mathrm{BC}(\iota\pi_v) = \iota\Pi_v$ for every embedding $\iota\colon L \hookrightarrow \mathbf{C}$.

2.4.2. *Hermitian representations.* We will say that a tempered representation Π_v of G_v' is *hermitian* if the space $\mathrm{Hom}_{H_{2,v}'}(\Pi_v, \eta_v^n \boxtimes \eta_v^{n-1})$ is nonzero. By the local Flicker–Rallis conjecture proved by Matringe, Mok, and others (see [Ana, §3.1] and references therein), a representation Π_v over $\mathbf{C}$ is hermitian if and only if it is in the image of base-change for some $V_v \in \mathscr{V}_v$.

2.4.3. *Distinction and the local Gan–Gross–Prasad conjecture.* Let v be a place of F_0 and let L be a field of characteristic zero; we restrict to $L = \mathbf{C}$ if v is archimedean and V_v is not definite. We say that a tempered representation π of $G_v^{V_v}$ over L is *distinguished* if the space $\mathrm{Hom}_{H_v^{V_v}}(\pi_v, L)$ is nonzero, and by the *multiplicity-one* result of [AGRS10], this space is one-dimensional if nonzero. It is clear that distinction is a $\mathrm{Gal}(\overline{L}/L)$-invariant property. We denote by

$$\mathrm{Temp}(H_v^{V_v}\backslash G_v^V)(L) \subset \mathrm{Temp}(G_v^V)(L)$$

the subset of orbits of distinguished tempered representations.

The following fundamental result is the local Gan–Gross–Prasad conjecture for unitary groups.

Proposition 2.4.1. *Let Π_v be a hermitian tempered representation of G_v' over a characteristic zero field L; we restrict to $L = \mathbf{C}$ if v is archimedean. There exists a unique pair (V_v, π_v) with $V_v \in \mathscr{V}_v$ and $\pi_v \in \mathrm{Temp}(H_v^{V_v}\backslash G_v^{V_v})(L)$ such that $\Pi_v = \mathrm{BC}(\pi_v)$.*

Proof. If $L = \mathbf{C}$, this is proved in [BP16, BP20]. In general, we may assume that there is an embedding $\iota\colon L \hookrightarrow \mathbf{C}$ and apply the result to $\iota\Pi_v$ to obtain a pair $(V_v, \pi_v^{\mathbf{C}})$. By uniqueness, $\pi_v^{\mathbf{C}}$ is isomorphic to its $\mathrm{Aut}(\mathbf{C}/\iota L)$-conjugates. $\square$

2.5. **Automorphic base-change.**

2.5.1. *Rational spaces of automorphic representations.* The following discussion is based on [Clo90, Théorème 3.1.3]. Let L be a field admitting embeddings into $\mathbf{C}$, and let $\Pi = \Pi^\infty \otimes \mathbf{1}_\infty$ be an absolutely irreducible representation of $G'(\mathbf{A})$ over an L-vector space. We say that Π is cuspidal automorphic of trivial weight if for every (equivalently, some) embedding $\iota\colon L \hookrightarrow \mathbf{C}$, the representation $\Pi^\iota := \iota\Pi^\infty \otimes \Pi_\infty^\circ$ is cuspidal and automorphic. Every cuspidal automorphic representation $\Pi_{\mathbf{C}}$ of $G'(\mathbf{A})$ such that $\Pi_{\mathbf{C},\infty} \cong \Pi_\infty^\circ$ arises as Π^ι for some Π defined over a number field; the smallest such number field $\mathbf{Q}(\Pi^\iota) =: \iota\mathbf{Q}(\Pi^\infty)$ depends only on $\Pi_{\mathbf{C}}$, and Π is unique up to $\mathbf{Q}(\Pi)$-isomorphism.

Denote by $\widetilde{\mathscr{C}}(G')(L)$ the set of isomorphism classes of trivial-weight cuspidal automorphic representations defined over L, and by $\mathscr{C}(G')(L) := \widetilde{\mathscr{C}}(G')(\overline{L})/G_L$, where we recall that $G_L :=$ $\mathrm{Gal}(\overline{L}/L)$. By [Car12], for every $\Pi \in \widetilde{\mathscr{C}}(G')$ and every finite place v, the representation Π_v of G'_v is tempered.

Lemma 2.5.1. *The natural map*

$$\widetilde{\mathscr{C}}(G') \longrightarrow \mathscr{C}(G')$$

is an isomorphism.

Proof. This follows from the above discussion and [Clo90, Proposition 3.1, Théorème 3.1.3]. □

2.5.2. *Base change.* Let $V \in \mathscr{V}$ or, if F_0 is totally real and F is CM, let $V \in \mathscr{V} \cup_{\mathscr{V}^\circ,+} \mathscr{V}^\circ$. Let $G = G^V$, $H = H^V$. For a field L admitting embeddings into $\mathbf{C}$, denote by

$$\widetilde{\mathscr{C}}(G)(L) \quad \supset \quad \widetilde{\mathscr{C}}(H\backslash G)(L)$$

the set of isomorphism classes of tempered[6] cuspidal automorphic representations of $G(\mathbf{A})$ that are trivial at infinity, and its subset of representations that are $H(\mathbf{A})$-distinguished. We also put

$$\mathscr{C}(G)(L) := \widetilde{\mathscr{C}}(G)(\overline{L})/G_L \quad \supset \quad \mathscr{C}(H\backslash G)(L) := \widetilde{\mathscr{C}}(H\backslash G)(\overline{L})/G_L.$$

We will view $\mathscr{C}(G')$, $\mathscr{C}(G)$ and $\mathscr{C}(H\backslash G)$ as ind-finite schemes over $\mathbf{Q}$.

Definition 2.5.2. Let $V \in \mathscr{V}$, and let $G = G^V$. Let π be a (complex) automorphic representation of $G(\mathbf{A})$ which is tempered everywhere, and let Π be an automoprhic representation of $G'(\mathbf{A})$. We say that Π is a *weak automorphic base-change of* π, and write $\Pi \cong \mathrm{BC}(\pi)$, if for all but finitely many places v of F_0 split in F, we have $\Pi_v \cong \mathrm{BC}(\pi_v)$ for the local base-change of (2.4.1). We say that Π is a *strong automorphic base-change* of π if $\Pi_v \cong \mathrm{BC}(\pi_v)$ for all places v.

Remark 2.5.3. By a result of Ramakrishnan [Ram], a weak automorphic base-change of π is unique up to isomorphism if it exists, which justifies the notation. Moreover by [Mok15, KMSW], if Π is a weak automorphic base-change of π, then Π is a strong base-change of π. From now we will simply write *the (automorphic) base-change* without adjectives.

Suppose now that F_0 is totally real and F is CM. Let $V \in \mathscr{V}^\circ$, let $G = G^V$, and let L be a characteristic-zero field. Let π be a cuspidal automorphic representation of $G(\mathbf{A})$ over L which

---

[6]That is, tempered at all finite places.

is trivial at infinity, and let Π be a trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over L. We say that Π is the cuspidal *automorphic base-change* of π, and write

$$\Pi \cong \mathrm{BC}(\pi),$$

if for every $\iota\colon L \hookrightarrow \mathbf{C}$ and every finite place v, we have $\iota\Pi_v \cong \mathrm{BC}(\iota\pi_v)$. We say that π is *stable* if it admits a cuspidal automorphic base-change over L; we denote by

$$\widetilde{\mathscr{C}}(\mathrm{G})(L)^{\mathrm{st}} \subset \widetilde{\mathscr{C}}(\mathrm{G})(L), \qquad \mathscr{C}(\mathrm{G})(L)^{\mathrm{st}} \subset \mathscr{C}(\mathrm{G})(L)$$

the subsets consisting of (orbits of) isomorphism classes of representations that are stable.

Note that by the definitions and the rationality of local base-change maps observed in § 2.4.1, the stability condition is Galois-invariant, so that the above definition makes sense.

2.5.3. *Hermitian automorphic representations as the image of base-change.*

Proposition 2.5.4. *Let Π be a cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ with $\Pi_\infty \cong \Pi_\infty^\circ$. The following are equivalent:*

(1) Π is hermitian;

(2) for every $V \in \mathscr{V}$ such that G^V is quasi-split at all places, there exists a cuspidal automorphic representation π' of $\mathrm{G}^V(\mathbf{A})$ such that $\Pi \cong \mathrm{BC}(\pi')$;

(3) for some $V \in \mathscr{V}^\circ$, there exists a cuspidal automorphic representation π of $\mathrm{G}^V(\mathbf{A})$ over $\mathbf{C}$, trivial at infinity and tempered everywhere, such that $\Pi^\infty \otimes \mathbf{1}_\infty \cong \mathrm{BC}(\pi)$.

(4) there exists a unique pair (V, π) with $V \in \mathscr{V}^\circ$ and π an $\mathrm{H}^V(\mathbf{A})$-distinguished cuspidal automorphic representation π of $\mathrm{G}^V(\mathbf{A})$ over $\mathbf{C}$, trivial at infinity and tempered everywhere, such that $\Pi^\infty \otimes \mathbf{1}_\infty \cong \mathrm{BC}^\circ(\pi)$.

Proof. That (1) implies (2) is the automorphic descent of [GRS11]. Assume (2) holds for the representation π' of $\mathrm{G}^{V'}(\mathbf{A})$, and let $V \in \mathscr{V}^\circ$ agree with V' at all finite places. If $V \in \mathscr{V}^{\circ,+}$, let $V'' = V$ and let $\pi'' = \pi'$; if $V \in \mathscr{V}^{\circ,-}$, let v be an archimedean place of F_0, let $V'' = V(v)$, and let $\pi'' = \pi'^v \otimes \pi_v^\circ$ for any $\pi_v^\circ \in \underline{\pi}_{\mathbf{R}}^\circ = (2.4.2)$, a representation of $\mathrm{G}^{V''}(\mathbf{A})$. Then $\Pi_v = \mathrm{BC}(\pi_v'')$ for all v, so that by [LTX$^+$22, Proposition C.3.1.1 (1)], π'' is automorphic with base-change Π. Let $\pi = \pi''^\infty \otimes \mathbf{1}_\infty$, which is a representation of $\mathrm{G}^V(\mathbf{A})$ over $\mathbf{C}$ trivial at infinity. Then by definition, $\mathrm{BC}(\pi) = \Pi^\infty \otimes \mathbf{1}_\infty$, so that (3) holds. The implication (3) $\Rightarrow$ (1) follows from [Mok15, KMSW] together with the special cases of base-change for real groups stated in § 2.4.1 (4)-(5); a simpler alternative proof, when Π is supercuspidal at some split places, is given in [BPLZZ21, Theorem 4.12 (2)].

Suppose now that (1) and (3) hold. By Proposition 2.4.1 and [LTX$^+$22, Proposition C.3.1.1 (1)], we can modify the pair (V, π) of part (3) locally at finitely many places so that the resulting representation satisfies the properties of (4). $\square$

Corollary 2.5.5. *There is a sub-ind-scheme*

$$\mathscr{C}(\mathrm{G}')^{\mathrm{her}} \quad \subset \quad \mathscr{C}(\mathrm{G}')$$

parametrising those trivial-weight cuspidal automorphic representation of Π *of* $\mathrm{G}'(\mathbf{A})$ *that are hermitian. Moreover, the base-change map gives an isomorphism of* $\mathbf{Q}$-*ind-schemes*

$$\mathrm{BC}\colon \bigsqcup_{V\in\mathscr{V}^\circ} \mathscr{C}(\mathrm{H}^V\backslash\mathrm{G}^V)^{\mathrm{st}} \longrightarrow \mathscr{C}(\mathrm{G}')^{\mathrm{her}}. \tag{2.5.1}$$

Proof. This follows from the equivalence $(1) \Leftrightarrow (4)$ in Proposition 2.5.4. $\qquad\square$

Remark 2.5.6. For $\epsilon \in \{\pm\}$, let $\mathscr{C}(\mathrm{G}')^{\mathrm{her},\epsilon} \subset \mathscr{C}(\mathrm{G}')^{\mathrm{her}}$ be the subset of those representations with $\epsilon(\Pi) := \varepsilon(\Pi_n \times \Pi_{n+1}, 1/2) = \epsilon$. By [GGP12, § 26, discussion of Question (1)], we have

$$\mathscr{C}(\mathrm{G}')^{\mathrm{her},\epsilon} = \mathrm{BC}\left(\bigsqcup_{V\in\mathscr{V}^{\circ,\epsilon}} \mathscr{C}(\mathrm{H}^V\backslash\mathrm{G}^V)^{\mathrm{st}} \right).$$

Remark 2.5.7. Similarly to the above, for a characteristic-zero field L we may define the notions of *discrete* (rather than tempered cuspidal), trivial-at-infinity automorphic representation of $\mathrm{G}^V(\mathbf{A})$ over L, and of *isobaric* (rather than cuspidal) trivial-weight automoprhic representation of $\mathrm{G}'(\mathbf{A})$ over L. Denote the corresponding sets of isomorphism classes by $\mathscr{C}^\sharp(\mathrm{G}^V)(L)$, $\mathscr{C}^\sharp(\mathrm{G}')(L)$. By the variant of Shin's result in [Gol14, Theorem A.1] stated in [LTX$^+$22, Proposition 3.2.8], we have a base-change map $\mathrm{BC}\colon \mathscr{C}^\sharp(\mathrm{G}^V)(L) \to \mathscr{C}^\sharp(\mathrm{G}')(L)$. By the Arthur classification proved in [KMSW], the map is injective.

2.6. **Relative traces.**

Definition 2.6.1. Let L be a normed field. Suppose given data $D = (\Pi_1, \Pi_2; \vartheta, \beta, T)$ consisting of:

— L-vector spaces Π_1, Π_2;

— a bilinear form $\vartheta\colon \Pi_1 \otimes \Pi_2 \to L$;

— a bilinear form $\beta\colon \Pi_1 \otimes \Pi_2 \to \Gamma$, where Γ is a finite-dimensional L-vector space;

— a map $T\colon \Pi_1 \to \Pi_1$,

satisfying:

— for $i = 1, 2$ we can write $\Pi_i = \varinjlim_{\lambda\in\Lambda} \Pi_{i,\lambda}$ as a filtered direct limit of finite-dimensional L-vector spaces and injective maps, in such a way that:

— for every $\lambda \in \Lambda$, $\vartheta_{|\Pi_{1,\lambda}\otimes\Pi_{2,\lambda}}$ is a perfect pairing.

Let us say that a basis $\{\phi\}$ of Π_1 is *admissible* if there is a presentation $\Pi_1 = \varinjlim_{\lambda\in\Lambda} \Pi_{1,\lambda}$ with the above properties, such that $\{\phi\} \cap \Pi_{1,\lambda}$ is a basis for all $\lambda \in \Lambda$; if this is the case we denote by $\{\phi^\vee\}$ the basis of Π_2 whose restriction to $\Pi_{2,\lambda}$ is the ϑ-dual basis of $\{\phi\} \cap \Pi_{1,\lambda}$.

We define the *trace of T relative to β, ϑ* to be

$$\mathrm{Tr}_\vartheta^\beta(T) := \sum_\phi \beta(T\phi, \phi^\vee), \tag{2.6.1}$$

provided the sum is absolutely convergent and is independent of the choice of an admissible basis $\{\phi\}$ of Π_1.

Remark 2.6.2. If $\Gamma = L$ and $\beta = \vartheta$, we recover the usual notion of trace. In the examples of interest to us:

− when L is not $\mathbf{C}$, the sum (2.6.1) will have only finitely many nonzero terms;

− we will have $\beta = h \circ (P_1 \boxtimes P_2)$ for some linear functionals $P_i \colon \Pi_i \to S_i$ valued in an L-vector space S_i, and some bilinear form $h \colon S_1 \otimes S_2 \to \Gamma$. (In fact, in the first part of the paper we will only consider $S_1 = S_2 = \Gamma = L$, and h equal to the multiplication map.)

2.6.1. *Relations between different relative traces.* We give a preliminary definition. In the situation of Definition 2.6.1, let $\alpha_2 \in \operatorname{End}_L(\Pi_2)$. Let $\mu \colon \Lambda \to \Lambda$ be a strictly increasing function with cofinal image such that $\alpha_2(\Pi_{2,\lambda}) \subset \Pi_{2,\mu(\lambda)}$. We define the *$\vartheta$-transpose* of α_2 to be the unique $\alpha_2^\vartheta \in \operatorname{End}_L(\Pi_1)$ whose restriction to $\Pi_{1,\mu(\lambda)}$ is the transpose of $\alpha_{2|\Pi_{2,\lambda}}$ for the restriction of ϑ.

Lemma 2.6.3. *Let $D = (\Pi_1, \Pi_2; \vartheta, \beta, T)$ and $D' = (\Pi_1', \Pi_2'; \vartheta', \beta', T')$ be data as in Definition 2.6.1. In each of the following, suppose that all the data in D, D' are equal except for the indicated differences.*

(1) Suppose that $\beta' = \beta \circ (1 \boxtimes \alpha_2)$ for some $\alpha_2 \in \operatorname{End}_L(\Pi_2)$. Then

$$\operatorname{Tr}_\vartheta^\beta(T) = \operatorname{Tr}_{\vartheta'}^{\beta'}(T'\alpha_2^\vartheta),$$

where $\alpha_2^\vartheta \in \operatorname{End}_L(\Pi_1)$ is the ϑ-transpose of α_2.

(2) Suppose that $\vartheta' = \vartheta \circ (\alpha_1 \boxtimes \operatorname{id})$ and $T' = T\alpha_1$ for some L-isomorphism $\alpha_1 \colon \Pi_1' \to \Pi_1$. Then

$$\operatorname{Tr}_\vartheta^\beta(T) = \operatorname{Tr}_{\vartheta'}^{\beta'}(T').$$

(3) Suppose that $\Pi_i' \subset \Pi_i$ are all direct summands, that $\vartheta' := \vartheta_{|\Pi_1'^* \otimes \Pi_2'}$ is a perfect pairing (in the sense that it satisfies the condition of Definition 2.6.1, and that $T(\Pi_1) \subset \Pi_1'$. If $\beta' = \beta_{|\Pi_1' \otimes \Pi_2'}$ and $T' = T_{|\Pi_1'}$, then

$$\operatorname{Tr}_\vartheta^\beta(T) = \operatorname{Tr}_{\vartheta'}^{\beta'}(T').$$

The proof is elementary linear algebra.

Part 1. p-adic L-functions and the analytic relative-trace formula

3. Jacquet–Rallis relative-trace formulas

We consider the traces of Hecke operators relative to two periods functionals and the Petersson inner product on automorphic forms for G', and compare (the resulting local terms) with a parallel relative-trace distribution for G. The substance of this section is not new. We omit detailed discussions of convergence issues, for which we refer to [Zha14b] or [BP21, Appendix A].

3.1. **Period functionals and the distribution.** Let $\mathscr{A}(\mathrm{G}')$ be the space of automorphic forms on $\mathrm{G}'(\mathbf{A})$, and let $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')$ be its cuspidal subspace. We endow $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')$ with the bilinear Petersson product

$$\vartheta(\phi_1, \phi_2) := \int_{[\mathrm{G}']} \phi_1(g)\phi_2(g)\, dg.$$

3.1.1. *Period functionals.* We define two functionals on $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}'(F_0)\backslash\mathrm{G}'(\mathbf{A}))$.

For $\chi \in Y(\mathbf{C})$, the (χ-twisted) *Rankin–Selberg period* is the functional

$$P_{1,\chi}(\phi) := \int_{[\mathrm{H}'_1]} \phi(h_1)\chi(h_1)\, dh_1.$$

where $\chi(h_1) := \chi(N_{F/F_0} \det h_1)$,

The *Flicker–Rallis period* is the functional

$$P_2(\phi) := \int_{[\mathrm{H}'_2]} \phi(h_2)\eta(h_2)\, dh_2,$$

where $\eta(h_2) := \eta(\det(h_n)^{n+1} \det(h_{n+1})^n)$ if $h_2 = ([h_n], [h_{n+1}])$.

3.1.2. *Relative-trace distribution.* We say that $f' \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$ is *quasicuspidal* if $R(f')$ sends $\mathscr{A}(\mathrm{G}')$ to $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')$ (cf. [BPLZZ21, Definition 3.2]), and we denote by $\mathscr{H}(\mathrm{G}'(\mathbf{A}))_{\mathrm{qc}}$ the space of quasicuspidal Hecke measures.

Definition 3.1.1. We define a relative-trace distribution on $\mathscr{H}(\mathrm{G}(\mathbf{A}))_{\mathrm{qc}} \times Y(\mathbf{C})$ by

$$I(f', \chi) := C \cdot \mathrm{Tr}_\vartheta^{P_{1,\chi} \otimes P_2}(R(f')),$$

where the constant

$$C := \frac{\Delta_{\mathrm{G}}^\circ}{\Delta_{\mathrm{H}}^2} \frac{\Delta_{\mathrm{H}'_1}^\circ \Delta_{\mathrm{H}'_2}^\circ}{\Delta_{\mathrm{G}'}^\circ} \tag{3.1.1}$$

is motivated by rationality considerations.[7]

We note that the above definition does fit within the setup of Definition 2.6.1: we may write

$$\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}') = \varinjlim_{(K,\mathfrak{a})} \mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')^{K,\mathfrak{a}=0}$$

as K varies among compact open subgroups of $\mathrm{G}'(\mathbf{A}^\infty)$ and $\mathfrak{a}$ among finite-codimension ideals in the center of the universal enveloping algebra of Lie G'_∞. The relative trace is well-defined by (the proof of) [Zha14a, Theorem 2.3]. (See also [BPCZ22, Proposition 2.8.4.1] for a more general result in a framework similar to ours.)

In the next two subsections we discuss the two expansions of I: a spectral expansion, in terms of automorphic representations, and a geometric expansion, in terms of double orbits.

3.2. **Spectral expansion.** Let Π be a cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$, which by multiplicity one we may and do identify with a subspace of $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')$. We define a distribution on $\mathscr{H}(\mathrm{G}'(\mathbf{A}))$ by

$$I_\Pi(f', \chi) := C \cdot \mathrm{Tr}_{\vartheta_\Pi}^{P_{1,\Pi,\chi} \otimes P_{2,\Pi^\vee}}(\Pi(f')),$$

where we use subscripts to indicate the restriction of period functionals and Petersson product to Π, $\Pi^\vee$, $\Pi \otimes \Pi^\vee$.

We define some local periods, in order to factorize I_Π.

---

[7][[In fact I suspect that C is the ratio between the measure on $\mathrm{B}(\mathbf{A})$ coming from the split groups and the measure on $\mathrm{B}(\mathbf{A})$ coming from unitary groups; anyway this remark does not seem necessary.]]

3.2.1. *Whittaker models and rational structures.* Let $\psi\colon F_0\backslash\mathbf{A}\to\mathbf{C}^\times$ be a nontrivial character, and let
$$\psi_F := \psi(\frac{1}{2}\operatorname{Tr}_{F/F_0}(\cdot))\colon F\backslash\mathbf{A}_F \longrightarrow \mathbf{C}^\times.$$
We inflate ψ_F to a character of $\mathrm{N}_n(\mathbf{A}_{F_0})$ by $\psi_{F,n}(u) = \psi_F(\sum_{i=1}^{n-1} u_{i,i+1})$. Let Π_ν be an automorphic representation of $\mathrm{G}_\nu(\mathbf{A})$. Its ψ-Whittaker model $\mathscr{W}_\psi(\Pi_\nu)$ is the image of the map

$$\mathscr{W}\colon \Pi \longrightarrow C^\infty(\mathrm{N}_\nu(\mathbf{A})\backslash\mathrm{GL}_\nu(\mathbf{A}), \psi_{F,\nu})$$
$$\phi \longmapsto W_\phi(g) := \int_{\mathrm{N}_\nu(\mathbf{A})} \phi(ug)\overline{\psi}_{F,\nu}(u)\,du. \tag{3.2.1}$$

The ψ-Whittaker model of $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ is $\mathscr{W}_\psi(\Pi) = \mathscr{W}_{\overline{\psi}}(\Pi_n) \boxtimes \mathscr{W}_\psi(\Pi_{n+1})$; it has a $\mathrm{G}'(\mathbf{A})$-factorization $\mathscr{W}_\psi(\Pi) = \bigotimes_v \mathscr{W}_{\psi_v}(\Pi_v)$.

We now consider rational structures, along the lines of [RS08, § 3.2]. Let v be a finite place of F_0 with underlying rational prime ℓ, and suppose that Π_v is a smooth irreducible admissible representation of G'_v over a subfield $L \subset \mathbf{C}$. For $\sigma \in \operatorname{Aut}(\mathbf{C}/L)$, let $a_\sigma \in \mathbf{Z}_\ell^\times$ be its image under the composition
$$\operatorname{Aut}(\mathbf{C}/L) \longrightarrow \operatorname{Gal}(L(\mu_{\ell^\infty})/L) \longrightarrow \mathbf{Z}_\ell^\times$$
of the restriction and the cyclotomic character. Let $t_{\sigma,\nu} := \operatorname{diag}(a_\sigma^{\nu-1}, \ldots, 1)$ and let $t_\sigma := (t_{\sigma,n}, t_{\sigma,n+1}) \in G'_v$. Then we may define an action of $\operatorname{Aut}(\mathbf{C}/L)$ on $\mathscr{W}_{\psi_v}(\Pi_v \otimes_L \mathbf{C})$ by

$$W^\sigma(g) := \sigma(W(t_\sigma^{-1}g)); \tag{3.2.2}$$

we will denote by $\mathscr{W}_{\psi_v}(\Pi_v)$ the space of $\operatorname{Aut}(\mathbf{C}/L)$-invariants; it is an $L[G'_v]$-module satisfying $\mathscr{W}_{\psi_v}(\Pi_v) \otimes_L \mathbf{C} \cong \mathscr{W}_{\psi_v}(\Pi_v \otimes_L \mathbf{C})$ (see [RS08, Lemma 3.2]).

3.2.2. *Factorizations of the periods and Petersson product.* For the following factorization results, see [Zha14b, § 3] and references therein. Let $\epsilon'_\nu(\tau) := \operatorname{diag}(\tau^{\nu+\epsilon-1}, \tau^{\nu+\epsilon-2}, \ldots, \tau^{\epsilon-1}) \in \mathrm{GL}_\nu(F_v)$, where $\epsilon \in \{0,1\}$ has the same parity as ν.

For $W = W_n \otimes W_{n+1} \in \mathscr{W}_{\psi,v}(\Pi_v)$, define[8]

$$P_{1,\Pi_v,\chi_v}(W) := \frac{\varepsilon(\frac{1}{2}, \chi_v^2, \psi_v)^{\binom{n+1}{2}}}{L(1/2, \Pi_v \otimes \chi_v)} \int_{\mathrm{N}_n(F_v)\backslash\mathrm{GL}_n(F_v)} W(j_1(h_1))\chi_v(h_1)\,d^\natural h_1,$$

$$P_{2,\Pi_v}(W) := \frac{\varepsilon(\frac{1}{2}, \eta_v, \psi_v)^{\binom{n+1}{2}}}{L(1, \Pi_v, \mathrm{As}^{-\star})} P^\natural_{2,\Pi_{n,v}}(W_n) P^\natural_{2,\Pi_{n+1,v}}(W_{n+1}),$$

$$P^\natural_{2,\Pi_{\nu,v}}(W_\nu) := \int_{\mathrm{N}_{\nu-1}(F_{0,v})\backslash\mathrm{GL}_{\nu-1}(F_{0,v})} W_\nu\left(\begin{pmatrix} \epsilon'_{\nu-1}(\tau)h_{2,\nu-1} & \\ & 1 \end{pmatrix}\right) \eta_v(\det h_{2,\nu-1})^{\nu-1}\,d^\natural h_{2,\nu-1}.$$
$$\tag{3.2.3}$$

where $L(1, \Pi_v, \mathrm{As}^{-\star}) = \prod_{\nu=n}^{n+1} L(1, \Pi_{\nu,v}, \mathrm{As}^{(-1)^{\nu-1}})$.

---

[8]Our definition of $P_{2,v}$ differs form the one of [Zha14b] by the factor $\varepsilon(\frac{1}{2}, \eta, \psi)^{\binom{n+1}{2}}$, cf. § 3.5.1 below.

For $W \in \mathscr{W}_{\psi_v}(\Pi_v)$, $W^\vee \in \mathscr{W}_{\overline{\psi}_v}(\Pi_v^\vee)$, define

$$\vartheta_{\Pi_v}(W, W^\vee) := L(1, \Pi_v \times \Pi_v^\vee)^{-1} \prod_{\nu=n}^{n+1} \int_{\mathrm{N}_{\nu-1}(F) \backslash \mathrm{GL}_{\nu-1}(F)} W_\nu \left(\begin{pmatrix} g_{\nu-1} & \\ & 1 \end{pmatrix} \right) W_\nu^\vee \left(\begin{pmatrix} g_{\nu-1} & \\ & 1 \end{pmatrix} \right) d^\natural g_{\nu-1}.$$

Remark 3.2.1. With our normalizations, when all the data are unramified and $W(1) = W^\vee(1) = 1$, we have

$$P_{1,\Pi_v,\chi_v}(W) = P_{2,\Pi_v}(W) = \vartheta_{\Pi_v}(W, W^\vee) = 1.$$

Moreover, the three functionals are rational in the following sense. If Π_v is defined over a subfield $L \subset \mathbf{C}$, by (3.2.2) and a change of variable we see that for every $\sigma \in \mathrm{Aut}(\mathbf{C}/L)$ we have

$$P_{1,\Pi_v,\chi_v^\sigma}(W^\sigma) = \sigma P_{1,\Pi_v,\chi_v}(W), \qquad P_{2,\Pi_v}(W^\sigma) = \sigma P_{2,\Pi_v}(W), \qquad \vartheta_{\Pi_v}(W^\sigma, W^{\vee,\sigma}) = \sigma \vartheta_{\Pi_v}(W, W^\vee).$$

We may now state the factorization (see [Zha14b, §3] and § 3.2.4 below): for any $\phi \in \Pi$ with factorizable ψ-Whittaker function $W = \otimes_v W_v \in \mathscr{W}_\psi(\Pi)$, and $\phi^\vee \in \Pi^\vee$ with factorizable $\overline{\psi}$-Whittaker function $W^\vee = \otimes_v W_v^\vee \in \mathscr{W}_{\overline{\psi}}(\Pi^\vee)$, we have

$$P_{1,\Pi,\chi}(\phi) = \frac{L(1/2, \Pi \otimes \chi)}{\Delta_{\mathrm{H}_1'}^\circ \cdot \varepsilon(\frac{1}{2}, \chi^2, \psi)^{\binom{n+1}{2}}} \prod_v P_{1,\Pi_v,\chi_v}(W_v)$$

$$P_{2,\Pi}(\phi^\vee) = \frac{n(n+1)L^*(1, \Pi, \mathrm{As}^{-\star})}{\Delta_{\mathrm{H}_2'}^\circ} \prod_v P_{2,\Pi_v}(W_v^\vee) \qquad (3.2.4)$$

$$\vartheta_\Pi(\phi, \phi^\vee) = \frac{4n(n+1)L^*(1, \Pi \times \Pi^\vee)}{\Delta_{\mathrm{G}'}^\circ} \prod_v \vartheta_{\Pi_v}(W_v, W_v^\vee),$$

where 4 is the Tamagawa number of G'. In the factorization of P_2, we have used that $\varepsilon(\frac{1}{2}, \eta, \psi) = 1$.

3.2.3. *Local spherical character.* We define a character

$$I_{\Pi_v}(f_v', \chi_v) = I_{\Pi_v}(f_v', \chi_v, \psi_v) = \mathrm{Tr}_{\vartheta_{\Pi_v}}^{P_{1,\Pi_v,\chi_v} \otimes P_{2,\Pi_v^\vee}}(R(f_v'))$$

on $\mathscr{H}(\mathrm{G}_v')$ [[same as in [Zha14b], except for Petersson on G and not G^{ad}, and our normalization of measures (so we have the same factors $\Delta_{\mathrm{H}_{i,v}'^{\mathrm{ad}}}$]].

3.2.4. *Comparison with the normalization of* [Zha14b]. Let

$$\widetilde{\mathrm{G}}' := \mathrm{G}_n' \times \mathrm{G}_{n+1}',$$

and let us identify representations of G_v' with representations of $\widetilde{G}_v'$ whose central character is trivial on $(F_{0,v}^\times)^2$. In [Zha14b], one considers a distribution $\widetilde{I}_{\Pi_v}$ on $\mathcal{S}(\widetilde{G}_v')$ (denoted there by I_{Π_v}), and a global distribution $\widetilde{I}_\Pi$ on $\mathcal{S}(\widetilde{\mathrm{G}}'(\mathbf{A}))$ (denoted there by I_Π).[9] If $f = \otimes_v f_v \in \mathscr{H}(\mathrm{G}_v')$ and

---

[9]Strictly speaking only $\chi_v = \mathbf{1}_v$ is considered in [Zha14b], but the definition remains valid in our more general case too. When this is again the case in the rest of the paper, we will simply cite [Zha14b] without repeating this remark.

$\dot{f} = \otimes_v \dot{f}_v \in \mathcal{S}(\widetilde{G}'_v)$ are related by (3.3.13), then

$$I_{\Pi_v}(f_v, \chi_v) = \frac{\Delta^\circ_{H_1'^{ad},v}\Delta^\circ_{H_2'^{ad},v}}{\Delta^\circ_{G^{ad},v}} \cdot \left(\varepsilon(\tfrac{1}{2}, \eta_v, \psi_v)\varepsilon(\tfrac{1}{2}, \chi_v^2, \psi_v)\right)^{\binom{n+1}{2}} \cdot \widetilde{I}_{\Pi_v}(\dot{f}_v, \chi_v),$$

$$I_\Pi(f, \chi) = \frac{1}{\mathrm{vol}([Z_{G'}], d^*z)} \frac{\zeta^*_{G'}(1)}{\zeta^*_{H_1'}(1)\zeta^*_{H_2'}(1)} \widetilde{I}_\Pi(\dot{f}, \chi) = \frac{1}{4} \frac{1}{\zeta^*_{H_1'}(1)\zeta^*_{H_2'}(1)} \widetilde{I}_\Pi(\dot{f}, \chi),$$

(3.2.5)

where the factor $\mathrm{vol}([Z_{G'}], d^*z) = 4L(1, \eta)^2 = 4\zeta^*_{G'}(1)$ accounts for the fact that the Petersson product in [Zha14b] is defined via integration on $[G'^{ad}] = [\widetilde{G}'^{ad}]$ and not $[G']$.

3.2.5. *Factorization of the spherical character.* Define

$$\mathscr{L}(1/2, \Pi, \chi) := \frac{\Delta^\circ_G}{\Delta^\circ_H} \frac{L(1/2, \Pi \otimes \chi)}{L(1, \Pi, \mathrm{As}^\star)},$$

which agrees with the definition made in the introduction as noted in § 2.2.1. We use the analogous notation relative to the constituents Π_v, χ_v for v a finite place of F_0 or $v = \infty$.

Proposition 3.2.2. *For all $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbf{A}))$, there is a factorization*

$$I_\Pi(f', \chi) = \frac{1}{4} \frac{\mathscr{L}(1/2, \Pi, \chi)}{\Delta^\circ_H \cdot \varepsilon(\tfrac{1}{2}, \chi^2)^{\binom{n+1}{2}}} \prod_v I_{\Pi,v}(f'_v, \chi_v)$$

Proof. Using (3.2.5), the factorization in [Zha14b, Proposition 3.6] is equivalent to

$$I_\Pi(f', \chi) = C\frac{1}{4} \frac{\Delta^\circ_{G'}}{\Delta^\circ_{H_1'}\Delta^\circ_{H_2'}} \cdot \varepsilon(\tfrac{1}{2}, \chi^2)^{-\binom{n+1}{2}} \cdot \frac{L(1/2, \Pi \otimes \chi)}{L(1, \Pi, \mathrm{As}^\star)} \cdot \prod_v I_{\Pi_v}(f'_v, \chi_v).$$

By the definition of C in (3.1.1) and of $\mathscr{L}$, this is equivalent to the asserted formula. (Equivalently, the factorization follows from (3.2.4).) $\square$

We will state the spectral expansion $I = \sum_\Pi I_\Pi$ as part of Proposition 3.3.3 below.

3.3. **Geometric expansion.** The distribution I also admits an expansion as a sum of orbital integrals, which we review.

3.3.1. *Orbit varieties.* Let

$$B' := H_1' \backslash G' / H_2'$$

be the categorical quotient, which is an affine variety over F_0, cf. [Zha14a]. We have isomorphisms $B' \cong G'_n \backslash G'_{n+1} / G'_{n+1,0}$ given by $[(g_{n+1}, g_n)] \mapsto [g_n^{-1}g_{n+1}]$, and

$$S := \{\gamma \in G'_{n+1} \mid \gamma\overline{\gamma} = 1_n\} \cong G'_{n+1}/G'_{n+1,0}$$

(3.3.1)

given by $g \mapsto gg^{c,-1}$; the latter map also gives a bijection on F'-points $G'_{n+1}(F')/G'_{n+1,0}(F') \to S(F')$ for any field $F' \supset F_0$.

Representing a point of B' by a matrix in S given in $(n, 1) \times (n, 1)$ block decomposition, the invariant map

$$\mathrm{inv} \colon B' \longrightarrow \mathrm{Res}_{F/F_0} \mathbb{A}^{2n+1}$$

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \longmapsto ((\mathrm{Tr}(\wedge^i A))_{i=1}^n, (cA^{j-1}b)_{j=1}^n, d)$$

(3.3.2)

gives an embedding into affine space. We define the Zariski open and dense subvariety of *regular semisimple orbits*

$$B'_{rs} := B' - V(D_B) \subset B',$$

where D_B is the invariant function given, on the orbit of $s \in S$, by

$$D_B([s]) := \det((e_{n+1}^t s^{i+j} e_{n+1})_{1 \le i,j \le n}), \tag{3.3.3}$$

where $e_{n+1} = (0, \ldots, 0, 1)^t \in F^{n+1}$. The preimage of B'_{rs} in G', respectively S, is denoted by G'_{rs}, respectively S_{rs}; their elements are also said to be regular semisimple. For any field $F' \supset F_0$, the F'-points of G'_{rs}, respectively S_{rs}, are those with trivial stabilizer and closed orbit for the action of $H'_1(F') \times H'_2(F')$, respectively the adjoint action of $G'_n(F')$.

3.3.2. *Local orbital integrals.* Let v a place of F_0, and let $\gamma' \in G'_{rs,v}$. Then for all $\chi_v \colon F_{0,v}^\times \to \mathbf{C}^\times$, we define the orbital integral

$$I_{\gamma'}(f'_v, \chi_v) := \int_{H'_{1,v}} \int_{H'_{2,v}} f'_v(h_{1,v}^{-1} \gamma' h_{2,v}) \chi(h_{1,v}) \eta(h_{2,v}) \frac{d^\natural h_{1,v} d^\natural h_{2,v}}{d^\natural g_v}, \tag{3.3.4}$$

where we recall that $f'_v/d^\natural g_v$ is a function.

For $h_1 \in H'_{1,v}$, $h'_2 \in H'_{2,v}$, we have $I_{h_1 \gamma' h_2} = \chi(h_1)\eta(h_2)I_{\gamma'}$. We then add a renormalization factor to the orbital integral so that it only depends on the orbit $\gamma = [\gamma'] \in B'_{rs,v}$ of γ'. Let $\eta' \colon F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ be a character such that $\eta'_{|\mathbf{A}^\times} = \eta$. For each place v of F_0 and $\gamma' = (\gamma'_n, \gamma'_{n+1}) \in G'_{v,rs}$, denote

$$\gamma_* := \gamma'_n \gamma'^{-1}_{n+1}, \qquad s = \gamma_* \gamma_*^{c,-1} \in S_v = S(F_{0,v}). \tag{3.3.5}$$

Then we define

$$\kappa_v(\gamma', \chi_v) := \chi_v^{-1}(\gamma'_n)\eta'\left(\det(\gamma_*)^\epsilon \det s^{-(n+\epsilon)/2} \det(e_{n+1}^t, e_{n+1}^t s, \ldots, e_{n+1}^t s^n)\right) \tag{3.3.6}$$

where $\epsilon := 0$ if n is even, $\epsilon := 1$ if n is odd.

Up to the dependence on χ_v, this is the transfer factor Ω_v of [Zha14b, (4.12)-(4.13)] (cf. § 3.5.3 below) – that is,

$$\kappa_v(\gamma', \mathbf{1}) = \Omega_v(\gamma_v); \tag{3.3.7}$$

it similarly satisfies $\kappa_v(h_1 \gamma' h_2, \chi_v) = \chi_v^{-1}(h_1)\eta_v(h_2)\kappa_v(\gamma', \chi_v)$, and

$$\prod_v \kappa_v(\gamma', \chi_v) = 1 \tag{3.3.8}$$

for all $\gamma' \in G'_{rs}(F_0)$. We also record the following rationality property.

Lemma 3.3.1. *Let $\gamma' \in G'_{v,rs}$. Then $\kappa_v(\gamma', \mathbf{1})$ is a square root of $\eta_v(-1)^{-\binom{n+1}{2}}$.*

Proof. With notation as above, write $n = 2m + \epsilon$ and let

$$a := \det(e_{n+1}^t, \ldots, e_{n+1}^t s^n) \det s^{-(m+\epsilon)} \det(\gamma_*)^\epsilon = \det(e_{n+1}^t s^{-m-\epsilon}, \ldots, e_{n+1}^t s^m) \det \gamma_*^\epsilon,$$

which satisfies $\kappa_v(\gamma', \mathbf{1}) = \eta'_v(a)$. Using $s^c = s^{-1}$ and $\gamma_*^c = s^{-1}\gamma_*$, we find

$$a^c = (-1)^{\binom{n+1}{2}} \det(e_{n+1}^t s^{-m-\epsilon}, \ldots, e_{n+1}^t s^m) \det \gamma_* = (-1)^{\binom{n+1}{2}} a$$

where $(-1)^{\binom{n+1}{2}}$ is the sign of the longest permutation on $n+1$ elements. The assertion of the lemma follows. $\qquad\square$

We may now define, for $\gamma \in B'_{\mathrm{rs},v}$,

$$I_\gamma(f'_v, \chi_v) := \kappa_v(\gamma', \chi_v)\, I_{\gamma'}(f'_v, \chi_v)$$

for any $\gamma' \in G'_{\mathrm{rs},v}$ with image γ.

3.3.3. *Global orbital integrals.* For any $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)$, any $f := \otimes_v f_v \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$, and any character $\chi \in Y(\mathbf{C})$, we define

$$I_\gamma(f', \chi) := C \cdot \int_{\mathrm{H}'_1(\mathbf{A})} \int_{\mathrm{H}'_2(\mathbf{A})} f'(h_1^{-1}\gamma' h_2)\chi(h_1)\eta(h_2)\,\frac{dh_1 dh_2}{dg}. \tag{3.3.9}$$

In fact, the definition (but not the following factorisation) makes sense for any locally constant function $\chi \colon F_0^\times \backslash \mathbf{A}^\times \to \mathbf{C}$.

Lemma 3.3.2. *Let* $\gamma \in \mathrm{B}'_{\mathrm{rs}}(\mathbf{A})$, *let* $f' = \otimes_v f'_v \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$, *and let* $\chi \in Y(\mathbf{C})$. *We have*

$$I_\gamma(f', \chi) := C \frac{\Delta^\circ_{\mathrm{G}'}}{\Delta^\circ_{\mathrm{H}_1}\Delta^\circ_{\mathrm{H}'_2}} \prod_v I_\gamma(f'_v, \chi_v) = \frac{\Delta^\circ_{\mathrm{G}}}{(\Delta^\circ_{\mathrm{H}})^2} \prod_v I_\gamma(f'_v, \chi_v) \tag{3.3.10}$$

where for almost all finite places v *of* F_0 *such that* $f'_v \in \mathscr{H}(G'_v)$ *equals the unit Hecke measure* (2.3.1) *and* χ_v *is unramified, we have*

$$I_\gamma(f'_v, \chi_v) = 1. \tag{3.3.11}$$

Proof. We only need to prove the last assertion, as the first one follows by the definitions. Let v be such that $\gamma_v \in \mathrm{B}'_{\mathrm{rs}}(\mathscr{O}_{F_{0,v}})$ and v is unramified in F; then $\kappa_v(\gamma'_v, \chi_v) = 1$ for any lift $\gamma'_v \in G'_{\mathrm{rs},v}$ of γ_v. Let $s = s(\gamma') \in S_v$ be as in (3.3.5). Then by [Zha14b, (4.22)] we have

$$I_\gamma(f'^\circ_v, \chi_v) = \int_{G'_{n,0}} \mathbf{1}_{\mathrm{S}(\mathscr{O}_{F_{0,v}})}(h^{-1}sh)\,dh.$$

Let D_B be the discriminant function (3.3.3). Since $\gamma \in \mathrm{B}'_{\mathrm{rs}}(\mathscr{O}_{F_{0,v}})$, we have $s \in \mathrm{S}(\mathscr{O}_{F_{0,v}})$ and $D_B(\gamma_v) = \det((e_{n+1}^{\mathrm{t}} s^{i+j} e_{n+1})_{1\le i,j\le n}) \in \mathscr{O}_{F_{0,v}}^\times$. This means that

$$\Lambda := \sum_{i=0}^n \mathscr{O}_{F_v} e_{n+1}^{\mathrm{t}} s^i = (\mathscr{O}_{F_v}^{n+1})^{\mathrm{t}}, \qquad \Lambda' := \sum_{j=0}^n \mathscr{O}_{F_v} s^j e_{n+1} = \mathscr{O}_{F_v}^{n+1}.$$

Now for $h \in H'_{1,v}$ we have $h^{-1}sh \in \mathrm{S}(\mathscr{O}_{F_{0,v}})$ if and only if we also have $\Lambda h = (\mathscr{O}_{F_v}^{n+1})^{\mathrm{t}} = \Lambda$ and $h^{-1}\Lambda' = \mathscr{O}_{F_v}^{n+1} = \Lambda'$. In other words, both h and h^{-1} belong to $M_n(\mathscr{O}_F)$. We conclude that

$$I_\gamma(f'^\circ_v, \chi_v) = \mathrm{vol}(\mathrm{GL}_n(\mathscr{O}_{F_{0,v}}), dh) = 1.$$

$\qquad\square$

3.3.4. *Comparison with the normalization of* [Zha14b]. In [Zha14b, §4], one considers the distribution on Hecke functions (and not measures) on $\widetilde{G}'_{\mathrm{rs},v}$, defined by

$$\widetilde{I}_{\gamma'}(\dot{f}'_v, \chi_v) := \int_{H_{1,v}} \int_{H_{2,v}} \dot{f}'_v(h_{1,v}^{-1}\gamma' h_{2,v})\chi(h_{1,v})\eta(h_{2,v})\,\frac{d^* h_{1,v} d^* h_{2,v}}{d^* g}$$

(and denoted there by $O(\gamma, f')$), and the global analogue

$$\widetilde{I}_{\gamma'}(\dot{f}', \chi) := \prod_v \widetilde{I}_{\gamma'}(\dot{f}'_v, \chi_v).$$

Let $\mathrm{p} \colon \widetilde{G}'_v \to G'_v$ be the projection, and let

$$\mathrm{p}_* \colon \mathcal{S}(\widetilde{G}'_v) \longrightarrow \mathcal{S}(G'_v)$$

$$\dot{f}' \longmapsto \left(g = [\widetilde{g}] \longmapsto \int_{F_{0,v}^{\times, 2}} \dot{f}'(z\widetilde{g}) \, d^*z \right). \tag{3.3.12}$$

Suppose that $f' = \otimes_v f'_v \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$ and $\dot{f}' = \otimes_v \dot{f}'_v \in \mathcal{S}(\widetilde{\mathrm{G}}'(\mathbf{A}))$ are related by

$$f'_v = \mathrm{p}_*(\dot{f}'_v) \, d^*g_v. \tag{3.3.13}$$

Then

$$I_\gamma(f'_v, \chi_v) = \kappa_v(\gamma', \chi_v) \frac{\Delta^{\circ}_{\mathrm{H}_1'^{\mathrm{ad}}, v} \Delta^{\circ}_{\mathrm{H}_2'^{\mathrm{ad}}, v}}{\Delta^{\circ}_{\mathrm{G}'^{\mathrm{ad}}, v}} \widetilde{I}_{\gamma'}(\dot{f}'_v, \chi_v),$$

$$I_\gamma(f', \chi) = C \frac{\zeta_{\mathrm{G}'}^*(1)}{\zeta_{\mathrm{H}_1'}^*(1) \zeta_{\mathrm{H}_2'}^*(1)} \prod_v \widetilde{I}_{\gamma'}(\dot{f}'_v, \chi). \tag{3.3.14}$$

We will also denote by

$$\mathrm{p}_* \colon \mathscr{H}(\widetilde{G}_v) \longrightarrow \mathscr{H}(G_v) \tag{3.3.15}$$

the pushforward map of Hecke measures.

3.3.5. *Relative-trace formula for I.* We describe the spectral and geometric expansions of I. For S a finite set of places of F_0, we say that $f'^S \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^S))$ is *regular-semisimply supported* (resp. *regularly supported*) if it is in the span of those $\otimes_{v \notin S} f'_v$ such that for some place v, f'_v is supported on $G'_{\mathrm{rs}, v}$ (resp. on the regular locus of G'_v).

Proposition 3.3.3. *Let $f' \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$ be regular-semisimply supported and quasicuspidal. Then for every character $\chi \in Y(\mathbf{C})$,*

$$\sum_\Pi I_\Pi(f', \chi) = I(f', \chi) = \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I_\gamma(f', \chi). \tag{3.3.16}$$

where both sums are absolutely convergent, the first one running over the cuspidal hermitian automorphic representations of $\mathrm{G}'(\mathbf{A})$ that are $\mathrm{H}'_1(\mathbf{A})$-distinguished. If $f' \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$ is regularly supported and quasicuspidal, then on the right hand side one needs to include the orbital integrals $I_\gamma(f', \chi)$ for all $\gamma \in \mathrm{B}'(F_0)$ defined by Chaudouard and Zydor [CZ21] (for $\chi = 1$, but their definition is easily generalized to finite order characters χ).

Proof. The proof of [BP21, Proposition A.2.1] applies. We briefly recall the formal steps for the geometric expansion. Defining the kernel measure

$$K_{f'}(x, y) := \sum_{\gamma \in \mathrm{G}'(F_0)} f'(x^{-1} \gamma y),$$

we have

$$I(f', \chi) = C \int_{\mathrm{H}_1(F_0) \backslash \mathrm{H}_1(\mathbf{A})} \int_{\mathrm{H}_2(F_0) \backslash \mathrm{H}_2(\mathbf{A})} K_{f'}(h_1, h_2) \chi(h_1) \eta(h_2) \frac{dh_1 dh_2}{dg}.$$

Then one can switch the integrations and the sum. □

Lemma 3.3.4. *Let* $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbf{A}))$ *be regularly supported at a split place* v_0. *Then, for all regular non-semisimple* $\gamma \in B'(F_0)$, *the orbital integral* $I_\gamma(f', \mathbf{1})$ *vanishes.*

Proof. By decent it suffices to consider the (most degenerate) case: γ is a regular element in the fiber $G'_{\mathrm{inv}=\mathrm{inv}(1)}$ under the invariant map (3.3.2) of the image of 1. The orbital integral of Chaudouard and Zydor

$$f_{v_0} \in \mathscr{H}_{v_0} \longmapsto I_\gamma(f_{v_0} \otimes (\otimes_{v \neq v_0} f^v))$$

defines an invariant distribution on $G'_{v_0}(F_{0,v_0})$, with support on $G'_{\mathrm{inv}=\mathrm{inv}(1)}(F_{0,v_0})$. We would like to show that its restriction to the regular locus $G'_{\mathrm{inv}=\mathrm{inv}(1)}(F_{0,v_0})$ vanishes. There are two regular orbits and each of them is an open subset of $G'_{\mathrm{inv}=\mathrm{inv}(1)}(F_{0,v_0})$. There is a unique (up to scalar) invariant distribution on each of the regular orbit, namely the one induced by the Haar measure and an isomorphism $(H'_1 \times H'_2)(F_{0,v_0})$ with a regular orbit. However this distribution does not extend to the entire space $G'_{\mathrm{inv}=\mathrm{inv}(1)}(F_{0,v_0})$. This can be seen by the divergence of the integral (for example, an explicit formula may be obtained from [Zha14b, §6.3] by replacing the character η by the trivial one). The proof is complete. □

3.4. Relative traces for unitary groups.

We review the Jacquet–Rallis RTF for unitary groups.

3.4.1. *Orbit spaces.* Let v be a finite place of F_0 or $v = \infty$, and recall from § 2.1.3 the set $\mathscr{V}_v$ of pairs of hermitian spaces. For $V \in \mathscr{V}_v$, let

$$B_{v,V} := H_v^V \backslash G_v^V / H_v^V, \tag{3.4.1}$$

which is isomorphic to the quotient of $U(V_{n+1})$ by the adjoint action of $U(V_n)$ via the map $[(g_n, g_{n+1})] \mapsto [g_n^{-1} g_{n+1}]$. Differently from the linear case, $B_{v,V}$ is a strict subset (open for the v-adic topology if v is non-archimedean) of the set of $F_{0,v}$-points of $B_v^V := H_v^V \backslash G_v^V / H_v^V$. We say that $g \in G_v^V$ is regular semisimple (for the $H_v^V \times H_v^V$-action) if its orbit is closed and its stabilizer is trivial. We denote by $G_{\mathrm{rs},v}^V \subset G_v^V$ the (Zariski-open) subset of regular semisimple elements and by $B_{\mathrm{rs},v}^V$ its image in B_v^V. When $v = \infty$ is archimedean and $V_\infty = V_\infty^\circ$ is the positive definite pair, we denote $B_{(\mathrm{rs}),\infty}^\circ := B_{(\mathrm{rs}),\infty,V_\infty^\circ}$.

Consider now the global case, and let $V \in \mathscr{V}$. We similarly define $G_{\mathrm{rs}}^V \subset G^V$ to be the subgroup-scheme of those g with closed orbit and trivial stabilizers for the $H^V \times H^V$-action. For uniformity of notation, we denote by

$$B_{\mathrm{rs}}(F_0)_V \subset B_{\mathrm{rs}}^V(F_0) \tag{3.4.2}$$

the image of $G_{\mathrm{rs}}^V(F_0)$.

3.4.2. *Local distributions.* Let $\delta \in B_{\mathrm{rs},v,V}$ and let $\delta' \in G_{\mathrm{rs},v}^V$ be a preimage of δ. We define a local orbital-integral distribution $J_{\delta,v} = J_{\delta,v}^V$ on the Hecke algebra of $G_v = G_v^V$ by

$$J_{\delta,v}(f_v) := \int_{H_v} \int_{H_v} f_v(x^{-1}\delta'y) \frac{d^\natural x \, d^\natural y}{d^\natural g}. \tag{3.4.3}$$

For π_v a representation of $G_v = G_v^V$, we define a spherical character $J_{\pi_v} = J_{\pi_v}^V$ on $\mathscr{H}(G_v)$ by

$$J_{\pi_v}(f_v) := \mathscr{L}(1/2, \mathrm{BC}(\pi_v))^{-1} \int_{H_v} \mathrm{Tr}_{\pi_v}(\pi_v(h)\pi_v(f)) \, d^\natural h. \tag{3.4.4}$$

By our choices of measures, for all finite v, if f_v is L-valued (for some subfield $L \subset \mathbf{C}$) then so are $J_{\delta,v}(f_v)$, $J_{\pi_v}(f_v)$.

3.4.3. *Comparison with the normalization of* [Zha14b]. Let $f_v \in \mathscr{H}(G_v)$ and $\dot{f}_v \in \mathcal{S}(G_v)$ be related by

$$f_v = \dot{f}_v d^* g. \tag{3.4.5}$$

(1) Let $\widetilde{J}_{\pi_v}$ be the spherical character on $\mathcal{S}(G_v)$ defined in [Zha14b, (1.8)] (using the measure $d^* h_v$ on H_v as in § 4 *ibid.*), and denoted there by $J_{\pi_v}^\natural$. Then

$$J_{\pi_v}(f_v) = D_v^{1/2} \Delta_{\mathrm{H}_v^{\mathrm{ad}}}^\circ \widetilde{J}_{\pi_v}(\dot{f}_v), \tag{3.4.6}$$

(2) Let $\widetilde{J}_\delta$ be the orbital integral distribution on $\mathcal{S}(G_v)$ defined in [Zha14b, (4.2)], and denoted there by $O(\delta, \cdot)$. Then

$$J_\delta(f_v) = \frac{(\Delta_{\mathrm{H}^{\mathrm{ad}},v}^\circ)^2}{\Delta_{\mathrm{G}^{\mathrm{ad}},v}^\circ} \widetilde{J}_\delta(\dot{f}_v). \tag{3.4.7}$$

3.4.4. *Global relative-trace formula.* Let now $V \in \mathscr{V}$ be coherent, and let $\mathrm{G} = \mathrm{G}^V$. Let $\vartheta \colon \mathscr{A}_{\mathrm{cusp}}(\mathrm{G}) \otimes \mathscr{A}_{\mathrm{cusp}}(\mathrm{G}) \to \mathbf{C}$ be the Petersson inner product (with respect to the Tamagawa measure on $[\mathrm{G}]$), and consider the H-period

$$P = P^V \colon \mathscr{A}_{\mathrm{cusp}}(\mathrm{G}) \longrightarrow \mathbf{C}$$

$$\phi \longmapsto \int_{[\mathrm{H}]} \phi(h) \, dh$$

We define the following distributions on (subspaces of) $\mathscr{H}(\mathrm{G}(\mathbf{A}))$:

− let $\mathscr{H}(\mathrm{G}(\mathbf{A}))_{\mathrm{qc}} \subset \mathscr{H}(\mathrm{G}(\mathbf{A}))$ be the quasicuspidal subspace (defined as in §3.1.2). For $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}))_{\mathrm{qc}}$, we define

$$J(f) := \mathrm{Tr}_\vartheta^{P \otimes P}(R(f));$$

− let π be a cuspidal automorphic representation of $\mathrm{G}(\mathbf{A})$. For $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}))$, we define

$$J_\pi(f) := \mathrm{Tr}_{\vartheta_\pi}^{P_\pi \otimes P_\pi}(\pi(f));$$

− let $\delta \in \mathrm{B}_{\mathrm{rs}}(F_0)$. For $f = \otimes_v f_v \in \mathscr{H}(\mathrm{G}(\mathbf{A}))$, we define

$$J_\delta(f) := \frac{\Delta_{\mathrm{G}}^\circ}{(\Delta_{\mathrm{H}}^\circ)^2} \prod_v J_{\delta,v}(f_v) = \prod_{v \nmid \infty} J_{\delta,v}(f_v) \cdot J_{\delta,\infty}^\circ(f_\infty). \tag{3.4.8}$$

Analogously to Proposition 3.3.3, we have the spectral and geometric expansions ([BP21, Proposition A.2.1])

$$\sum_\pi J_\pi(f) = J(f) = \sum_{\delta \in \mathrm{B}_{\mathrm{rs}}(F_0)} J_\delta(f),$$

valid whenever $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}))$ is quasicuspidal with regular support, where the second sum runs over cuspidal representations of $\mathrm{G}(\mathbf{A})$.

However, unlike the factorization

$$I_\Pi(f',\chi) = \frac{1}{4}\frac{\mathscr{L}(1/2,\Pi,\chi)}{\Delta_{\mathrm{H}}^\circ \cdot \varepsilon(\frac{1}{2},\chi^2)^{\binom{n+1}{2}}} \prod_v I_{\Pi,v}(f'_v,\chi_v)$$

of Proposition 3.2.2, the analogous factorization

$$J_\pi = \frac{1}{4\Delta_{\mathrm{H}}^\circ}\mathscr{L}(1/2,\Pi,\mathbf{1}) \prod_v J_{\pi_v}$$

for a stable cuspidal tempered representation π of $\mathrm{G}(\mathbf{A})$ is highly nontrivial, and equivalent to the Ichino–Ikeda conjecture for unitary groups [Zha14b, Conjecture 1.1], whose proof is completed in [BPLZZ21]. The proof, reviewed in § **??** below, goes through a comparison of local orbital integrals $I_{\gamma,v}$ and $J_{\delta,v}$ and of local spherical characters I_{Π_v} and I_π. We first review the main results on the local comparison, which are equally important in the arithmetic setting.

3.5. **Comparison of the local distributions.**

3.5.1. *Spectral matching.* Let v be a place of F_0. For $V \in \mathscr{V}_v$ and $\pi_v^V \in \mathrm{Temp}(G_v^V)$, define a spectral transfer factor

$$\kappa(\pi_v^V) = \kappa(\pi_v^V,\psi_v,\tau) := \eta'_v((-1)^{n+1}\tau)^{\binom{n+1}{2}} \cdot \eta_v(\mathrm{disc}(V_n))^n \cdot \omega_{\pi_v^V}(-1); \qquad (3.5.1)$$

this is the same as in [Zha14b, Conjecture 4.4] with the correction of [BP21a, Remark 5.52], up to a factor $\varepsilon(\frac{1}{2},\eta_v,\psi)^{\binom{n+1}{2}}$.[10]

Let S be a finite set of places of F_0. Denote $\mathscr{V}_S := \prod_{v\in S}\mathscr{V}_v$; for $V = (V_v)_{v\in S} \in \mathscr{V}_S$, denote $\mathrm{Temp}((H_S^V)\backslash G_S^V) = \prod_{v\in S}\mathrm{Temp}((H_v^{V_v})\backslash G_v^{V_v})$; for $\pi_S^V \in \mathrm{Temp}(G_S^V)$, set $\kappa(\pi_S^V) := \prod_{v\in S}\kappa(\pi_v)$ and $J_{\pi_S} := \otimes_{v\in S}J_{\pi_v}$. For $\Pi_S \in \mathrm{Temp}(G'_S) := \prod_{v\in S}\mathrm{Temp}(G'_v)$, let $I_{\Pi_S} := \otimes_{v\in S}I_{\Pi_v}$.

Definition 3.5.1. We say that Hecke measures $f'_S \in \mathscr{H}(G'_S)$ and $(f_S^V)_{V\in\mathscr{V}_S} \in \prod_{V\in\mathscr{V}_S}\mathscr{H}(G_S^V)$ *match spectrally* if for all $V \in \mathscr{V}_S$ and all $\pi_S^V \in \mathrm{Temp}(H_S^V\backslash G_S^V)$, we have

$$I_{\mathrm{BC}(\pi_S^V)}(f'_S,\mathbf{1}) = \kappa(\pi_S^V)J_{\pi_S}(f_S^V). \qquad (3.5.2)$$

3.5.2. *Geometric matching.* Let us first recall the matching of orbits for G' and G; for the details, see [Zha12, § 2.1]. Let $V \in \mathscr{V}_v$. Orbits $\gamma \in B'_{\mathrm{rs},v}$ and $\delta \in B_{\mathrm{rs},v,V}$ are said to *match* if a lift $\gamma' \in S_v \subset \mathrm{GL}_{n+1}(F_v)$ of γ and a lift $\delta' \in U(V_{n+1}) \subset \mathrm{GL}_{n+1}(F)$ of δ are conjugate for the adjoint action of $\mathrm{GL}_n(F)$. This notion is independent of the choices of the lifts and of the basis of V_{n+1} giving the embedding $U(V_{n+1}) \subset \mathrm{GL}_{n+1}(F)$. The matching relation defines a bijection (an isomorphism of $F_{0,v}$-manifolds if v is non-archimedean)

$$\underline{\delta}\colon B'_{\mathrm{rs},v} \cong \bigsqcup_{V\in\mathscr{V}_v} B_{\mathrm{rs},v,V}. \qquad (3.5.3)$$

If S is a finite set of places of F_0, by taking products we obtain a matching bijection

$$\underline{\delta}\colon B'_{\mathrm{rs},S} \cong \bigsqcup_{V\in\mathscr{V}_S} B_{\mathrm{rs},S,V}.$$

---

[10]To compare the last factor in (3.5.1) with [Zha14b], recall that $\omega_{\Pi_v}(z) = \omega_{\pi_v}(z/z^c)$, so that $\omega_{\pi_v}(-1) = \omega_{\Pi_v}(\tau)$. The absence of the factor $(\varepsilon(\frac{1}{2},\eta_v,\psi)^{\binom{n+1}{2}}$, which cancels out its presence in our local Flicker–Rallis period P_{2,Π_v}, is helpful in Lemma 4.1.1.

where $B'_{\mathrm{rs},S} := \prod_{v \in S} B'_{\mathrm{rs},v}$, $B_{\mathrm{rs},S,V} := \prod_{v \in S} B_{\mathrm{rs},v,V_v}$.

For the number field F_0 and for $V \in \mathscr{V}$, with the notation of (3.4.2) we have an analogous bijection

$$\underline{\delta} \colon \mathrm{B}'_{\mathrm{rs}}(F_0) \cong \bigsqcup_{V \in \mathscr{V}} \mathrm{B}_{\mathrm{rs}}(F_0)_V. \tag{3.5.4}$$

compatible with (3.5.3)

Definition 3.5.2. Let S be a finite set of places of F_0. We say that Hecke measures $f'_S \in \mathscr{H}(G'_v)$ and $(f^V_S)_V \in \prod_{V \in \mathscr{V}_S} \mathscr{H}(G^V_S)$ *match geometrically* if

$$I_{\gamma,S}(f'_S, \mathbf{1}_S) = J_{\delta,S}(f^V_S) \tag{3.5.5}$$

whenever $\gamma \in B'_{\mathrm{rs},S}$ and $\delta \in B_{\mathrm{rs},S,V}$ match.

3.5.3. *Comparison with the normalization of* [Zha14b]. Let v be a place of F_0. Suppose that f'_v is related to $\dot{f}'_v$ as in (3.3.13) and f_v is related to $\dot{f}_v$ as in (3.4.5). Let

$$c_v := \frac{\Delta^\circ_{\mathrm{H}'^{\mathrm{ad}}_1,v} \Delta^\circ_{\mathrm{H}'^{\mathrm{ad}}_2,v}}{\Delta^\circ_{\mathrm{G}'^{\mathrm{ad}},v}} \cdot \frac{\Delta^\circ_{\mathrm{G}^{\mathrm{ad}},v}}{(\Delta^\circ_{\mathrm{H}^{\mathrm{ad}},v})^2}. \tag{3.5.6}$$

(1) By (3.2.5), (3.4.6), the Hecke measures f'_v and (f^V_v) match spectrally if and only if $c_v \dot{f}'_v$ and $(\dot{f}^V_v)$ match spectrally in the sense ([Zha14b, Conjecture 4.4 and last equation on p. 566]) that

$$\widetilde{I}_{\Pi_v}(c_v \dot{f}'_v) = \kappa(\pi^V_v) \frac{\Delta_{\mathrm{G},v}}{\Delta_{\mathrm{H},v} \zeta_{\mathrm{H},v}(1)} \cdot \widetilde{J}_{\pi_v}(\dot{f}^V_v).$$

(2) By (3.3.7), (3.3.14), (3.4.7), the Hecke measures f'_v and (f^V_v) match geometrically if and only if $c_v \dot{f}'$ and $(\dot{f}^V_v)$ match geometrically in the sense of [Zha14b, (4.14)], namely

$$\kappa_v(\gamma', \mathbf{1})^{-1} \widetilde{I}_{\gamma'}(c_v \dot{f}', \mathbf{1}) = \widetilde{J}_\delta(\dot{f}_v)$$

for all matching pairs of orbits (γ, δ).

3.5.4. *Main results on the local comparisons.* Each of the following is a deep result.

Proposition 3.5.3 (Equivalence of spectral and geometric matching)**.** *Let S be a finite set of places of F_0. The pairs $f'_S \in \mathscr{H}(G'_S)$ and $(f^V_S) \in \prod_{V \in \mathscr{V}_S} \mathscr{H}(G^V_S)$ match spectrally if and only if they match geometrically.*

Proof. The proof of [BPLZZ21, Lemma 4.9], based on [BP21a, BP21], applies. (As noted in [BPLZZ21, Remark 4.10], in general this relies on [Mok15, KMSW].) Note that by § 3.5.3, the comparisons of matchings in *locc. citt.*, whose conventions are inherited from [Zha14b], are compatible with ours. □

From now on we will simply say that f'_S and (f^V_S) *match* when they match spectrally and geometrically. For a fixed $V \in \mathscr{V}_S$, we say that f'_S *purely matches* f^V_S if it matches $(f^V_S, (0^{V'})_{V' \neq V})$.

Proposition 3.5.4 (Fundamental Lemma [Yun11, BP21b])**.** *Let v be a finite place of F_0 that is unramified in F, let $V \in \mathscr{V}_v$ be the unramified pair of hermitian spaces, and recall the unit Hecke measures from (2.3.1).*

The unit Hecke measure f'°_v on G'_v purely matches the unit measure f°_v on G^V_v.

Proposition 3.5.5 (Existence of matching [Zha14a, Theorem 2.6]). *Let v be a finite place of F_0. For every $f' \in \mathscr{H}(G'_v)$, a matching $(f^V) \in \prod_{V \in \mathscr{V}_v}$ exists; conversely, for every $(f^V) \in \prod_{V \in \mathscr{V}_v}$, a matching $f' \in \mathscr{H}(G'_v)$ exists.*

A matching result for archimedean places will be proved in § 4.3.2. We will also need to note the following (relatively easy) fact.

Lemma 3.5.6 ([Zha14a, Proposition 2.5]). *Let $v = w\overline{w}$ be a split place of F_v, and identify $G_v \cong G'_{n,0} \times G'_{n+1,0}$. Then $f'_v = \mathrm{p}_*(f'_w \otimes f'_{\overline{w}}) \in \mathscr{H}(G'_v)$ matches $f_v := f_w * f^\vee_{\overline{w}} \in \mathscr{H}(G_v)$.*

4. RATIONALITY

This section is dedicated to establishing the rationality of our L-values and a rational relative-trace formula. From now on, we assume that F_0 is totally real and F is CM.

4.0.1. *Notation.* For a locally compact and totally disconnected group G with a fixed Haar measure dg, from now on we denote by $\mathscr{H}(G)$ the sheaf of smooth compactly supported $\mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}}$-multiples of dg; we will write $\mathscr{H}(G, R) := \mathscr{H}(G)(R)$. (Thus the object denoted by $\mathscr{H}(G)$ in the previous section will henceforth be denoted by $\mathscr{H}(G, \mathbf{C})$).

4.1. **Archimedean theory.** We define some rational variant of the archimedean distributions of the previous section. Denote $G^\circ_\infty := G^{V^\circ_\infty}_\infty$, $H^\circ_\infty := H^{V^\circ_\infty}_\infty$, $B^\circ_\infty := B_{\infty, V^\circ_\infty}$.

4.1.1. *A product of transfer factors.* Let

$$\kappa(\mathbf{1}_\infty) := \prod_{v | \infty} \kappa(\mathbf{1}_v)$$

be the product of (3.5.1) for the trivial representation of the positive-definite group G°_∞.

Lemma 4.1.1. *For each $\gamma' \in \mathrm{G}'_{\mathrm{rs}}(F_{0,\infty})$, we have $\kappa_\infty(\gamma', \mathbf{1})\kappa(\mathbf{1}_\infty) \in \{\pm 1\}$.*

Proof. By Lemma 3.3.1, the first factor is a square root of $(-1)^{-\binom{n+1}{2}[F_0:\mathbf{Q}]}$; so are $\eta'_\infty(\tau)^{\binom{n+1}{2}}$ and, hence, the second factor. $\square$

4.1.2. *Distributions on $\mathscr{H}(G'_\infty, \mathbf{C})$.* For any tempered representation Π_∞ of G'_∞ and any $\gamma \in B'_{\mathrm{rs},\infty}$, we define

$$I^\circ_{\Pi_\infty}(f_\infty, \chi_\infty) := \frac{1}{\kappa(\mathbf{1}_\infty)\Delta^\circ_{\mathrm{H}}} \mathscr{L}(1/2, \Pi_\infty, \chi_\infty)\, I_{\pi_\infty}(f_\infty, \chi_\infty),$$

$$I^\circ_\gamma(f'_\infty, \chi_\infty) := \frac{\Delta^\circ_{\mathrm{G}}}{(\Delta^\circ_{\mathrm{H}})^2} I_\gamma(f'_\infty, \chi_\infty).$$

Then the factorizations of Proposition 3.2.2 and of (3.3.10) are equivalent to

$$\kappa(\mathbf{1}_\infty)^{-1} I_\Pi(f', \chi) = \frac{1}{4} \frac{\mathscr{L}^\infty(1/2, \Pi, \chi)}{\varepsilon(\frac{1}{2}, \chi^2)^{\binom{n+1}{2}}} \prod_{v \nmid \infty} I_{\Pi_v}(f'_v, \chi_v) \cdot I^\circ_{\Pi_\infty}(f_\infty, \chi_\infty),$$

$$I_\gamma(f', \chi) = \prod_{v \nmid \infty} I_\gamma(f'_v, \chi_v) \cdot I^\circ_\gamma(f'_\infty, \chi_\infty).$$

$$(4.1.1)$$

4.1.3. *Distributions and special elements in* $\mathscr{H}(G^\circ_\infty, \mathbf{C})$. For any $V \in \mathscr{V}_\infty = \prod_{v|\infty} \mathscr{V}_v$, every representation π^V_∞ of G^V_∞, and every $\delta \in B_{\mathrm{rs},\infty,V}$, we define variants of $J_{\pi^V_\infty}$ and $J_{\delta,\infty}$ by

$$J^\circ_{\pi_\infty}(f_\infty) := \int_{H^V_\infty} \mathrm{Tr}_{\pi_\infty}(\pi_\infty(h)\pi_v(f_\infty))\, dh = \frac{1}{\Delta^\circ_{\mathrm{H}}} \mathscr{L}(1/2, \mathrm{BC}(\pi_\infty)) \cdot J_{\pi_\infty}(f_\infty),$$

$$J^\circ_{\delta,\infty}(f_\infty) := \int_{H^V_\infty} \int_{H^V_\infty} f_v(x^{-1}\delta'y)\, \frac{dxdy}{dg} = \frac{\Delta^\circ_{\mathrm{G}}}{(\Delta^\circ_{\mathrm{H}})^2} J_{\delta,\infty}(f_\infty).$$

Then the matching relations (3.5.2) and, respectively, (3.5.5) for $S = \{v|\infty\}$ are equivalent to

$$I^\circ_{\mathrm{BC}(\pi^V_\infty)}(f'_\infty) = \frac{\kappa(\pi^V_\infty)}{\kappa(\mathbf{1}_\infty)} J^\circ_{\pi^V_\infty}(f_\infty) \qquad\qquad (4.1.2)$$

$$I^\circ_{\gamma,\infty}(f'_\infty, \mathbf{1}_\infty) = J^\circ_{\delta,\infty}(f^V_\infty).$$

Lemma 4.1.2. *Let*

$$f^\circ_\infty := \mathrm{vol}(G^\circ_\infty, dg)^{-1} dg \quad \in \mathscr{H}(G^\circ_\infty, \mathbf{Q}). \qquad\qquad (4.1.3)$$

Then:

(1) for all $\pi_\infty \in \mathrm{Temp}(G^\circ_\infty)$, *we have*

$$J^\circ_{\pi_\infty}(f^\circ_\infty) = \begin{cases} \mathrm{vol}(H^\circ_\infty) := \mathrm{vol}(H^\circ_\infty, dh_\infty) & \text{if } \pi_\infty \cong \mathbf{1} \\ 0 & \text{otherwise}; \end{cases} \qquad\qquad (4.1.4)$$

(2) for all $\delta \in \mathrm{G}_{\mathrm{rs}}(F_{0,\infty})$, *we have*

$$J^\circ_\delta(f^\circ_\infty) = \mathrm{vol}(B^\circ_\infty)^{-1} := \frac{\mathrm{vol}(H^\circ_\infty, dh_\infty)^2}{\mathrm{vol}(G^\circ_\infty, dg_\infty)}.$$

Moreover, both of the above values are rational.

Proof. The calculation is immediate. The rationality follows from Lemma 2.2.1. $\square$

4.1.4. *Gaussians.* Let $f^\circ_\infty = (4.1.3)$ be the standard Hecke measure on $G^\circ_\infty = G^{V_\circ}_\infty$. For each characteristic-zero field L, we put $\mathscr{H}(G^\circ_\infty, L)^\circ := Lf^\circ_\infty$.

For L a subfield of $\mathbf{C}$, we denote by

$$\mathscr{H}(G'_\infty, L)^{\circ,\sharp} \subset \mathscr{H}(G'_\infty, \mathbf{C})$$

the preimage of $Lf_\infty \subset \mathscr{H}(G^\circ_\infty, L)^\circ$ under pure matching. By Proposition 4.1.3 below, the pure matching map

$$\mathrm{tr} \colon \mathscr{H}(G'_\infty, L)^{\circ,\sharp} \longrightarrow \mathscr{H}(G^\circ_\infty, L)^\circ$$

is surjective. We put

$$\mathscr{H}(G'_\infty, L)^\circ := \mathscr{H}(G'_\infty, L)^{\circ,\sharp} / \mathrm{Ker}\, \mathrm{tr}$$

extend the definition to any characteristic-zero field L by $\mathscr{H}(G'_\infty, L)^\circ := \mathscr{H}(G'_\infty, \mathbf{Q})^\circ \otimes_\mathbf{Q} L$, and we extend the notion of matching by linearity. Elements of $\mathscr{H}(G'_\infty, L)^\circ$ are called L-rational *Gaussians.* If L can be embedded into $\mathbf{C}$, we also refer to an $f'_\infty \in \mathscr{H}(G'_\infty, L)^{\circ,\sharp}$ as a Gaussian; we say that f'_∞ is nontrivial if its image in $\mathscr{H}(G'_\infty, L)^\circ$ is nonzero.

If S is a finite set of non-archimedean places of F_0, we put

$$\mathscr{H}(G'_{S\infty}, L)^\circ := \mathscr{H}(G'_S, L) \otimes_L \mathscr{H}(G'_\infty, L)^\circ, \qquad \mathscr{H}(\mathrm{G}'(\mathbf{A}^S), L)^\circ := \mathscr{H}(\mathrm{G}'(\mathbf{A}^{S\infty}), L) \otimes_L \mathscr{H}(G'_\infty, L)^\circ,$$

and refer to the elements of those spaces as Gaussians too.

Proposition 4.1.3 (Existence of Gaussians)**.** *The space $\mathscr{H}(G'_\infty, \mathbf{Q})^\circ$ is nonzero.*

Proof. This follows from [BPLZZ21, Proposition 4.11].   □

4.2. **Rationality statements.** We state the main results of this section, whose proofs will be completed in § 4.5.

4.2.1. *Rationality of L-values.* The following is Theorem A from the introduction.

Theorem 4.2.1. *Let L be a number field and let Π be a trivial-weight hermitian cuspidal automorphic representation of $G'(\mathbf{A})$ defined over L. There is a function*

$$\mathscr{L}(M_\Pi, \cdot) \in \mathscr{O}(Y_L)$$

such that for each $\chi \in Y_L(\mathbf{C})$ with underlying embedding $\iota \colon L \hookrightarrow \mathbf{C}$,

$$\mathscr{L}(M_\Pi, \chi) = \frac{\mathscr{L}^\infty(1/2, \Pi^\iota, \chi)}{\varepsilon(\tfrac{1}{2}, \chi^2)^{\binom{n+1}{2}}}.$$

4.2.2. *Special Hecke algebras.* Let L be a field that is embeddable in $\mathbf{C}$ and let S be a finite set of finite places of F_0. We denote by

$$\mathscr{H}(G'(\mathbf{A}^S), L)^\circ_{K_S,(\mathrm{rs}),\mathrm{qc}} \subset \mathscr{H}(G'(\mathbf{A}^S), L)^\circ$$

the space of Gaussian measures f'^S (that are regularly supported and) such that for every $\iota \colon L \hookrightarrow \mathbf{C}$, $f'^{S,\iota} e_{K_S}$ is quasicuspidal.

If $\Pi \in \mathscr{C}(G')(L)$ and $\chi \in Y(L)$, we say that a Hecke measure $f' \in \mathscr{H}(G'(\mathbf{A}^S), L)^\circ$ is *adapted to* (Π, χ, K_S) if $(\otimes_{v \notin SS} I_{\Pi_v})(f'^S, \chi^S) \neq 0$ and for every $\iota \colon L \hookrightarrow \mathbf{C}$, the operator $R(f'^{S,\iota} e_{K_S})$ sends $\mathscr{A}(G')$ into (the image in $\mathscr{A}(G')$ of) Π. We denote by

$$\mathscr{H}(G'(\mathbf{A}^S), L)^\circ_{K_S,(\mathrm{rs}),\Pi,\chi}$$

the space of (regularly supported) Gaussians that are adapted to (Π, χ, K_S). When $\chi = \mathbf{1}$ we omit it from the notation.

4.2.3. *Rational relative-trace formula.* We introduce a variant of the distribution I and its expansions. From now on, we change the notation for the distributions $I_?$ of the previous section by appending a superscript '$\mathbf{C}$', thus writing $I^\mathbf{C}_?$ in place of $I_?$.

We introduce some further notation. For a finite place v of F_0 and an ideal $m \subset \mathscr{O}_{F_0,v}$, let $Y_v(m) = \operatorname{Spec} \mathbf{Q}[F^\times_{0,v}/(\mathscr{O}^\times_{F_0,v} \cap 1 + m\mathscr{O}_{F_0,v})]$, viewed as the space of characters of the group within square brackets. Let $Y_v := \varinjlim_r Y_v(v^r)$. For the sake of uniformity, we will denote $\mathscr{H}(G'_v, L)^\circ := \mathscr{H}(G'_v, L)$ if $v \nmid \infty$, and $Y_\infty := \operatorname{Spec} \mathbf{Q}$.

In the rest of the paper, unless otherwise noted all products '$\prod_v$' run over the union of the set of finite places v of F_0 and $\{v = \infty\}$. If $\mathscr{H}$ is a Hecke algebra over a field L and Y is an ind-scheme over L, an L-linear functional $D \colon \mathscr{H} \to \mathscr{O}(Y)$ will be called a *distribution*.

Proposition 4.2.2. *Let L be a field that can be embedded in $\mathbf{C}$. There exist:*

(1) (a) *for each finite place v of F_0 and each tempered irreducible admissible representation Π_v of G'_v over L, a distribution*

$$I_{\Pi_v} \colon \mathscr{H}(G'_v, L) \longrightarrow \mathscr{O}(Y_{v,L})$$

characterized by

$$I_{\Pi_v}(f'_v, \chi_v) = I^{\mathbf{C}}_{\iota\Pi_v}(\iota f'_v, \chi_v)$$

for all $\chi_v \in Y_{v,L}(\mathbf{C})$ with underlying embedding $\iota\colon L \hookrightarrow \mathbf{C}$;

   (b) *for each tempered irreducible admissible representation Π_∞ of G'_∞, a distribution*

$$I_{\Pi_\infty}(\cdot, \mathbf{1}) \colon \mathscr{H}(G'_\infty, L)^\circ \longrightarrow L$$

characterized by

$$I_{\Pi_\infty}(f'_\infty, \mathbf{1}) = I^{\circ,\mathbf{C}}_{\Pi_\infty}(f'_\infty, \mathbf{1}).$$

(2) *for each representation $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her}}$ over L as in Theorem 4.2.1, a distribution*

$$I_\Pi \colon \mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ \longrightarrow \mathscr{O}(Y_L)$$

defined on factorizable elements $f' = \otimes_{v\nmid\infty} f'_v \otimes f'_\infty$ by

$$I_\Pi(f', \chi) = \frac{1}{4}\mathscr{L}(\mathrm{M}_\Pi, \chi) \cdot \prod_v I_{\Pi_v}(f'_v, \chi_v). \tag{4.2.1}$$

(3) *for each finite place v of F_0 and for $v = \infty$, and each $\gamma \in B'_{\mathrm{rs},v}$, a distribution*

$$I_{\gamma,v} \colon \mathscr{H}(G'_v, L)^\circ \longrightarrow \mathscr{O}(Y_{v,L[\sqrt{-1}]})$$

characterized by

$$I_{\gamma,v}(f'_v, \chi_v) = \begin{cases} I^{\mathbf{C}}_{\gamma,v}(\iota f'_v, \chi_v) & \text{if } v \nmid \infty \\ I^{\circ,\mathbf{C}}_{\gamma,v}(\iota f'_v, \chi_v) & \text{if } v = \infty. \end{cases}$$

for each $\chi_v \in Y_{v,L[\sqrt{-1}]}(\mathbf{C})$ with underlying embedding $\iota\colon L[\sqrt{-1}] \hookrightarrow \mathbf{C}$.

(4) *for each $\gamma \in \mathrm{B}'_{\mathrm{rs}}(\mathbf{A})$, a distribution*

$$I_\gamma = \kappa(\mathbf{1}_\infty)^{-1} \prod_v I_{\gamma,v} \colon \mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ_{\mathrm{rs}} \longrightarrow \mathscr{O}(Y_L)$$

where the product is locally finite;

(5) *a distribution*

$$I \colon \mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ_{\mathrm{rs,qc}} \longrightarrow \mathscr{O}(Y_L)$$

admitting the spectral and geometric expansions

$$\sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her}}} I_\Pi = I = \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I_\gamma$$

where both sums are locally finite.

Remark 4.2.3. It should be possible to interpret the rational distribution I as the inner product of analogues of $P_{1,\chi}$, P_2 in the rational Betti homology (in complementary degrees) of the symmetric space of G'.

Remark 4.2.4. In [DZ], we plan to prove the analogue of (4) in the case of regular, non-semisimple orbits (cf. Proposition 3.3.3).

We prove the local parts of Proposition 4.2.2

Proof of Proposition 4.2.2 (1)-(3). We need to show the existence of various distributions.

Archimedean distributions. Suppose that f'_∞ is an L-rational Gaussian matching $f_\infty = cf^\circ_\infty \in \mathscr{H}(G^\circ_\infty, L)$. Then by Lemma 4.1.2 and (4.1.2), we may define

$$I_{\Pi_\infty}(f'_\infty, \mathbf{1}) := \begin{cases} c\,\mathrm{vol}(H^\circ_\infty) & \text{if } \Pi_\infty \cong \Pi^\circ_\infty \\ 0 & \text{otherwise,} \end{cases}$$

$$I_\gamma(f'_\infty) := \begin{cases} c\,\mathrm{vol}(B^\circ_\infty)^{-1} & \text{if } \gamma \in B'^\circ_{\mathrm{rs},\infty} \\ 0 & \text{otherwise,} \end{cases}$$

where $B'^\circ_{\mathrm{rs},\infty}$ is the image of $B^\circ_{\mathrm{rs},\infty}$ in $B'_{\mathrm{rs},\infty}$.

Orbital integrals. Suppose v is non-archimedean. We observe that for all f'_v of a given level and support and all χ_v of a given level, the value of the orbital integral

$$I^{\mathbf{C}}_\gamma(f'_v, \chi_v)$$

is a polynomial, with $\mathbf{Q}$-coefficients, in a fixed finite set of values of f'_v and χ_v. Together with Lemma 3.3.1, this implies part (3) of Proposition 4.2.2. Part (4) follows from part (3) and (3.3.11), together with (3.3.8) and Lemma 4.1.1 for the elimination of $\sqrt{-1}$ from the field of rationality.

Local spherical character. It suffices to show that $P_{1,\Pi_v,\chi_v}(W_v)$, $P_{2,\Pi_v}(W'_v)$ and $\vartheta_{\Pi_v}(W_v, W'_v)$ are polynomials, with L-coefficients, in the values of W_v and χ_v. The rationality in W_v is observed in Remark 3.2.1. Then we only need to consider the function $\chi_v \mapsto P_{1,\Pi_v,\chi_v}(W_v)$. Let Y'_v be the ind-finite scheme over L of smooth characters of $\mathscr{O}^\times_{F_{0,v}}$; then $\chi \mapsto \chi_{v|\mathscr{O}^\times_{F,0,v}}$ gives an exact sequence of ind-group-schemes $1 \to Y^\circ_v \to Y_v \to Y'_v \to 1$ where $Y^\circ_v \cong \mathbf{G}_{m,L}$ parametrizes unramified characters of $F^\times_{0,v}$. Thus locally we may reduce to proving the desired result when χ_v is restricted to Y°_v at the cost of replacing Π_v by (one of locally finitely many) ramified twists. In this case, that P_{1,Π_v} is a polynomial in Y°_v and the values of W_v is one of the main results of [JPSS83], whose proof considers unramified characters of the form $|\cdot|^s_v$ but goes through in our context. $\quad\square$

4.3. **Test Hecke measures.** We now give some results asserting the existence of suitable Hecke measures.

4.3.1. *Test measures at finite places.* Let v be a finite place of F_0 and let L be a field that can be embedded into $\mathbf{C}$. A character $\xi' = \xi'_1 \boxtimes \cdots \boxtimes \xi'_\nu \colon (F^\times_w)^\nu \to \mathbf{C}^\times$ is called *regular* if the characters ξ'_i are pairwise distinct. A *regular principal series* representation of G'_v is a representation $\Pi_v = \Pi_{n,v} \boxtimes \Pi_{n+1,v}$ such that for $\nu = n, n+1$, all places $w|v$, and any $\iota\colon L \hookrightarrow \mathbf{C}^\times$ the representation $\Pi_{\nu,w} := \Pi_{\nu,v|\mathrm{GL}_\nu(F_w)}$ is unitarily induced from a regular character of the diagonal torus.

Lemma 4.3.1. *Let Π_v be a hermitian (§ 2.4.1) tempered representation of G'_v over L, and let χ_v be a smooth character of $F^\times_{0,v}$ with values in some finitely generated extension L' of L. For a*

compact open $K_v \subset G'_v$, denote by

$$\mathscr{H}(G'_v, L)_{K_v, \Pi_v, \chi_v}$$

the set of those $f'_v \in \mathscr{H}(G'_v, L)$ that are right-K_v-invariant and satisfy $I_{\Pi_v}(f'_v, \chi_v) \neq 0$.

(1) There exists some K_v such that $\mathscr{H}(G'_v, L)_{K_v, \Pi_v, \chi_v} \neq \emptyset$.

(2) If Π_v and χ_v are unramified, then $f'^\circ_v \in \mathscr{H}(G'_v, L)_{K_v, \Pi_v, \chi_v}$.

(3) If v splits in F and Π_v is a regular principal series, there exist a deeper Iwahori subgroup $K_v \subset G'_v$ and a regularly supported $f' \in \mathscr{H}(G'_v, L)_{K_v, \Pi_v, \chi_v}$; if moreover Π_v is unramified, we can take K_v to be an Iwahori subgroup.

The proof of part (3) relies on some results from later sections. (In fact, see § 5.1.4 for a definition of Iwahori and deeper Iwahori subgroups.)

Proof. We omit all subscripts v.

(1) Let $K \subset G'$ be an open compact subgroup. The restriction of $I_\Pi(\cdot, \chi)$ to $\mathscr{H}(G')_K$ is the inner product, for the natural pairing, of the elements

$$P_{1, \Pi, \chi | \Pi^K} \circ \Pi(\cdot) \in \Pi^{K, \vee} \otimes_L L', \qquad P_{2 | \Pi^{\vee}, K} \in (\Pi^{\vee, K})^\vee \cong \Pi^K.$$

Now if K is sufficiently small, both $P_{1, \Pi, \chi | \Pi^K}$ and $P_{2 | \Pi^{\vee}, K}$ are nonzero – the former by the theory of [JPSS83], the latter because Π, hence $\Pi^\vee$, is hermitian. Since Π^K is irreducible as an $\mathscr{H}(G', L)_K$-module, there exists an $f'_{L'} \in \mathscr{H}(G', L')_K$ such that $P_{1, \Pi, \chi | \Pi^K} \circ \Pi(f'_{L'})$ and $P_{2 | \Pi^{\vee}, K}$ do not pair to zero. Fix an embedding $\iota \colon L' \hookrightarrow \mathbf{C}$. If $\iota(L)$ is not contained in $\mathbf{R}$, then it is dense in $\mathbf{C}$, and any $f' \in \mathscr{H}(G', L)_K$ that is sufficiently close to $f'_{L'}$ in the topology induced from $\mathbf{C}$ by ι will also have the desired nonvanishing property. If $\iota(L)$ is contained in $\mathbf{R}$, note that one of $\operatorname{Re} \iota f'_{L'}$, $\operatorname{Im} \iota f'_{L'}$ has the nonvanishing property, and then so does any sufficiently close $f' \in \mathscr{H}(G', L)_K$ (for the topology induced from $\mathbf{R}$ by ι).

(2) This follows from Remark 3.2.1.

(3) This follows from Proposition 5.2.12 and Corollary 7.1.3 below.

$\square$

4.3.2. *Test Gaussians.* For a pure tensor $f'_{S\infty} = f'_S f'_\infty \in \mathscr{H}(G'_{S\infty}, L)^\circ$, we define $f''_{S\infty} := \iota f'_S f''^\iota_\infty$, and extend this definition to all of $\mathscr{H}(G'_{S\infty}, L)^\circ$ by linearity.

Proposition 4.3.2. *Let Π be a trivial-weight hermitian cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a field L admitting embeddings into $\mathbf{C}$, let $K = \prod_{v \nmid \infty} K_v \subset \mathrm{G}'(\mathbf{A}^\infty)$ be an open compact subgroup such that $\Pi^K \neq 0$, and let P be a finite set of non-archimedean places of F_0 containing all those for which K_v is not maximal.*

There exist a finite set S of split non-archimedean places of F_0 disjoint from P, and Gaussians

$$(f''_{S\infty})_\iota \in \prod_{\iota \colon L \hookrightarrow \mathbf{C}} \mathscr{H}(G'_{S\infty}, \iota L)^{\circ, \sharp}_{K_S}, \qquad f'_{S\infty} \in \mathscr{H}(G'_{S\infty}, L)^\circ_{K_S}$$

satisfying that for every $\iota \colon L \hookrightarrow \mathbf{C}$:

(1) the image of $f''_{S\infty}$ in $\mathscr{H}(G'_{S\infty}, \iota L)^\circ_{K_S}$ equals $\iota f_{S\infty}$;;

(2) $I_{\Pi^\iota_{S\infty}}(f''^\iota_{S\infty}, \chi_{S\infty}) \neq 0$ for every unramified character $\chi_{S\infty} \colon F^\times_{0,S} F^\times_{0,\infty}/F^\times_{0,\infty} \to \mathbf{C}^\times$;

(3) $R(f''^\iota_{S\infty})$ maps $\mathscr{A}(\mathrm{G}')^K$ into $(\Pi^\iota)^K$. (In particular, for any $f'^{S\infty} \in \mathscr{H}(G', \iota L)_{K^S}$, the Hecke measure $f'^{S\infty} f''^\iota_{S\infty}$ is quasicuspidal.)

The proof will be given in § 4.4

Lemma 4.3.3. *Let Π be a representation in $\mathscr{C}$. Then there exist infinitely many places v of F_0 that are split in F such that Π_v is an unramified regular principal series.*

Proof. This follows from the similar observation about Π_ν made in the proof of [CH13, Proposition 3.2.5]. $\square$

Corollary 4.3.4. *Let Π be a trivial-weight hermitian cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a field L admitting embeddings into $\mathbf{C}$, and let $\chi \in Y_L$. There exists an L-rational Gaussian $f' \in \mathscr{H}(G'(\mathbf{A}), L)^{\circ,\sharp}$ that is regularly supported and adapted to (Π, χ) (in the sense of § 4.2.2).*

Proof. Let P' be the set of all finite places of F_0 at which Π or χ is ramified. Let v_0 be a finite place of F_0 that is split in F, and such that Π_{v_0} is a regular principal series. Let f'_{v_0} be as in Lemma 4.3.1 (3), and for $v \in P'$ let f'_v be as in Lemma 4.3.1 (1). Let $f'_{S\infty}$ be as given by Proposition 4.3.2 with $P = P' \cup \{v_0\}$, and any K that is maximal away from P and sufficiently small at the places in P. Then $f' = f'_P f'_{S\infty} \prod_{v \nmid PS\infty} f'^\circ_v$ is as desired. $\square$

4.4. Proof of Proposition 4.3.2.

We will refine the arguments of [BPLZZ21], of which the reader is invited to open a copy. Briefly, in order to construct the desired $f'_{S\infty}$ we will start from a Gaussian $f'_{1,S'\infty}$ constructed in a simple way as a pure tensor, and then correct $f'_{1,S'\infty}$ by acting on it by a carefully chosen *multiplier* of the Hecke algebra for $\mathrm{G}'(\mathbf{A})$.

The substance of this subsection was generously provided to us by Yifeng Liu. Of course, any defects in the following pages are to be attributed to the authors only.

4.4.1. *Archimedean multipliers annihilating non-strongly typical cuspidal data.* We momentarily consider the more general situation of [BPLZZ21, § 3.2]. Consider a connected reductive algebraic group G over a number field F_0. We freely adopt notation from [BPLZZ21, § 3], up to cosmetic modifications to adapt to our conventions (for instance, in *loc. cit.* the algebraic group is denoted by G rather than G). Take a unitary automorphic character $\omega \colon Z(\mathbf{A}) \to \mathbf{C}^\times$. We fix

- a subset P of primes of F_0 containing S_G and a P-character $\xi = (\xi_\infty, \xi^{\infty,P})$,

- a finite set S of primes of F satisfying $S_G \subseteq S \subseteq P$, and

- a subgroup $K \subseteq K^\infty_0$ of finite index of the form $K = K_S \times \prod_{v \notin S} K_{0,v}$.

The following definition is modified from [BPLZZ21, Definition 3.11].

Definition 4.4.1. Let $\mathrm{M} \subset \mathrm{G}$ be a standard Levi subgroup. We say that a $\sigma \in \mathfrak{C}(\mathrm{M}, \omega)^\heartsuit$ is *strongly ξ_∞-typical* if $\gamma_\mathrm{M}(\xi_{\sigma\infty}) \subseteq \gamma_\mathrm{M}(\xi_\infty)$. Denote by $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty!}$ the subset of $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit$ consisting of strongly ξ_∞-typical elements.

It is clear that set of strongly ξ_∞-typical elements, $\mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty!}$, is a subset of $\mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty}$, the set of ξ_∞-typical elements defined in *loc. cit.* The following lemma slightly strengthens [BPLZZ21, Lemma 3.14], whose notation we simplify by putting

$$\mathcal{M}_\infty := \mathcal{M}^{\sharp}_\theta(\mathfrak{h}'^*_{\mathbb{C}})^{\mathsf{W}}.$$

We fix an element μ^0_∞ and a finite set $\mathfrak{T}$ of $K^{\mathrm{G}}_{0,\infty}$-types as described after [BPLZZ21, Definition 3.11].

Lemma 4.4.2. *For every standard Levi subgroup* $\mathrm{M} \subset \mathrm{G}$*, there exists an element*

$$\mu^{\mathrm{M}}_\infty \in \mathcal{M}_\infty$$

satisfying:

− $\mu^{\mathrm{M}}_\infty(\xi_\infty) \neq 0$

− *for every open compact* $K_{\mathrm{M}} \subset \mathrm{M}(\mathbf{A}^\infty)$ *and every finite set* $\mathfrak{T}_{\mathrm{M}}$ *of* $K^{\mathrm{M}}_{0,\infty}$*-types satisfying the conditions following* [BPLZZ21, Definition 3.11] *with respect to* $(\mu^0_\infty, \mathfrak{T})$*, for every non-strongly typical*

$$\sigma \in \mathfrak{C}(\mathrm{M},\omega; K_{\mathrm{M}}, \mathfrak{T}_{\mathrm{M}})^{\heartsuit} - \mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty!}$$

and every $s \in \mathfrak{a}^*_{\mathrm{M},\mathbb{C}}$*, we have*

$$\mu^{\mathrm{M}}_\infty(\xi^{\mathrm{G}}_{\sigma_s,\infty}) = 0.$$

Here, $\xi^{\mathrm{G}}_{\sigma_s,\infty}$ *is the infinitesimal character of* $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{P_M}}(\sigma_{s,\infty})$*.*

Proof. By Definition 4.4.1, it is easy to see that for each element $\sigma \in \mathfrak{C}(\mathrm{M},\omega; K_{\mathrm{M}}, \mathfrak{T}_{\mathrm{M}})^{\heartsuit} - \mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty!}$, there exists a W-invariant polynomial function ν_σ on $\mathfrak{h}^*_{\mathbb{C}}$ satisfying $\nu_\sigma(\xi_\infty) \neq 0$ and $\nu_\sigma(\xi^{\mathrm{G}}_{\sigma_s,\infty}) = 0$ for every $s \in \mathfrak{a}^*_{\mathrm{M},\mathbb{C}}$. By [BPLZZ21, Lemma 3.14], we have an element $\nu^{\mathrm{M}}_\infty \in \mathcal{M}^{\sharp}_\theta(\mathfrak{h}^*_{\mathbb{C}})^{\mathsf{W}}$ satisfying the similar property but with $\mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty!}$ replaced by $\mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty}$. Now by [BPLZZ21, Lemma 3.13], the set $\mathfrak{C}(\mathrm{M},\omega; K_{\mathrm{M}}, \mathfrak{T}_{\mathrm{M}})^{\heartsuit} \cap (\mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty} - \mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty!})$ is finite. Thus, we may take

$$\mu^{\mathrm{M}}_\infty := \nu^{\mathrm{M}}_\infty \cdot \prod_{\sigma \in \mathfrak{C}(\mathrm{M},\omega; K_{\mathrm{M}}, \mathfrak{T}_{\mathrm{M}})^{\heartsuit} \cap (\mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty} - \mathfrak{C}(\mathrm{M},\omega)^{\heartsuit}_{\xi_\infty!})} \nu_\sigma.$$

$\square$

The following is a direct analogue of [BPLZZ21, Proposition 3.15] in terms of Lemma 4.4.2.

Proposition 4.4.3. *There exists* $\mu_\infty \in \mathcal{M}_\infty$ *such that*

− $\mu_\infty(\xi_\infty) = 1$*;*

− *for every cuspidal datum* (M,σ) *for* G' *that does not belong to* $\mathfrak{D}(\mathrm{G},\omega,K,\mathfrak{T})^{\heartsuit}_{\xi_\infty!}$ *and for every* $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}),\mathbf{C})_K$*, the endomorphism* $R(\mu_\infty \star f)$ *of* $L^2(\mathrm{G}(F_0)\backslash\mathrm{G}(\mathbf{A})/K,\omega)$ *annihilates the subspace* $L^2_{(\mathrm{M},\sigma)}(\mathrm{G}(F_0)\backslash\mathrm{G}(\mathbf{A})/K,\omega)$*.*

4.4.2. *Multipliers annihilating strongly typical cuspidal data for a proper Levi subgroup.* We now specialize back to the setup of Proposition 4.3.2. We denote by ξ°_∞ the infinitesimal character of Π°_∞. We still freely use terminology and notation from [BPLZZ21, § 3] where not in conflict with ours.

Lemma 4.4.4. *Let* $(\mathrm{M}, \sigma) \in \mathfrak{C}(\mathrm{M}, 1)$ *satisfy*

$$\xi_{\sigma,\infty}^{\mathrm{G}'} = \xi_\infty^\circ.$$

We have:

(1) σ *is an exterior product of regular algebraic representations;*

(2) σ^∞ *is defined over a number field* $\mathbf{Q}(\sigma) \subset \mathbf{C}$, *and for each* $\tau \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$, *the representation* $\sigma^\infty \otimes_{\mathbf{C},\tau} \sigma$ *is the finite part of a cuspidal datum* $^\tau\sigma$;

(3) $\xi_{^\tau\sigma,\infty}^{\mathrm{G}'} = \xi_\infty^\circ$ *as well.*

Proof. All references in this proof point to [Clo90] (where the group considered is GL_N; the modifications needed to treat G′ are trivial). Part (1) follows from Lemme 3.9 (ii) and the regularity of ξ_∞°. Part (2) follows from Théorème 3.13. Since ξ_∞° is invariant under the action of $\mathrm{Aut}(\mathbf{C})$ on infinity types defined in § 3.3, part (3) follows from the rationality of induction (Lemme 3.9 (i)). $\qquad\square$

For a characteristic-zero field L, define $\mathbb{T}_L^{\mathrm{spl},P} \subset \mathscr{H}(\mathrm{G}'(\mathbf{A}^{p\infty}, L))_{\prod_{v\nmid p} K_v^\circ}$ to be the spherical Hecke algebra of elements supported at a set of places of F_0 split in F and disjoint from P. If L' is a subfield of $\mathbf{C}$, define $\mathcal{M}_{\infty,L'}$ to be the L'-linear subspace of $\mathcal{M}_\infty$ consisting of those μ such that $\mu(\xi_\infty)^\circ \in L'$. We put

$$\mathcal{M}_{L'}^{\mathrm{spl},P} := \mathscr{H}_{L'}^{\mathrm{spl},P} \otimes_L \mathcal{M}_{\infty,L'},$$

which is stable under multiplication and preserves $\mathscr{H}(\mathrm{G}'(\mathbf{A}, L')_K^\circ$. We have a surjective map

$$[-]^\circ : \mathcal{M}_{L'}^{\mathrm{spl},P} \longrightarrow \mathbb{T}_{L'}'^{\mathrm{spl},P}$$

$$\mu \longmapsto [\mu]^\circ$$

given by the evaluation at ξ_∞°. It is clear that the action of $\mathcal{M}_{L'}^{\mathrm{spl},P}$ on $\mathscr{H}(\mathrm{G}'(\mathbf{A}), L')^\circ$ factors through $[-]^\circ$.

We denote by $\mathscr{C}_K \subset \mathscr{C}$ the subset consisting of those Π' with $\Pi'^K \neq 0$.

Lemma 4.4.5. *Let* $\Pi' \in \mathscr{C}_K(\mathbf{C})$ *and let* (M, σ) *be a strongly* ξ_∞°-*typical cuspidal datum for* G′ *with*

$$\mathrm{M} \neq \mathrm{G}'.$$

Denote by $\overline{\mathbf{Q}}$ *the algebraic closure of* $\mathbf{Q}$ *in* $\mathbf{C}$.

There exists an element $\mu \in \mathcal{M}_{\overline{\mathbf{Q}}}^{\mathrm{spl},P}$ *satisfying:*

– for every $f' \in \mathcal{S}(\mathrm{G}'(\mathbf{A}), \mathbf{C})_K$, *the endomorphism* $R(\mu \star f')$ *of* $L^2_{(\mathrm{M},\sigma)}(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A})/K)$ *annihilates the subspace* $L^2_{(\mathrm{M},\sigma)}(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A})/K')$;

– $\mu(\xi_{\Pi'}^P) = 1$.

Proof. We refine the argument in the proof of [BPLZZ21, Proposition 3.17]. [11] Denote by $L' \subset \overline{\mathbf{Q}}$ the field of definition of Π'. Note that the subspace $\mathfrak{a}_M^* \subseteq \mathfrak{h}'^*$ has a natural model $\mathfrak{a}_{M,\mathbf{Q}}^* \subseteq \mathfrak{h}_\mathbf{Q}'^*$ over $\mathbf{Q}$. We fix a rational splitting map $\ell : \mathfrak{h}_\mathbf{Q}'^* \to \mathfrak{a}_{M,\mathbf{Q}}^*$ and an element $\alpha \in \xi_\infty^\circ$. By Ramakrishnan's

---

[11]With respect to the notation of *loc. cit.*, we omit the central character ω, which in our setup is necessarily trivial.

automorphic Tchebotarev theorem [Ram], for every $w \in \mathsf{W}'$, there is a finite place $v[w] \notin P$ of F_0, split in F, such that $\xi^{G'}_{\sigma_{s_w},v[w]} \neq \xi_{v[w]}$ where $s_w := \ell(w\alpha) - \ell(\alpha) \in \mathfrak{a}^*_{M,\mathbf{Q}}$.

This allows us to choose an element $\nu_w \in \mathscr{H}(\mathrm{G}'_{v[w]}, L')_{K'_{v[w]}}$ such that

$$\nu_w(\xi_{v[w]}) \neq \nu_w(\xi^{G'}_{\sigma_{s_w},v[w]}).$$

By the process in the proof of [BPLZZ21, Proposition 3.17], it suffices to show that for every $w' \in \mathsf{W}'$, $\nu_w(\xi^{G'}_{\sigma_{s_{w'}},v[w]})$ is algebraic. By Lemma 4.4.4, the Satake parameters of $\sigma_{w'}$ are algebraic numbers. Since $s_{w'} \in \mathfrak{a}^*_{M,\mathbf{Q}}$, it follows that the Satake parameters of $\sigma_{s_{w'},v[w]}$ are all algebraic numbers as well, which implies that $\mu_w(\xi^{G'}_{\sigma_{s_{w'}},v[w]})$ is algebraic. $\qquad \square$

We now extend the result to a finite set of cuspidal data and descend it to $\mathbf{Q}$. For $\mu \in \mathcal{M}^{\mathrm{spl},P}_{\overline{\mathbf{Q}}}$ and $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, we denote by $\tau.\mu \in \mathcal{M}^{\mathrm{spl},P}_{\overline{\mathbf{Q}}}$ a chosen lift of $\tau([\mu]^\circ)$.

Proposition 4.4.6. *Let $\mathfrak{D}$ be a finite set of strongly ξ°_∞-typical cuspidal data for G' such that*

$$\mathrm{M} \neq \mathrm{G}'$$

for every $(M, \sigma) \in \mathfrak{D}$. Then there exists a collection

$$(\mu^\iota_{\mathfrak{D}})_\iota \in \prod_{\iota:\, L \hookrightarrow \mathbf{C}} \mathcal{M}^{\mathrm{spl},P}_{\iota L})$$

satisfying

(1) for every $f' \in \mathcal{S}(\mathrm{G}'(\mathbf{A}), \mathbf{C})_K$ and every ι, the endomorphism $R(\mu \star f'^\iota)$ of $L^2_{(M,\sigma)}(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A})/K)$ annihilates the subspace $L^2_{(M,\sigma)}(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A})/K)$;

(2) there exists a $[\mu_{\mathfrak{D}}]^\circ \in \rightarrow \mathbb{T}'^{\mathrm{spl},P}_L$ such that $[\mu^\iota_{\mathfrak{D}}]^\circ = \iota[\mu_{\mathfrak{D}}]^\circ$ for every $\iota: L \hookrightarrow \mathbf{C}$;

(3) $\mu^\iota_{\mathfrak{D}}(\xi^P_\Pi) = 1$ for every $\iota: L \hookrightarrow \mathbf{C}$.

Proof. Up to enlarging $\mathfrak{D}$, we may assume it is stable under the $G_\mathbf{Q}$-action given by Lemma 4.4.4. We denote its elements simply by σ in order to lighten the notation. For each $\iota: L \hookrightarrow \mathbf{C}$ and each $\sigma \in \mathfrak{D}$, let $\mu_{\sigma,\iota}$ be as provided by Lemma 4.4.5 applied to σ and Π^ι. Let L' be a Galois extension of $\mathbf{Q}$ in $\mathbf{C}$ containing ιL and the fields of definition of $\mu_{\sigma,\iota}$ for every $\sigma \in \mathfrak{D}$ and every $\iota: L \hookrightarrow \mathbf{C}$. Now take the collection

$$\mu^\iota_{\mathfrak{D}} := \prod_{\tau \in \mathrm{Gal}(L'/\mathbf{Q})} \prod_{\sigma \in \mathfrak{D}} \tau.\mu_{\sigma,\tau^{-1}\iota}.$$

It satisfies the desiderata: the first one is enforced by the factors with $\tau = 1$; the second and third properties follow by construction and elementary Galois theory. $\qquad \square$

4.4.3. *Proof of Proposition 4.3.2.* Let $f'_{1,\infty} \in \mathscr{H}(G'_\infty, \mathbf{Q})^{\circ,\sharp}$ be a nontrivial rational Gaussian, which exists by Proposition 4.1.3. By [Ram], we can find a finite set S_1 of split places of F_0, disjoint from P and the ramification set of Π, and an

$$f'_{1,S_1} \in \mathscr{H}(G_{S_1}, L)_{K_{S_1}},$$

such that $\Pi(f_{1,S_1}) \neq 0$ and $\Pi'(f_{1,S_1}) = 0$ for every $\Pi' \in \mathscr{C}_{K,L} - \Pi$. For each $\iota: \hookrightarrow \mathbf{C}$, let $f'^\iota_1 = \iota f'_{1,S_1} \otimes f_{1,\infty} \otimes \otimes_{v \notin S\infty} f^\circ_v$.

Let μ_∞ and $\mathfrak{D} = \mathfrak{D}(\mathrm{G}, \omega, K, \mathfrak{T})^\heartsuit_{\xi_\infty!}$ be as provided by Proposition 4.4.3; the set $\mathfrak{D}$ is finite and it consists of of strongly ξ°_∞-typical cuspidal data for G'. Let $(\mu^\iota_\mathfrak{D})_\iota$, $[\mu_\mathfrak{D}]^\circ$ be as provided by Proposition 4.4.6 for $\mathfrak{D}$. Let

$$(f'^\iota) := \mu^\iota_\mathfrak{D} \star \mu_\infty \star f'^\iota_1, \qquad f' := [\mu_\mathfrak{D}]^\circ \star [\mu_\infty]^\circ \star f'_1$$

By construction, there is a set of split places $S \supset S_1$ disjoint from P such that for $? = \iota, \emptyset$, we have $f'^? = f'^?_{S\infty} \otimes \otimes_{v \notin S} f^\circ_v$ for some

$$f'^\iota_{S\infty}, \qquad f'_{S\infty}$$

that satisfy the desired properties.

The proof is complete.

4.5. Proofs of the rationality statements.

Recall that the local assertions of Propositions 4.2.2 were proved at the end of § 4.2. We will prove the global assertions of Proposition 4.2.2 and, as an interlude, Theorem 4.2.1.

4.5.1. *Global distribution.* The global orbital-integral distributions I_γ of part (4) are well-defined by Lemma 3.3.2. Then may define the distribution I of part (5) by its asserted geometric expansion:

$$I := \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I_\gamma.$$

We show the sum is locally finite. We may assume that f' factors as $f' = f'^\infty \otimes f'_\infty$. By definition, the sum is supported in $\mathrm{B}'_{\mathrm{rs}}(F_0) \cap \mathrm{B}'^\circ_\infty$. The invariant map (3.3.2) sends $\mathrm{B}'_{\mathrm{rs}}$ isomorphically to an open subset of the affine space $\mathrm{Res}_{F/F_0} \mathbb{A}^{2n+1}$. Let $\Omega^\infty \subset (\mathbf{A}^\infty_F)^{2n+1}$ be the image of the support of $f'^\infty \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^\infty))$, which is compact. Let $\Omega_\infty \subset F^{2n+1}_\infty$ be the image of B'°_∞. By definition, this is contained in the image of the positive-definite unitary group G°_∞ under the invariant map, which is compact. Therefore the support of the sum is in bijection with a subset of the set $F^{2n+1} \cap \Omega^\infty \Omega_\infty$; as the first intersecting set is discrete and second one is compact, this is finite.

By construction, I has the geometric expansion asserted in part (5); it satisfies

$$I(f', \chi) = \kappa(\mathbf{1}_\infty)^{-1} I^{\mathbf{C}}(\iota f', \chi) \tag{4.5.1}$$

for any $f' \in \mathscr{H}(G'(\mathbf{A}), L)^\circ_{\mathrm{rs,qc}}$ and any $\chi \in Y_L(\mathbf{C})$ with underlying embedding $\iota \colon L \hookrightarrow \mathbf{C}$.

Remark 4.5.1. By linearity, we may extend the distributions I, I_Π, I_γ to distributions (defined over $\mathbf{Q}$ or, for I_Π, the field of definition of Π) on the space of locally constant functions $\ell \colon F^\times_0 \backslash \mathbf{A}^\times / F^\times_{0,\infty} \to L$, in such a way that for every $\gamma' \in \mathrm{G}'_{\mathrm{rs}}(F_0)$ with image $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)$ and every $f'^\infty \otimes f'_\infty \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^\circ$, we have

$$I_\gamma(f', \ell) = \frac{I_\gamma(f'_\infty)}{\kappa(\mathbf{1}_\infty) \kappa_\infty(\gamma', \mathbf{1})} \int_{\mathrm{H}_1(\mathbf{A}^\infty)} \int_{\mathrm{H}_2(\mathbf{A}^\infty)} f'^\infty(h_1^{-1} \gamma' h_2) \ell(h_1) \eta(h_2) \frac{d^\natural h_1 d^\natural h_2}{d^\natural g},$$

where $d^\natural x := \prod_{v \nmid \infty} d^\natural x_v$, and the integral reduces to a finite sum.

4.5.2. *L-function.* We are now ready to prove the rationality of $\mathscr{L}$.

Proof of Theorem 4.2.1. For $\chi \in Y$, consider the set $\mathscr{H}(\mathrm{G}'(\mathbf{A}^\infty), L)^\circ_{\mathrm{rs},\Pi,\chi}$ of regularly supported Gaussians that are adapted to (Π, χ) (§ 4.2.2). It is non-empty by Corollary 4.3.4. For any

$\chi \in Y_L$ and $f' \in \mathscr{H}(G'(\mathbf{A}^\infty), L)^\circ_{\mathrm{rs},\Pi,\chi}$, we define

$$\mathscr{L}(M_\Pi, \cdot)_{f'} := \frac{4 \cdot I(f', \cdot)}{(\otimes_v I_{\Pi_v})(f'_v, \cdot)}$$

away from the zeros of the denominator. Then for any $\chi \in Y_L(\mathbf{C})$ with underlying $\iota: L \hookrightarrow \mathbf{C}$, we have

$$\mathscr{L}(M_\Pi, \chi)_{f'} = \frac{4 \cdot I^{\mathbf{C}}_{\Pi^\iota}(f', \chi)}{\kappa(\mathbf{1}_\infty)(\otimes_{v \nmid \infty} I^{\mathbf{C}}_{\Pi^\iota_v} \otimes I^{\circ,\mathbf{C}}_{\Pi^\iota_\infty})(f', \chi)} = \frac{\mathscr{L}^\infty(1/2, \Pi^\iota, \chi)}{\varepsilon(\frac{1}{2}, \chi^2)^{\binom{n+1}{2}}}$$

where the first equality is (4.5.1), and the second one is (4.1.1). Thus the functions $\mathscr{L}(M_\Pi, \cdot)_{f'}$ glue to the desired $\mathscr{L}(M_\Pi, \cdot)$. $\qquad\square$

4.5.3. *Spectral expansion.* We define

$$I_\Pi := \frac{1}{4}\mathscr{L}(M_\Pi) \cdot \prod_v I_{\Pi_v}$$

Then the spectral expansion of part (5) of Proposition 4.2.2 follows from the definition, Proposition 3.3.3, and (4.5.1). This completes the proof of the proposition.

5. p-ADIC RELATIVE-TRACE FORMULA: LOCAL THEORY

Throughout this subsection, we fix a place $v | p$ of F_0 and work in a local situation, dropping all subscripts v. We denote by $\mathscr{O}$ the ring of integers of the étale F_0-algebra F, by $\mathscr{O}_0$ the ring of integers of F_0, by $\varpi \in \mathscr{O}_0$ a chosen uniformizer, and we let $q_0 := |\mathscr{O}_0/\varpi\mathscr{O}_0|$, $q := |\mathscr{O}/\varpi\mathscr{O}|$.

5.1. **Group-theoretic preliminaries.** We introduce some notation and the group-theoretic foundations for the construction of the p-adic distribution.

5.1.1. *Notation.* If v splits in F, we fix an isomorphism $F \cong F_0 \times F_0$ and we expand our list of groups to include

$$\widetilde{G}'_0 := G'_{n,0} \times G'_{n+1,0}, \qquad H'_{1,0} := G'_{n,0},$$

so that $\widetilde{G}' = \widetilde{G}'_0 \times \widetilde{G}'_0$ and $H'_1 = H'_{1,0} \times H'_{1,0}$. We may then write elements of $G' = \widetilde{G}'/(F_0^\times)^2$ as $[g_1; g_2]$ with $g_i \in \widetilde{G}'_0$.

We will denote all conjugation actions by

$$x^g := g^{-1}xg.$$

Convention. Throughout this section, for $\nu \in \{n, n+1, \emptyset\}$ and $* = \{\emptyset, 0\}$ we will define various subgroups and elements $\Box_{\nu,*}$ of $G'_{\nu,*}$ (or $\widetilde{G}'_0$ for this 'pair' of subscripts). Unless otherwise specified, *we will define $\Box_{\nu,*}$ in a way that makes sense for $\nu = n, n+1$, and tacitly stipulate that $\Box_*$ is the product of $\Box_{n,*}$ and $\Box_{n+1,*}$, if $* = 0$, or its image via $\widetilde{G}' \to G'$ if $* = \emptyset$.* For the sake of uniformity, we introduce the notation

$$\dot{G}'_0 := \widetilde{G}'_0, \qquad \dot{G}' := G'.$$

5.1.2. *Some subgroups.* The lattice $\mathscr{O}^\nu_* \subset F^n_*$ induces an integral model for $G'_{\nu,*}$ over $\mathscr{O}_0$, still denoted by $G'_{\nu,*}$. Let $T_{\nu,*} \subset G'_{\nu,*}$ denote the diagonal torus, and let $W_{\nu,*}$ be the associated Weyl

group, identified with the permutation matrices in $G'_{\nu,*}$. We denote by

$$w_{\nu,*} \in W_{\nu,*}$$

the antidiagonal matrix $(w_{\nu,*})_{ij} = \delta_{i,\nu+1-j}$.

5.1.3. *On the torus in $G'_{\nu,*}$.* We denote by $N_{\nu,*} \subset G'_{\nu,*}$ the set of upper-triangular unipotent matrices and by

$$N^\circ_{\nu,*} := N_{\nu,*} \cap \mathrm{G}'_{\nu,*}(\mathscr{O}_0).$$

Let $T^+_{\nu,*} \subset T_{\nu,*}$ be the sub-monoid consisting of those t such that $N^{\circ,t}_{\nu,*} := (N^\circ_{\nu,*})^t \subset N^\circ_{\nu,*}$, and $T^{++}_{\nu,*} \subset T^+_{\nu,*}$ the multiplicative subset ot those t such that

$$\bigcap_{r \geq 1} N^{\circ,t^r}_{\nu,*} = \{1\}.$$

Concretely, $T^+_{\nu,*}$ (respectively $T^{++}_{\nu,*}$) consists of matrices $\mathrm{diag}(t_1,\ldots,t_\nu)$ with $t_i \in F^\times_*$ and $v(t_i/t_{i+1}) \geq 0$ (respectively > 0) for all $1 \leq i \leq \nu - 1$.

The group $T_{\nu,*}$ is equipped with the involution

$$\iota\colon t \longmapsto w^{-1}_{\nu,*} t^{-1} w_{\nu,*},$$

which preserves $T^+_{\nu,*}$ and $T^{++}_{\nu,*}$. We still denote by ι the resulting involution on $\mathbf{Q}_p[T_{\nu,*}]$.

We identify $\mathbf{Z}^\nu$ with the space of cocharacters of $T_{\nu,*}$ via

$$\lambda \longmapsto [x \longmapsto x^\lambda := \mathrm{diag}(x^{\lambda_1},\ldots,x^{\lambda_\nu})] \in T_{\nu,0} \subset T_{\nu,*},$$

where the inclusion is diagonal.

We fix the elements

$$t_{\nu,*} := \varpi^{(\nu-1,\ldots,0)} \in T^{++}_{\nu,*}, \qquad z_{\nu,*} = \varpi^{\nu-1} 1_\nu \in G'_{\nu,*}. \tag{5.1.1}$$

Then

$$t^\iota_{\nu,*} = z^{-1}_{\nu,*} t_{\nu,*}, \qquad t_{\nu,*} t^\iota_{\nu,*} = \varpi^{2\rho_\nu}$$

where $\rho_\nu \in \mathbf{Z}^\nu$ denotes half the sum of positive roots (with respect to $N_{\nu,*}$); concretely,

$$\rho_\nu := \frac{1}{2}(\nu - 1, \nu - 3, \ldots, 1 - \nu) \in \frac{1}{2}\mathbf{Z}^\nu.$$

5.1.4. *Iwahori and deeper Iwahori subgroups.* The standard *Iwahori subgroup*

$$\mathrm{Iw}_{\nu,*} \subset G'_{\nu,*}$$

is the set of matrices in $\mathrm{G}'_{\nu,*}(\mathscr{O}_0)$ whose reduction modulo ϖ belongs to the image of the upper-triangular matrices in $\mathrm{G}'_{\nu,*}(\mathscr{O}_0)$. A *deeper Iwahori subgroup* of $G'_{\nu,*}$ is an open subgroup $K \subset G'_{\nu,*}$ satisfying $N^\circ_{\nu,*} \subset K \subset \mathrm{Iw}^g_{\nu,*}$ for some $g \in N_{G'_{\nu,*}}(T_{\nu,*})$, the normalizer of $T_{\nu,*}$ in $G'_{\nu,*}$; it is said to be *standard* if we can take $g = 1_\nu$.

For $r \in \mathbf{Z} - \{0\}$, we define three families of subgroups

$$K^{[r]}_{\nu,*} \subset K^{\langle r \rangle}_{\nu,*} \subset K^{(r)}_{\nu,*} \tag{5.1.2}$$

of $G'_{\nu,*}(\mathscr{O}_0)$ by

$$K_{\nu,*}^{(r)} := G'_{\nu,*}(\mathscr{O}_0) \cap t_{\nu,*}^{-r} G'_{\nu,*}(\mathscr{O}_0) t_{\nu,*}^r,$$

$$K_{\nu,*}^{\langle r \rangle} := \{g \in K_{\nu,*}^{(r)} \mid g_{ii} \in 1 + \varpi^{|r|-1}\mathscr{O}_*, \ 1 \le i \le \nu\}$$

$$K_{\nu,*}^{[r]} := \{g \in K_{\nu,*}^{(r)} \mid g_{ii} \in 1 + \varpi^{|r|}\mathscr{O}_*, \ 1 \le i \le \nu\}.$$

They are standard deeper Iwahori subgroups whenever $r \ge 1$. For $r \ge 1$, we say that a standard deeper Iwahori $K_{\nu,*}$ has *level* $\le r$ if

$$K_{\nu,*} \supset K_{\nu,*}^{\langle r \rangle}.$$

5.1.5. *Iwahori–Weyl symmetries.* For $c \ge 1$, define

$$w_{*,\nu,c} := w_{\nu,*} t_{\nu,*}^c \in N_{G'_{\nu,*}}(T_{\nu,*}) \subset G'_{\nu,*}.$$

Let $K \subset G'_{\nu,*}$ be a deeper Iwahori subgroup. We say that K is *symmetric* if $K^{w_{\nu,*,c}} = K$ for some $c \ge 1$ such that $K_*^{\langle c \rangle} \subset K$. If v splits in F and $* = \emptyset$, we say that K is *conjugate-symmetric* if $K = K_0 \times K_0^{w_{\nu,0,c}}$ for some $c \ge 1$ and some standard[12] deeper Iwahori subgroup $K_0 \subset G'_{\nu,0}$ containing $K_{\nu,0}^{\langle c \rangle}$.

Remark 5.1.1. For $r \ge 1$, the subgroups $K_{\nu,*}^{[r]} \subset K_{\nu,*}^{\langle r \rangle} \subset K_{\nu,*}^{(r)} \subset G'_{\nu,*}$ are all symmetric, whereas for $\nu \ge 3$ Iwahori subgroups are not symmetric. On the other hand, conjugate-symmetric deeper Iwahori subgroups of G'_ν are obviously in bijection with standard deeper Iwahori subgroups of $G'_{\nu,0}$.

5.1.6. *Iwahori–Hecke algebras and the operators U_t.* Let $K \subset G'_{\nu,*}$ be a standard deeper Iwahori subgroup. Define sheaves of $\mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p}$-algebras by

$$\mathscr{H}_{K,*}^{\dagger,+} := C_c^\infty(K \backslash K T_{\nu,*}^+ K / K, \mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p})\, dg \quad \subset \quad \mathscr{H}_{K,*} := C_c^\infty(K_\nu \backslash G'_{\nu,*} / K_\nu, \mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p})\, dg.$$

The involution ι extends to $\mathscr{H}_{K,*}^{\dagger,+}$ by linearity. For $x \in G'_{\nu,*}$, we abusively denote by KxK both the coset in $KG'_{\nu,*}K$ and the Hecke element

$$KxK := \mathrm{vol}(K, dg)^{-1} \mathbf{1}_{KxK}\, dg \in \mathscr{H}_{K,*}.$$

The map

$$\mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p}[T_*^+ / T_* \cap K] \longrightarrow \mathscr{H}_{K,*}^{\dagger,+}$$
$$[t] \longmapsto U_{t,K} := KtK. \tag{5.1.3}$$

is an $\mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p}$-algebra isomorphism. We define

$$\mathscr{H}_{K,*}^\dagger := \mathscr{H}_{K,*}^{\dagger,+}[(U_t^{-1})_{t \in T^+}] \cong \mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p}[T_* / T_* \cap K].$$

If $K' \subset K$, then $U_{t,K'}$ maps to $U_{t,K}$ under the natural map $\mathscr{H}_{K',*}^{\dagger,+} \to \mathscr{H}_{K,*}^{\dagger,+}$. Thus for $? = +, \emptyset$, we define $\mathscr{H}_{\nu,*}^{\dagger,?} := \varprojlim_K \mathscr{H}_{K,*}^{\dagger,?}$, where the limit runs over the standard deeper Iwahori subgroups, and denote $U_t := \lim U_{t,K}$

5.1.7. *Multiplication rules in Iwahori–Hecke algebras.* The following result is standard and easy to verify.

---

[12]Requiring this condition may not be entirely natural but will be convenient.

Lemma 5.1.2. *Let $K \subset G'_{\nu,*}$ be a deeper Iwahori subgroup, and define $\ell \colon K \backslash G'_{\nu,*}/K \to \mathbf{N}$ by $q_*^{\ell(g)} := |KgK| = |K/K \cap gKg^{-1}|$. Then:*

(1) We have $KgKg'K = Kgg'K$ in $\mathscr{H}_K$ if and only if $\ell(gg') = \ell(g) + \ell(g')$.

(2) Assume that K is standard. Then for all $t' \in T^+_{\nu,*}$,

$$\ell(t'w_{\nu,*}) = \ell(t') + \ell(w_{\nu,*}), \qquad \ell(w_{\nu,*}t'^{-1}) = \ell(w_{\nu,*}) + \ell(t'^{-1}).$$

(3) Assume that K is standard, and let $K'' \subset K$ be an open subgroup. Then for all $t' \in T^+_{\nu,*}$,

$$Kt'wK'' = Kt'wK;$$

If moreover K is of level $\leq c$ and $t't_{\nu,*}^{-c} \in T^+_{\nu,*}$, then

$$K''t'K = Kt'K.$$

Proof. Part (1) is standard. Consider the first equality of part (2), and drop all subscripts. By part (1), it is equivalent to prove $Kt'KwK = Kt'wK$. Since the quotient $K \backslash Kt'K$ is represented by lower-triangular matrices in K, it suffices to show that for such a matrix k, we have $t'kw \in Kt'wK$; fact, since $K \supset N^\circ$ we even have $kw \in wK$. The second equation follows from taking inverses in the identity $Kt'KwK = Kt'wK$.

Consider now part (3). For the first equality, It suffices to prove that for any lower-triangular $k \in K$ we have $t'wk \in Kt'w$, which is clear since $t'wkw^{-1}t'^{-1} \in N^\circ \subset K$. For the second one, it suffices to prove that for any lower-triangular $kt' \in Kt'$ we have $kt' \in t'K$. In fact, by the assumptions we have $t'^{-1}tk \in K^{(c)} \cap K \subset K$. □

5.1.8. *Twisting matrices.* Let $u \in (\mathcal{O}_{F,p}^\times)^n$; we will take $u = (1,\ldots,1)^{\mathrm{t}}$ to fix ideas in computations. Then we define the *twisting matrices*[13]

$$m_{n,*} := 1_n, \qquad m_{n+1,*} := \begin{pmatrix} w_n & u \\ & 1 \end{pmatrix}, \tag{5.1.4}$$

and for $r \geq 1$ we let

$$m_{*,r} := m_{\nu,*,r}t_{\nu,*}$$

5.1.9. *Subgroups of H'_1.* Recall that by the convention introduced at the beginning ot this subsection, $\square_*$ denotes the (image of the) product of $\square_{n,*}$ and $\square_{n+1,*}$ in $\dot{G}'_*$. For $r \in \mathbf{Z}_{>0}$, let

$$\begin{aligned} K_{H,*}^{(r)} &:= m_* K_*^{(-r)} m_*^{-1} \cap G'_{n,*}(\mathcal{O}_0) \\ &= m_* K_*^{[-r]} m_*^{-1} \cap G'_{n,*}(\mathcal{O}_0) \qquad \subset G'_{n,*}(\mathcal{O}_0) \subset H'_{1,*}, \end{aligned} \tag{5.1.5}$$

where the intersections are with respect to the usual diagonal embedding $H'_{1,*} \hookrightarrow \dot{G}'_*$.

Remark 5.1.3. A simple computation shows that $K_{H,*}^{(r)}$ consists of the matrices h satisfying

$$\begin{cases} h_{ij} \in \varpi^{r|i-j|}\mathcal{O}_* \\ \sum_{j=1}^n h_{ij} \in 1 + \varpi^{ir}\mathcal{O}_* \end{cases} \tag{5.1.6}$$

---

[13]For their history, see [Jan] and references therein.

for all $1 \le i, j \le n$. This description also shows the equality in (5.1.5). We may then compute that

$$\mathrm{vol}^\circ(K_{H,*}) := q_*^{d(n)s}\,\mathrm{vol}(K_{H,*}^{(s)}) = \prod_{i=1}^{n} \frac{1 - q_*^{-1}}{1 - q_*^{-i}} \tag{5.1.7}$$

is a rational number independent of $s \ge 1$, and a p-unit.

We record the following easily checked property, for a later use: for all $r \ge 1$, we have

$$m_{*,r}^{-1} K_{H,*}^{(r)} m_{*,r} \subset K_*^{(2r)} \cap K_*^{[r]} \subset K_*^{\langle r+1 \rangle}. \tag{5.1.8}$$

5.1.10. *Twisting identity.* We come to the key result of this subsection, which refines [Jan, Lemma 5.2] in the spirit of [Loe21, Lemma 4.4.1]. Denote $N_*^{\circ,(r)} := t^r N_*^\circ t^{-r}$.

Lemma 5.1.4 (Twisting identity). *Let $r \ge 1$ and let $K \subset \dot{G}'_*$ be a subgroup containing $K_*^{\langle r+1 \rangle}$. For all $x \in N_*^\circ$, there exists $h_x \in K_{H,*}^{(r)}$ such that*

$$m_{*,r} x t K = h_x m_{*,r+1} K \tag{5.1.9}$$

Moreover, the map

$$N_*^{\circ,(1)} \backslash N_*^\circ \longrightarrow K_{H,*}^{(r+1)} \backslash K_{H,*}^{(r)}$$
$$[x] \longmapsto [h_x]$$

is well-defined and a group isomorphism.

Proof. We omit the subscript '$*$' from the notation. It suffices to take $K = K^{\langle r+1 \rangle}$. Consider the diagram

$$K_H^{(r+1)} \backslash K_H^{(r)} \xrightarrow{\alpha} K^{\langle -r-1 \rangle} \backslash K^{[-r]} \xleftarrow{\beta} N^{\circ,(r+1)} \backslash N^{\circ,(r)} \xleftarrow{\gamma} N^{\circ,(1)} \backslash N^\circ$$

where $\alpha \colon h \mapsto m^{-1}hm$, β is induced by the inclusion $N^{\circ,(r)} \subset K^{[-r]}$, and γ is the isomorphism $x \mapsto t^r x t^{-r}$. All four quotients have cardinality $q^{d(n)}$ where $d(n) = (5.1.11)$, and by (5.1.5), α is well-defined and injective. Hence all three maps are isomorphisms, and the second statement of the lemma is proved with $[h_x] = \alpha^{-1} \circ \beta \circ \gamma([x])$. The first statement is then easily verified using $t^{-r-1} K^{\langle -r-1 \rangle} t^{r+1} = K^{\langle r+1 \rangle}$. $\qquad\square$

Corollary 5.1.5. *Let $r \ge 1$, and let $K \subset \dot{G}'_*$ be a deeper Iwahori of level $\le r+1$. For all $s \ge r$, we have the identities*

$$m_{*,s} U_{t_*,K} = \sum_{h \in K_{H,*}^{(s+1)} \backslash K_{H,*}^{(s)}} h m_{*,s+1} K \qquad\qquad in\ C_c(\dot{G}'_*/K),$$

$$q^{sd(n)} \cdot m_{*,s} U_{t_*,K}^{-s} = q_*^{(s+1)d(n)} \cdot \sum_{h \in K_{H,*}^{(s+1)} \backslash K_{H,*}^{(s)}}^{\mathrm{avg}} h m_{*,s+1} U_{t_*,K}^{-(s+1)} \quad in\ C_c(\dot{G}'_*/K) \otimes_{\mathscr{H}_K^{\dagger,+}} \mathscr{H}_K^\dagger,$$

$$\tag{5.1.10}$$

where $\sum^{\mathrm{avg}}$ denotes average.

5.1.11. *Volumes.* The volumes of $K_{\nu,*}^{(r)}$ and $K_{H,*}^{(r)}$ are constant multiples of $q_*^{-c(\nu)r}$, respectively $q_*^{-d(n)r}$, where

$$c(\nu) := \frac{1}{6}(\nu - 1)\nu(\nu + 1)$$

$$d(n) := \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) = c(n) + c(n+1). \tag{5.1.11}$$

5.2. p-adic spherical characters.

Let $\Pi_{\nu,*}$ be a tempered representation of $G'_{\nu,*}$ over L. Denote by $B_{\nu,*} \subset G'_{\nu,*}$ the upper-triangular Borel and by $\delta_{B_{\nu,*}} : T_{\nu,*} \to \mathbf{Q}^\times$ its modulus character.

5.2.1. *Finite-slope subspace.* Let

$$\Pi_{\nu,*}^\dagger \subset \Pi_{\nu,*}^{N^\circ},$$

be the subspace where $T_{\nu,*}^+$ acts invertibly. It has a structure of $\mathscr{H}_{\nu,*}^\dagger(L)$-module, and it is isomorphic as $L[T_{\nu,*}]$-module to the twisted Jacquet module $\delta_{B_{\nu,*}} \otimes (\Pi_{\nu,*})_{N_{\nu,*}}$ of $\Pi_{\nu,*}$ (see e.g. [Eme06, Proposition 4.3.4]). We define $c(\Pi^\dagger)$ to be the minimal $c \in \mathbf{Z}_{\geq 1}$ such that $\Pi_{\nu,*}^\dagger \subset \Pi_{\nu,*}^K$ for some deeper Iwahori subgroup K of level $\leq c$.

Denote by $\widehat{T}$ the dual torus (as a scheme over L). For a subgroup $K \subset G'(\mathscr{O}_{F_0})$ containing N°, let $\Pi^{K,\dagger}$ be the image of Π^K in $\Pi^\dagger$ under the U_t-eigenprojection, for any sufficiently positive t. Then there are decompositions into generalized $\mathscr{H}_K^\dagger$-eigenspaces

$$\Pi_{\overline{L}}^{K,\dagger} = \bigoplus_{\xi \in \widehat{T}_L(\overline{L})} \Pi_{\overline{L}}^{K,\dagger}[\xi],$$

and similarly $\Pi_{\overline{L}}^\dagger = \bigoplus \Pi_{\overline{L}}^\dagger[\xi]$.

If Π is a subquotient of a regular principal series (as defined in § 4.3.1) and ξ is a character ot $\widehat{T}$ occurring in $\Pi_{\overline{L}}^\dagger$, then by [Jan, Proposition 1.3 (ii)] (or its proof, applied to Π_n, Π_{n+1}), any Whittaker model of $\Pi_{L(\xi)}$ contains a a unique vector

$$W_\xi \tag{5.2.1}$$

satisfying $W_\xi(1) = 1$ and $U_t W = \xi(t)W$ for all $t \in T^+$.

5.2.2. *Ordinary representations.* Suppose for this paragraph only that L is a finite extension of $\mathbf{Q}_p$, with algebraic closure denoted $\overline{\mathbf{Q}}_p$.

Definition 5.2.1. Let $N^\circ \subset K \subset G'(\mathscr{O}_{F_0})$. We say that the tempered representation Π is K-*ordinary* (with respect to Π_∞°) if there is a character $\xi^\circ \in \widehat{T}_L(\overline{\mathbf{Q}}_p)$ occurring in $\Pi^{K,\dagger}$ (that is, such that $\Pi^{K,\dagger}[\xi^\circ] \neq 0$) satisfying

$$|\xi^\circ(t')| = 1$$

for all $t' \in T^+$ and the absolute value on $\overline{\mathbf{Q}}_p$. We say that Π is ordinary if it is K-ordinary for sufficiently small $K \supset N^\circ$.

We call a character ξ° as above an *ordinary refinement* of Π. By the following proposition, an ordinary refinement is unique and defined over the field of definition of Π. We will then denote

$$\Pi^{\mathrm{ord}} := \Pi_{\overline{\mathbf{Q}}_p}[\xi^\circ] \cap \Pi.$$

Proposition 5.2.2. *Let Π be an ordinary tempered representation of G' over L. Then Π is a subquotient of a regular principal series, the space $\Pi^\dagger$ is T_L-semisimple, and every $\xi \in \widehat{T}_L(\overline{\mathbf{Q}}_p)$ occurring in $\Pi^\dagger_{\overline{\mathbf{Q}}_p}$ satisfies $\dim_{\overline{\mathbf{Q}}_p} \Pi^\dagger_{\overline{\mathbf{Q}}_p}[\xi] = 1$ and it is defined over L. Moreover the ordinary refinement ξ° is unique.*

Proof. This is essentially [Hid98, Corollary 8.3]. We recall the argument, working over $\overline{\mathbf{Q}}_p$ without signaling this in the notation. Let $W_{G'}$ be the Weyl group of G. Recall form § 5.2.1 that $\Pi^\dagger \cong \delta_B \otimes \Pi_N$, the δ_B-twisted Jacquet module of Π. By Frobenius reciprocity, ξ occurs in $\Pi^\dagger$ if and only if Π embeds into the normalized induction $\mathrm{Ind}_B^G(\widetilde{\xi})$ where $\widetilde{\xi} := \delta_B^{-1/2}\xi$. Now $\mathrm{Ind}_B^{G'}(\widetilde{\xi}) \cong \mathrm{Ind}_B^{G'}(\widetilde{\xi}^w)$ for all $w \in W_G$. If $\xi^\circ_{|T^+}$ is valued in units, then the stabilizer of $\widetilde{\xi}^\circ$ in $W_{G'}$ is trivial, therefore its orbit consists of $|W_{G'}|$ distinct characters $\widetilde{\xi}$, and $\mathrm{Ind}_B^{G'}(\widetilde{\xi})$ is regular. By [BZ76, Theorem 5.21], we have $\dim \Pi_N \leq |W_{G'}|$, hence all the characters ξ occur with multiplicity one. The rationality assertion follows from the fact that the $\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$-action on the set of occurring ξ preserves valuations. $\qquad\square$

Denote by
$$\mathrm{e}^{\mathrm{ord}} \colon \Pi^{N^\circ} \longrightarrow \Pi^{\mathrm{ord}}$$
the $\mathscr{H}^\dagger$-eigenprojector Let $e_K^{\mathrm{ord}} := e^{\mathrm{ord}}e_K$. Thus Π is K-ordinary if $e_K^{\mathrm{ord}}\Pi = \Pi^{\mathrm{ord}}$. If K is a deeper Iwahori, then $\Pi^{K,\dagger} = \Pi^\dagger \cap \Pi^K$ so that it is clear that Π is K-ordinary if and only if Π is ordinary and $\Pi^K \neq 0$. By the following lemma, the same is true if $K = G'(\mathscr{O}_{F_0})$.

Lemma 5.2.3. *Suppose that Π is ordinary and unramified, and let $K := G'(\mathscr{O}_{F_0}) \subset G'$. Then $e_K^{\mathrm{ord}}\Pi = \Pi^{\mathrm{ord}}$.*

Proof. By Proposition 5.2.2, the operator e^{ord} acts as the eigenprojection (with respect to the action of $\mathbf{Q}_p[T]$) onto Π^{ord}. Write a nonzero spherical vector $\phi_K \in \Pi^K$ as
$$\phi_K = \sum_{w \in W_G} c_w \phi_w \tag{5.2.2}$$
with $c_w \in L$ and $\phi_w \in \Pi^\dagger[\xi^\circ.w] - \{0\}$, where we define $\xi.w$ by $\widetilde{\xi.w} = \widetilde{\xi}^w$ (with notation as in the proof of Proposition 5.2.2). Then we need to show $c_1 \neq 0$. Now by [Cas80, Lemma 3.9], the expansion of [Cas80, Lemma 3.8] (where $\chi = \widetilde{\xi}^\circ$) is of the form (5.2.2), and there one has (see Theorem 3.1 *ibid.*) that $c_1 = 1$. $\qquad\square$

5.2.3. *p-adic Rankin–Selberg period.* Let $\chi \in Y_L$. We define a functional on $\Pi^\dagger$ by
$$P^\dagger_{1,\Pi,\chi} := \lim_{s \to \infty} P^\dagger_{1,\Pi,\chi,s}, \qquad P^\dagger_{1,\Pi,\chi,s} := q^{d(n)s} P_{1,\Pi,\chi} \circ m_s U_t^{-s} \colon \Pi^\dagger \longrightarrow L(\chi). \tag{5.2.3}$$

Let $c(\chi)$ to be the conductor of χ in the usual sense: $c(\chi) = 0$ if χ is unramified and otherwise $c(\chi)$ is the minimal $c \in \mathbf{Z}_{\geq 1}$ such that $\chi_{|1+\varpi^c \mathscr{O}_0} = 1$.

Lemma 5.2.4. *The sequence in the limit (5.2.3) stabilizes as soon as $s \geq s_0 := \max\{1, c(\Pi^\dagger) - 1, c(\chi)\}$.*

Proof. In the definition of $P^\dagger_{1,\Pi,\chi,s}(W)$ in § 3.2.2, we may first integrate over $K_H^{(s_0)}$; observing that χ is $(\det K_H^{(s_0)})$-invariant, the lemma results from (5.1.10). $\qquad\square$

5.2.4. *p-adic pairing.* We define a (non-degenerate) pairing

$$\vartheta_\Pi^\dagger := \lim_{r\to\infty} \vartheta_{\Pi,r}^\dagger, \qquad \vartheta_{\Pi,t}^\dagger(\cdot,\cdot) := q^{d(n)}\vartheta_\Pi(w_r U_t^{-r}\cdot,\cdot)\colon \Pi^\dagger \times \Pi^{\vee,\dagger} \longrightarrow L.$$

It is easy to show, using the symmetry of $K^{\langle c\rangle}$, that the sequence in the limit stabilizes as soon as $r \geq c(\Pi^\dagger)$, and that for all $t' \in T^+$ the $\vartheta_\Pi^\dagger$-adjoint of $U_{t'}$ is $U_{t'^\iota}$.

5.2.5. *p-adic Flicker–Rallis period.* Suppose that v splits in F and that Π_ν is in the image of the local base change map (2.4.1); in other words, we may write $\Pi_\nu \cong \Pi_{\nu,0} \boxtimes \Pi_{\nu,0}^\vee$ for some representation Π_ν of $\widetilde{G}_0$. We define

$$P_{2,\Pi}^\dagger := \lim_{r\to\infty} P_{2,\Pi,r}, \qquad P_{2,\Pi,r} := q_0^{d(n)r} P_2 \circ [w_{0,r};1]U_{[t_0;1]}^{-r}\colon \Pi^\dagger \longrightarrow L.$$

The sequence in the limit stabilizes as soon as $r \geq c(\Pi^\dagger)$.

5.2.6. *Finite-slope spherical character.* We say that a subgroup $K \subset G'$ is *convenient* if either $K = \mathrm{G}'(\mathscr{O}_0)$, or v splits in F and K is a conjugate-symmetric deeper Iwahori (henceforth: a CSDI).

Let $K \subset G'$ be a convenient subgroup. We define a distribution

$$I_{\Pi,K}^\dagger \in \mathscr{O}(\mathscr{H}_L^\dagger \times Y_v)$$

by

$$I_{\Pi,K}^\dagger(f^\dagger,\chi) := \begin{cases} \mathrm{Tr}_{\vartheta_\Pi}^{P_{1,\Pi,\chi}^\dagger \otimes P_{2,\Pi}}(\Pi(f^\dagger e_K)) & \text{if } K = \mathrm{G}'(\mathscr{O}_0), \\[2em] \mathrm{Tr}_{\vartheta_\Pi^\dagger}^{P_{1,\Pi,\chi}^\dagger \otimes P_{2,\Pi}^\dagger}(\Pi(f^\dagger e_K)) & \text{if } K \text{ is a CSDI.} \end{cases}$$

Remark 5.2.5. The second definition is the 'correct' one from the p-adic point of view. The first one is made because, first, in the arithmetic side the geometry will compel us to work at spherical level; and second, we have not investigated the analogue of the notion of 'conjugate-symmetric' in the nonsplit case.

5.2.7. *Decomposition according to refinements.* Identify Π_{n+1} (respectively Π_n) with its ψ- (respectively $\overline{\psi}$-) Whittaker model, and Π with their product. Suppose that Π is a subquotient of a regular principal series.

Let $\xi \in \widehat{T}$ be a character occurring in $\Pi_L^\dagger$; by the argument in the proof of Proposition 5.2.2, we have $\dim_{L(\xi)} \Pi_{L(\xi)}^\dagger \Pi^\dagger[\xi] = 1$. We denote by $W_\xi \in \Pi_{L(\xi)}$ the element of (5.2.1).

Define

$$e(\Pi,\xi,\chi) := P_{1,\Pi,\chi}^\dagger(W_\xi) \in L(\xi,\chi). \tag{5.2.4}$$

We state our working hypothesis on an explicit formula for this term. Write $\widetilde{\xi} = \widetilde{\xi}_n \boxtimes \widetilde{\xi}_{n+1}$, and for $1 \leq i \leq \nu$, let $\widetilde{\xi}_{\nu,i}\colon F^\times \to L(\xi)^\times$ be the restriction of $\widetilde{\xi}_\nu$ to the ith component of $T_\nu = (F^\times)^\nu/F_0^\times$. For any character ξ' of $F_v^\times$ and any place $w|v$ of F, denote by $\xi'_w := \xi_{|F_w^\times}$; denote by $N_w\colon F_w^\times \to F_0^\times$ the norm map. Finally, we denote by

$$\gamma(s,\xi'_{F,w},\psi_{F,w})^{-1} := L(s,\xi'_w)/\varepsilon(s,\xi'_w,\psi_{F,w})L(1-s,\xi'^{-1}_w)$$

the inverse Deligne–Langlands γ-factor of a character of $\xi'_w \colon F_w^\times \to \mathbf{C}^\times$. If

$$|\cdot|^{1/2}\xi'_k, \quad |\cdot|^{1/2}\xi''_k \colon F_w^\times \hookrightarrow L'^\times \subset \mathbf{C}$$

(for $1 \leq k \leq N$) are characters with $\prod_{k=1}^N |\cdot|^{1/2}\xi'_k = \prod_{k=1}^N |\cdot|^{1/2}\xi''_k$, then it is easy to see that $\prod_{k=1}^N \gamma(1/2, \xi'_k, \psi_{F,w})/\gamma(1/2, \xi''_k, \psi_{F,w})$ belongs to L'. Thus the following expression gives an element of $L(\xi, \chi)$ (unless some division by zero has occurred).

Define

$$\hat{e}(\Pi, \xi, \chi) \coloneqq \frac{\varepsilon(\tfrac{1}{2}, \chi^2, \psi)^{\binom{n+1}{2}}}{L(\tfrac{1}{2}, \Pi \otimes \chi)} \prod_{i+j \leq n} \prod_{w|v} \gamma(\frac{1}{2}, \chi \circ N_w \cdot \widetilde{\xi}_{n,i,w}\widetilde{\xi}_{n+1,j,w}, \psi_{F,w})^{-1}.$$

Hypothesis 5.2.6. *We have*

$$e(\Pi, \xi, \chi) = \hat{e}(\Pi, \xi, \chi).$$

The key consequences for us are Propositions 5.2.12 and Remark 5.2.9 below, both derived from the following lemma. We temporarily restore the notation of the rest of the paper.

Lemma 5.2.7. *Suppose that Π_v is a regular irreducible principal series that is the local component of a representation Π as in Theorem A. For every character ξ_v of T_v occurring in $\Pi_v^\dagger$ and every finite-order character χ_v of $F_{0,v}^\times$, we have*

$$\hat{e}(\Pi_v, \xi_v, \chi_v) \in L(\Pi, \xi, \chi)^\times.$$

Proof. By [Car14, Theorem 1.1], for each place w of F, the semisimple Weil–Deligne representation attached to $\rho_{\Pi|G_{F_w}}$ (cf. (1.2.1)) is

$$r_{\Pi,w} = \bigoplus_{1 \leq i \leq n, 1 \leq j \leq n+1} |\cdot|^{1/2}\widetilde{\xi}_{n,i,w}\widetilde{\xi}_{n+1,j,w},$$

and it is strictly pure of some weight that is independent of w (here we identify a character of $F_w^\times$ with its correspondent on the Weil group of F_w via class field theory). By considering $\det r_{\Pi,w}$ at an inert place w we then see that the weight must be -1. Thus for each (i,j), the character $|\cdot|^{1/2}\widetilde{\xi}_{n,i,w}\widetilde{\xi}_{n+1,j,w}$ is either ramified (so that its γ-factor is an ε-factor, hence nonzero), or it is an unramified character whose value at a uniformizer of F_w is a Weil q_w-number of weight -1, which again implies the nonvanishing of each term in the γ-factors of Hypothesis 5.2.6. $\square$

We go back to the notation of the rest of this section. If $K = \mathrm{G}'(\mathscr{O}_{F_0})$ and Π is unramified, define

$$c_K(\Pi, \xi) \coloneqq c_K(\Pi) \coloneqq 1 = P_{2,\Pi}(W_0^\vee)/\vartheta_\Pi(W_0, W_0^\vee), \tag{5.2.5}$$

where $W_0^{(\vee)} \in \Pi^{(\vee),K}$ is the generator normalized by $W_0^{(\vee)}(1) = 1$. If K is a conjugate-symmetric deeper Iwahori, define

$$c_K(\Pi, \xi) \coloneqq c(\Pi, \xi) \coloneqq \frac{P_{2,\Pi}^\dagger(W_{\xi^\iota})}{\vartheta_\Pi^\dagger(W_\xi, W_{\xi^\iota})} \in L(\xi). \tag{5.2.6}$$

Here, the denominator is nonvanishing since U_t is ϑ-adjoint to U_{t^ι}. Similarly, the numerator is nonvanishing if and only if Π_v is hermitian.

Then by the definitions we have a decomposition

$$I^\dagger_{\Pi,K}(f^\dagger, \chi) = \sum_{\xi \in \Xi_K(\Pi)} I^\dagger_{\Pi,K,\xi}(f^\dagger, \chi) \tag{5.2.7}$$

where

$$I^\dagger_{\Pi,K,\xi}(f^\dagger, \chi) = \xi(f^\dagger) c_K(\Pi, \xi) e(\Pi, \xi, \chi). \tag{5.2.8}$$

5.2.8. *Ordinary spherical character.* Suppose for this paragraph only that L is a finite extension of $\mathbf{Q}_p$ and that there is an $\mathscr{O}_L$-lattice $\Pi_{\mathscr{O}_L} \subset \Pi$ that is stable under $\mathscr{H}^\dagger$. Then we have Hida's description

$$e^{\mathrm{ord}} = \lim_{N \to \infty} U_t^{N!}$$

for the action of the ordinary projector on Π.

Remark 5.2.8. The above assumption holds whenever Π is a local component of a global representation in $\mathscr{C}_L$. Indeed, representations in $\mathscr{C}_L$ can be realised in the Betti cohomology of the locally symmetric space attached to G', and the cohomology with coefficients in $\mathscr{O}_L$ gives a natural integral structure stable under the Hecke operators; see [Hid98] for more details.

For any convenient $K \subset G'$, we then define

$$I^{\mathrm{ord}}_{\Pi,K}(\chi) := \lim_{N \to \infty} I^\dagger_{\Pi,K}(U_t^{N!}, \chi).$$

If Π is ordinary, we denote

$$\begin{aligned}
e(\Pi, \chi) &:= e(\Pi, \xi^\circ, \chi) \in L(\chi), \\
c_K(\Pi) &:= c_K(\Pi, \xi^\circ) \in L^\times
\end{aligned} \tag{5.2.9}$$

where the right-hand sides are defined in (5.2.4), (5.2.5), (5.2.6).

Remark 5.2.9. If Π_v, χ_v are as in Lemma 5.2.7 and moreover Π_v is ordinary, it follows from that lemma and Hypothesis 5.2.6 that $e(\Pi_v, \chi_v)$ and $c_{K_v}(\Pi)$ are nonzero.

Corollary 5.2.10. *Suppose that Π admits an $\mathscr{O}_L$-stable lattice. Then for every $\chi \in Y_L$ and every convenient $K \subset G'$, we have*

$$I^{\mathrm{ord}}_{\Pi,K}(\chi) = \begin{cases} c_K(\Pi)e(\Pi, \chi) & \text{if } \Pi \text{ is } K\text{-ordinary} \\ 0 & \text{otherwise,} \end{cases}$$

where $e(\Pi, \xi^\circ) = (5.2.9)$.

Proof. This follows from (5.2.8). $\qquad\square$

5.2.9. *Relation to the character I_Π.* Let Π be a tempered irreducible representation of G', let $\chi \colon F_0^\times \to L^\times$ be a smooth character, $K \subset G'$ be a convenient subgroup, and let $s \geq 1$. We say that s is sufficiently positive for χ (respectively for K) if $s \geq \max\{1, c(\chi)\}$ (respectively K contains a deeper Iwahori of level c with[14] $s \geq 2c$). We say that $f^\dagger \in \mathscr{H}^\dagger$ is *sufficiently positive*

---

[14]In fact, at least if K is an Iwahori subgroup or one of the subgroups (5.1.2) with $r = c \geq 1$, the weaker condition $s \geq c$ will suffice; this is only used in the application of Lemma 5.1.2 (3) in the proof of Lemma 7.1.1.

for Π *(respectively for s_0, for χ, for K)* if $f^\dagger \Pi \subset \Pi^\dagger$ *(respectively if $U_t^{-s} f^\dagger$ belongs to $\mathscr{H}^{\dagger,+}$ for $s = s_0$ or some s that is sufficiently positive for χ, respectively for K).*

It is clear that if $f^\dagger$ is in the span of $\{U_t \mid t \in T^{++}\}$ and s and Π are given, then some power of $f^\dagger$ is sufficiently positive for both s and Π.

Lemma 5.2.11. *For every s that is sufficiently positive for K and χ and every $f^\dagger$ that is sufficiently positive for s and Π, we have*

$$I_{\Pi,K}^\dagger(f^\dagger, \chi) = I_\Pi(f', \chi)$$

where

$$f' = f'_{K,s} := \begin{cases} q^{d(n)s} \cdot m_s U_t^{-s} f^\dagger e_K & \text{if } K = G(\mathscr{O}_0), \\ q_0^{d(n)(2s-c)} \cdot m_s U_t^{-s} f^\dagger U_{[1;t_0]}^c e_K[1; w_{0,c}^{-1}] & \text{if } K \text{ is a CSDI of level } \leq c. \end{cases} \tag{5.2.10}$$

Proof. The first case is clear. Consider the second case, dropping the subscripts Π and K from the notation. Let $\Pi_{\dagger,K} := w_c \Pi^{\dagger,K}$, and let $\vartheta_| \colon \Pi_{\dagger,K} \otimes \Pi^{\vee,\dagger,K} \to L$ be the restriction of $\vartheta \colon \Pi \otimes \Pi^\vee \to L$, which is still a perfect pairing. By Lemma 2.6.3 (using, in order, part (1), part (2), and part (1) together with part (3)),

$$\begin{aligned} I^\dagger(f^\dagger) &= q_0^{d(n)c} \operatorname{Tr}_{\vartheta^\dagger}^{P_1^\dagger \otimes P_2[w_{0,c};1]}(\Pi(f^\dagger e_K U_{[t_0^t;1]}^{-c})) \\ &= q_0^{-d(n)c} \operatorname{Tr}_{\vartheta_|}^{P_1^\dagger \otimes P_2[w_{0,c};1]}(\Pi(f^\dagger e_K U_{[t_0/t_0^t;t_0]}^c w_c^{-1})) \\ &= q_0^{d(n)(2s-c)} \operatorname{Tr}_\vartheta^{P_1 \otimes P_2}(\Pi(m_s U_t^{-s} f^\dagger U_{[z_0;t_0]}^c e_K w_c^{-1}[w_{0,c}^{-1};1])) = I(f'), \end{aligned}$$

where

$$f' = q_0^{d(n)(2s-c)} \cdot m_s U_t^{-s} f^\dagger U_{[z_0;t_0]}^c e_K[z_0^{-c}; w_{0,c}^{-1}],$$

the same as asserted. $\square$

5.2.10. *A non-vanishing result.* Unlike the rest of this section, the following result is not used for the p-local theory of the p-adic relative-trace formula, but rather as an input to Lemma 4.3.1 (3).

Proposition 5.2.12. *Let Π, χ, K be as in § 5.2.9. Suppose that v is split, K is a conjugate-symmetric Iwahori, and Π is a regular principal series. Assume Hypothesis 5.2.6 holds. Then there exists an $f^\dagger \in \mathscr{H}^\dagger$ that is sufficiently positive for Π, χ, K, such that the Hecke measure $f' := f'_{K,s} = (5.2.10)$ satisfies*

$$I_\Pi(f'_{K,s}, \chi) \neq 0.$$

Proof. Let f'_N correspond to $f^\dagger = U_t^N$ for some sufficiently large integer N. We may and do extend scalars from L to $\mathbf{C}$; we do not alter the notation. By (5.2.7), we have

$$I_\Pi(f'_N, \chi) = \sum_\xi \xi(t)^N \hat{c}_K(\Pi, \xi) \hat{e}(\Pi, \xi, \chi).$$

Order the characters ξ occurring in $\Pi^\dagger$ as $\xi_1, \ldots, \xi_r$; then we may write $I_\Pi(f'_N, \chi) = a_N x$ where $x = (m_{\xi_i} c_K(\Pi, \xi_i) e(\Pi, \xi_i, \chi))_i \in \mathbf{C}^r$ and $a_N^t = (\xi_i(t)^N)_i \in \mathbf{C}^r$. Now all entries of the vector x are nonzero by Lemma 5.2.7, and the Vandermonde matrix A with rows $a_N, \ldots a_{2N}, \ldots, a_{rN}$ is invertible. Hence there is some $1 \leq i \leq r$ such that $0 \neq a_{iN} x = I_\Pi(f'_{iN}, \chi)$, as desired. $\square$

5.3. **p-adic orbital integrals.** Let $K \subset G'$ be a convenient subgroup. For $f^\dagger$ sufficiently positive (depending on χ) and $\gamma \in B'_{\mathrm{rs}}$, define

$$I^\dagger_{\gamma,K}(f^\dagger, \chi) := I_\gamma(f'_{K,s}, \chi) \tag{5.3.1}$$

where s and $f'_{K,s}$ are as in Lemma 5.2.11.

Lemma 5.3.1. *We have:*

(1) *the right hand side of* (5.3.1) *is independent of the choices of an s that is sufficiently positive for K and χ, so long as $f^\dagger$ is sufficiently positive for s.*

(2) *For any s_0 that is sufficiently positive for K, any finite extension L of $\mathbf{Q}_p$, and any $f^\dagger \in \mathscr{H}^\dagger(\mathscr{O}_L)$ that is sufficiently positive for s_0, the map*

$$\chi \longmapsto I^\dagger_{\gamma,K}(f^\dagger, \chi)$$

extends by linearity to a functional $C^\infty(F_0^\times/(1 + \varpi^{s_0}\mathscr{O}_0), \mathscr{O}_L) \to \mathscr{O}_L$.

Proof. For a Hecke operator $T' = \sum_i c_i \operatorname{vol}(A_i)^{-1}\mathbf{1}_{A_i}dg \in \mathscr{H}_K$, denote $\mathbf{1}[\gamma' \in T'] := \sum c_i \mathbf{1}_{A_i}(\gamma')$. Let s_0 be sufficiently positive for K and χ. The integrand in the explicit expression for $I^\dagger_{\gamma,K}(f^\dagger)$ equals

$$q^{d(n)s}\chi(h_1)\mathbf{1}[\gamma h_2 \in h_1 m_s U_t^{-s} f^\dagger e_K], \tag{5.3.2}$$

and it is $K_H^{(s_0)}$-invariant by (5.1.8). Integrating first over over $K_H^{(s_0)} \subset H'_1$, the relation (5.1.10) shows that (5.3.2) is independent of $s \geq s_0$. We also see that the functional $I^\dagger_{\gamma,K}(f^\dagger, -)$ sends $C^\infty(F_0^\times/(1 + \varpi^{s_0}\mathscr{O}_0), \mathscr{O}_L)$ to $\operatorname{vol}^\circ(K_H)\mathscr{O}_L = \mathscr{O}_L$ (Remark 5.1.3). $\qquad\square$

The p-adic orbital integral may be explicitly computed when $f^\dagger$ is sufficiently positive and K is a conjugate-symmetric deeper Iwahori. A remarkable fact is that it is independent of χ. By linearity, it suffices to consider the case $f^\dagger = U_{t'}$ for some $t' \in T^{++}$.

Proposition 5.3.2. *Let $K \subset G'$ be a conjugate-symmetric deeper Iwahori (in particular, v splits in F). Assume that $f^\dagger = U_{t'} \in \mathscr{H}^\dagger$ for some $t' \in T^{++}$. There exists a compact subset*

$$B^\dagger_K(f^\dagger) \subset B'_{\mathrm{rs}}$$

such that for every smooth character χ of $F_0^\times$ such that $f^\dagger$ is sufficiently positive for χ and K,

$$I^\dagger_{\gamma,K}(f^\dagger, \chi) = \begin{cases} \operatorname{vol}^\circ(K_H) & \text{if } \gamma \in B^\dagger_K(f^\dagger) \\ 0 & \text{if } \gamma \notin B^\dagger_K(f^\dagger). \end{cases}$$

The proof of Proposition 5.3.2 will occupy all of § 7, which may be skipped on a first reading.

6. p-ADIC RELATIVE-TRACE FORMULA AND p-ADIC L-FUNCTION: GLOBAL THEORY

Throughout this section, we fix a rational prime p.

6.1. **Statements.** Recall that we denote $\Gamma = \Gamma_{F_0} := F_0^\times \backslash \mathbf{A}^{\infty,\times}/\widehat{\mathscr{O}}_{F_0}^{p,\times}$, and $\mathscr{Y} := \operatorname{Spec} \mathbf{Z}_p[\![\Gamma_{F_0}]\!] \otimes \mathbf{Q}_p$ We say that $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her}}_{\mathbf{Q}_p}$ is *ordinary* if for all $v|p$, the representation Π_v is ordinary in the

sense of Definition 5.2.1. The ordinary representations form an ind-subscheme

$$\mathscr{C}(G')^{\mathrm{her,ord}} \subset \mathscr{C}(G')^{\mathrm{her}}_{\mathbf{Q}_p}.$$

For $K_p = \prod_v K_v$, we let $\mathscr{C}(G')^{\mathrm{her,ord}}_{K_p}$ be the subscheme of those Π which are K_v-ordinary for all $v|p$.

6.1.1. *p-adic L-function.* The following is Theorem B from the introduction

Theorem 6.1.1. *Let L be a finite extension of $\mathbf{Q}_p$, and let Π be an ordinary hermitian trivial-weight cuspidal automorphic representation of $G'(\mathbf{A})$ over L.*

Assume that for each place $v|p$ of F_0, v splits in F or Π_v is unramified. Then there exists a unique function

$$\mathscr{L}_p(M_\Pi) \in \mathscr{O}(\mathscr{Y}_L)$$

whose restriction to $Y(p^\infty)_L$ satisfies

$$\mathscr{L}_p(M_\Pi)(\chi) = e_p(M_{\Pi \otimes \chi})\, \mathscr{L}(M_\Pi)(\chi) \tag{6.1.1}$$

where $\mathscr{L}(M_\Pi)$ is the function in Theorem 4.2.1, and $e_p(M_{\Pi \otimes \chi}) := \prod_{v|p} e(\Pi_v, \chi_v)$ for the factors of (5.2.9).

6.1.2. *Generalized Radon measures.* We make the first of two preparations which will be the relevant to the the p-adic relative-trace formula.

Recall that a *Banach ring* is a topological ring equipped with a norm $|\cdot|$ for which it is complete; the relevant examples for us are the finite extensions of $\mathbf{Q}_p$ (with the p-adic norm) and $\mathscr{O}(\mathscr{Y})$ (with the Gauss norm).

Definition 6.1.2. Let X be a locally compact topological space and let R be a Banach ring. A generalized bounded Radon measure with values in R is a pair $(\mu, L^{1,\infty}(X, \mu))$, where

− $L^{1,\infty}(X, \mu) \subset L^\infty(X)$ is a closed subspace of the R-Banach space of bounded R-valued function on X;

− $\mu \colon L^{1,\infty}(X, \mu) \to R$ is a bounded R-linear functional.

We will usually denote such measures simply by μ, and for $\Phi \in L^{1,\infty}(X, \mu)$, we will use the notation

$$\mu(\Phi) =: \int_X \Phi(x)\, d\mu(x).$$

When $L^{1,\infty}(X, \mu)$ contains $C_c(X)$, the functional μ is a (bounded) Radon measure in the sense of Bourbaki. When $R' \supset R$ is an extension of Banach rings, an R-valued generalized bounded Radon measure μ gives rise to an R'-valued generalized bounded Radon measure by extension of scalars, which we will still denote by μ. We say that a function $\Phi \in L^\infty(X)$ is μ-integrable if it belongs to $L^{1,\infty}(X, \mu)$. When we make an assertion regarding $\int_X \Phi\, d\mu$, it implicitly includes the assertion that Φ is μ-integrable.

6.1.3. *Local distributions at p.* Let $K_p = \prod_{v|p} K_v \subset G'(F_{0,p})$ be a compact open subgroup that is *convenient* in the sense that each K_v is (as defined in § 5.2.6). We will say that K_p is a *conjugate-symmetric deeper Iwahori* (CSDI) if each K_v is (as defined in § 5.1.4).

For $\chi \in Y(p^\infty)$ and $f_p^\dagger = \otimes_{v|p} f_v^\dagger \in \mathscr{H}_p^\dagger = \bigotimes_{v|p} \mathscr{H}_v^\dagger$ that is sufficiently positive for Π_p, χ_p, and K_p (in the obvious sense derived from § 5.2.9 for each $v|p$), and for $\gamma \in B'_{\mathrm{rs},p}$, Π_p a tempered irreducible representation of G'_p, define

$$I_{\Pi_p,K_p}^\dagger(f_p^\dagger, \chi_p) := \prod_{v|p} I_{\Pi_v,K_v}^\dagger(f_v^\dagger, \chi_v), \qquad I_{\gamma,p,K_p}^\dagger(f_p^\dagger, \chi_p) := \prod_{v|p} I_{\gamma,v,K_v}^\dagger(f_v^\dagger, \chi_v),$$

where the last factors are as in (5.3.1).

6.1.4. *p-adic relative-trace formula.* For K_p as above, define the Hecke subspace

$$\mathscr{H}(\mathrm{G}'(\mathbf{A}^p))_{K_p,\mathrm{rs},\mathrm{qc}}^\circ \subset \mathscr{H}(\mathrm{G}'(\mathbf{A}^p))_{\mathrm{rs}}^\circ$$

to consist of those f'^p such that $f'^p \otimes e_{K_p}$ is quasicuspidal. We also denote $U_{t_p} = \otimes_{v|p} U_{t_v}$.

Theorem 6.1.3 (*p-adic analytic RTF*). *Let $K_p = \prod_{v|p} K_v \subset \mathrm{G}'(F_{0,p})$ be a convenient subgroup, and let L be a finite extension of $\mathbf{Q}_p$.*

There exist:

(1) *For each finite place $v \nmid p$ of F_0 and for $v = \infty$, for each $\gamma \in B'_{\mathrm{rs},v}$, and for each tempered irreducible representation Π_v of G'_v over L, distributions*

$$\mathscr{I}_{\Pi_v} : \mathscr{H}(G'_v, L) \longrightarrow \mathscr{O}(\mathscr{Y}_L),$$

$$\mathscr{I}_{\gamma,v} : \mathscr{H}(G'_v, L) \longrightarrow \mathscr{O}(\mathscr{Y}_{L[\sqrt{-1}]}),$$

obtained from the corresponding distributions of Proposition 4.2.2 (1), (3), by pullback via the restriction maps $\mathscr{Y} \ni \chi \mapsto \chi_v \in Y_v(1)_{\mathbf{Q}_p}$. (If $v = \infty$, $Y_v(1) := \mathrm{Spec}\,\mathbf{Q}$.)

(2) *For each representation Π over L as in Theorem 6.1.1, a distribution*

$$\mathscr{I}_\Pi := \frac{1}{4}\mathscr{L}_p(\mathrm{M}_\Pi) \cdot \prod_{v\nmid p} \mathscr{I}_{\Pi_v} : \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ \longrightarrow \mathscr{O}(\mathscr{Y}_L).$$

(3) *A bounded orbital-integral function*

$$\mathscr{I}^p : B'_{\mathrm{rs}}(\mathbf{A}^p) \times \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ \longrightarrow \mathscr{O}(\mathscr{Y}_L) \qquad (6.1.2)$$

defined by

$$(\gamma, f'^p) \longmapsto \mathscr{I}_\gamma^p(f'^p) := \kappa(\mathbf{1}_\infty)^{-1} \prod_{v\nmid p} \mathscr{I}_{\gamma,v}.$$

(4) *For every $\chi \in Y(p^\infty)$, a $\mathbf{Q}_p(\chi)$-valued generalized bounded Radon measure $I_{\gamma,p,K_p}^{\mathrm{ord}}(\chi_p)$ on $B'_{\mathrm{rs}}(\mathbf{A}^p)$, which:*

(a) *is defined by the limit of weighted samplings*

$$\int_{B'_{\mathrm{rs}}(\mathbf{A}^p)} \Phi(\gamma)\, dI_{\gamma,p,K_p}^{\mathrm{ord}}(\chi_p) := \lim_{N\to\infty} \sum_{\gamma \in B'_{\mathrm{rs}}(F_0)} I_{\gamma,p,K_p}^\dagger(U_{t_p}^{N!}, \chi_p) \cdot \Phi(\gamma) \qquad (6.1.3)$$

on the space of bounded functions $\Phi \in L^\infty(B'_{\mathrm{rs}}(\mathbf{A}^p))$ for which the sums over γ converge and the limit converges;

(b) if K_p is a CSDI, is independent of χ_p (thus omitted from the notation) and equal to

$$\int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} \Phi(\gamma)\, dI^{\mathrm{ord}}_{\gamma,p,K_p} = \mathrm{vol}^\circ(K_{H,p}) \cdot \lim_{N\to\infty} \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0) \cap B^\dagger_{p,N}} \Phi(\gamma); \qquad (6.1.4)$$

here, $B^\dagger_{p,N} = \prod_{v|p} B^\dagger_{K_v}(U^{N!}_{t_v,K_v})$ and $\mathrm{vol}^\circ(K_{H,p}) = \prod_{v|p} \mathrm{vol}^\circ(K_{H,v})$, with the factors defined in Proposition 5.3.2.

(5) A distribution

$$\mathscr{I}_{K_p} \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^p),L)^\circ_{K_p,\mathrm{rs,qc}} \longrightarrow \mathscr{O}(\mathscr{Y}_L)$$

which admits the spectral expansion

$$\mathscr{I}_{K_p} = \sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}_{K_p}} \mathscr{I}_\Pi$$

and:

(a) for each finite-order $\chi \in \mathscr{Y}_L$, the geometric expansion in $L(\chi)$

$$\mathscr{I}_{K_p}(f'^p,\chi) = \int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} I^p_\gamma(f'^p,\chi)\, dI^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p);$$

(b) if K_p is a CSDI, the geometric expansion in $\mathscr{O}(\mathscr{Y}_L)$

$$\mathscr{I}_{K_p}(f'^p) = \int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} \mathscr{I}^p_\gamma(f'^p)\, dI^{\mathrm{ord}}_{\gamma,p,K_p}.$$

6.2. **Proofs.** We will prove Theorem 6.1.3 and, as an interlude, Theorem 6.1.1.

6.2.1. *Finite-slope distributions and boundedness of the associated orbital integrals.* For $\gamma \in \mathrm{B}'_{\mathrm{rs}}(\mathbf{A})$, and Π as in Theorem 6.1.1, we first define the following distributions on the subspace of $\mathscr{H}(\mathrm{G}'(\mathbf{A}^p),L)^\circ \otimes \mathscr{H}^\dagger_p$ of elements that are sufficiently positive for all the relevant data:

$$I^\dagger_{\Pi,K_p}(f'^p f^\dagger_p,\chi) := \frac{1}{4}\mathscr{L}(\mathrm{M}_\Pi,\chi) \cdot I^\dagger_{\Pi_p,K_p}(f^\dagger_p,\chi) \prod_{v\nmid p} I_{\Pi_v}(f_v,\chi_v) = I_\Pi(f'^p f'_p),$$

$$I^\dagger_{\gamma,K_p}(f'^p f^\dagger_p,\chi_p) := \kappa(\mathbf{1}_\infty)^{-1} \cdot I^\dagger_{\gamma,p}(f^\dagger_p,\chi_v) \prod_{v\nmid p} I_{\gamma,v}(f'_v,\chi_v). \tag{6.2.1}$$

Then we may define and expand

$$I^\dagger_{K_p}(f'^p f^\dagger_p,\chi) := \sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her}}_{\mathbf{Q}_p}} I^\dagger_{\Pi,K_p}(f'^p f^\dagger_p,\chi) \;\; = \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I^\dagger_{\gamma,K_p}(f'^p f^\dagger_p,\chi) \tag{6.2.2}$$

where the geometric expansion follows from Proposition 4.2.2 (5) and the definition (5.3.1).

Lemma 6.2.1. *Let s_0 be an integer that is sufficiently positive for K_v for all $v|p$, and denote $\varpi_p := \prod_{v|p} \varpi_v \in \mathscr{O}_{F_0,p}$. For any $f^\dagger = f'^p \otimes \otimes_{v|p} f^\dagger_v \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^p),L)^\circ \otimes \mathscr{H}^\dagger_p$ such that each $f^\dagger_v$ is sufficiently positive for s_0, the map*

$$Y(\varpi^{s_0}_p) \ni \chi \longmapsto I^\dagger_{\gamma,K}(f^\dagger,\chi)$$

extends by linearity to a functional $C(\Gamma_{F_0}/(1+\varpi^{s_0}_p \mathscr{O}_{F_0,p}),L) \to L$ that is bounded by a constant depending on $f^\dagger$ but not on s_0 or γ.

Proof. This follows by the same argument in the proof of Lemma 5.3.1 (2), applied to the possibly non-Eulerian global orbital integrals deduced by linearity from (6.2.1) as in Remark 4.5.1.   □

6.2.2. *Proof of Theorem 6.1.3 / I.* Part (1) and the definitions in part (3) and (4) of Theorem 6.1.3 are self-explanatory. The boundedness of the measures $I^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p)$ (which is, importantly, uniform in χ_p) follows from Lemma 5.3.1 (2). To see that $\gamma \mapsto \mathscr{I}^p_\gamma$ is bounded, it suffices to notice that the denominators arising from the orbital integration are bounded in terms of the level of f'^p only.

Corollary 5.2.10 shows that in the limit

$$I^{\mathrm{ord}}_{K_p}(f'^p, \chi) := \lim_{N \to \infty} I^\dagger_{K_p}(f'^p U^{N!}_{t,K^p}, \chi) = \sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}} e_p(\mathrm{M}_\Pi, \chi) \mathscr{L}(\mathrm{M}_\Pi, \chi) \cdot (\otimes_{v \nmid p} I_{\Pi_v})(f'^p, \chi^p). \tag{6.2.3}$$

The existence of the limit and (6.2.2) prove that the orbital-integral functions

$$I^p_{(-)}(f'^p, \chi) \colon \gamma \longmapsto \kappa(\mathbf{1}_\infty)^{-1} \cdot (\otimes_{v \nmid p} I_{\gamma,v})(f'^p, \chi^p)$$

are $I^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p)$-integrable, and that

$$I^{\mathrm{ord}}_{K_p}(f'^p, \chi) = \int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} I^p_\gamma(f'^p, \chi) \, dI^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p). \tag{6.2.4}$$

We have already seen that the function

$$(\gamma, \chi) \longmapsto I_\gamma(f'^p, \chi)$$

is bounded on $\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p) \times Y(p^\infty)$. By Lemma 6.2.1, the map $\chi \mapsto I^{\mathrm{ord}}_{K_p}(f'^p, \chi)$ extends to a *bounded* distribution $C^{\mathrm{lc}}(\Gamma_{F_0}, L') \to L'$ on L'-valued locally constant functions on Γ_{F_0}, for any L'-vector space L. It therefore extends uniquely to a bounded Radon measure

$$\mathscr{I}_{K_p}(f'^p) \colon C(\Gamma_{F_0}, L') \longrightarrow L' \tag{6.2.5}$$

corresponding to the element $\mathscr{I}_{K_p}(f'^p) \in \mathscr{O}(\mathscr{Y}_L)$ of part (5).

The geometric expansion in part (5a) follows from the definitions and the geometric expansion of I. Then by (6.2.4) and Lemma 6.2.2 below applied to (6.1.3), the distribution $\mathscr{I}_{K_p}$ has the geometric expansion described in part (5b).

Lemma 6.2.2. *Let $(\mathscr{I}_N)_{N \in \mathbf{N}}$, $\mathscr{I}_\infty \in \mathscr{O}(\mathscr{Y})$. Suppose that for all $\chi \in Y(p^\infty)$ we have $\lim_{N \to \infty} \mathscr{I}_N(\chi) = \mathscr{I}_\infty(\chi)$. Then $\lim_{N \to \infty} \mathscr{I}_n = \mathscr{I}_\infty$.*

Proof. Recall that $Y(p^\infty) = \varinjlim Y(p^s)$; then observe that the ideals $J_s := \mathrm{Ker}[\mathscr{O}(\mathscr{Y}) \to \mathscr{O}(Y(p^s))] \subset \prod_{\chi \in Y(p^s)} \mathbf{Q}_p(\chi)]$ form a fundamental system of neighbourhoods of 0 in $\mathscr{O}(\mathscr{Y})$.   □

We now turn to the p-adic L-function and the spectral expansion of $\mathscr{I}^{\mathrm{ord}}_{K_p}$.

6.2.3. *Proof of Theorem 6.1.1.* Let $K_p = \prod_v K_v$ be a convenient subgroup such that for every place $v|p$ of F_0, the representation Π_v is K_v-ordinary. Similarly to the proof ot Theorem 4.2.1, we prepare with a variant of Corollary 4.3.4, with a similar proof.

Lemma 6.2.3. *For any $\chi \in \mathscr{Y}_L$, the set $\mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ_{\mathrm{rs},K_p,\Pi^p,\chi^p}$ of regularly supported Gaussians adapted to (Π, χ, K_p) is non-empty.*

(Note that as $\chi_{|\mathbf{A}^{p,\times}}$ is smooth, the definition of 'adapted to (Π, χ, K_p)' in § 4.2.2) still makes sense.)

Proof. Let P' be the set of all finite places of F_0 not above p at which Π is ramified. Let v_0 be a finite place of F_0 that is split in F and not above p, and such that Π_{v_0} is a regular principal series, which exists by Lemma 4.3.3. Let f'_{v_0} be as in Lemma 4.3.1 (3), and for $v \in P'$ let f'_v be as in Lemma 4.3.1 (1). Let $f'_{S\infty}$ be as given by Proposition 4.3.2 with $P = P' \cup \{v_0\} \cup \{v|p\}$, and any K that is maximal away from P, sufficiently small at the places in $P' \cup \{v_0\}$, and equal to K_p at p. Then

$$f'^p = f'_{P'v_0} f'_{S\infty} \otimes \otimes_{v \nmid P'v_0 S\infty} f'^\circ_v$$

is as desired.   □

For any $\chi \in Y(p^\infty)_L$ and any $f'^p \in \mathscr{H}(G'(\mathbf{A}^{p\infty}), L)^\circ_{\mathrm{rs}, K_p, \Pi^p, \chi^p}$, we define

$$\mathscr{L}_p(\mathrm{M}_\Pi, \cdot)_{f'^p} := \frac{4 \cdot \mathscr{I}_{K_p}(f'^p, \cdot)}{(\otimes_{v \nmid p} I_{\Pi_v})(f'_v, \cdot)}$$

away from the zero set $\mathscr{Z}(f'^p)$ of the denominator. As $\mathscr{I}_{K_p}$ restricts to $I^{\mathrm{ord}}_{K_p}$, it follows from (6.2.3) that the functions $\mathscr{L}_p(\mathrm{M}_\Pi, \cdot)_{f'^p}$ glue to a function $\mathscr{L}_p(\mathrm{M}_\Pi, \cdot)$ with the desired interpolation properties, on the complement of the polar locus $\mathscr{Z} := \bigcap_{f' \in \mathscr{H}(G'(\mathbf{A}^{p\infty}), L)^\circ_{\mathrm{rs}, K_p, \Pi^p, \chi^p}} \mathscr{Z}(f'^p) \subset \mathscr{Y}_L$. By Lemma 6.2.3, the closed subset $\mathscr{Z}$ is empty. This completes the proof of Theorem 6.1.1.

6.2.4. *Proof of Theorem 6.1.3 / II.* Part (2) of the theorem is now clear. The spectral expansion of $\mathscr{I}^{\mathrm{ord}}_{K_p}$ in part (5) then follows from the definitions and (6.2.4). This completes the proof of Theorem 6.1.3.

6.3. Derivative of the analytic RTF. We study the derivative of the distribution $\mathscr{I}_{K_p}$.

6.3.1. *Notation.* Denote by $\mathfrak{m} \subset \mathscr{O}_{\mathscr{Y}}$ the ideal of functions vanishing at $\chi = \mathbf{1}$. For a $\mathscr{Y}$-scheme $\mathscr{Y}'$ and a function $\Phi \in \mathfrak{m}\mathscr{O}(\mathscr{Y}')$, we say that Φ *vanishes at* $\chi = \mathbf{1}$ and we denote by $\partial\Phi$ be the image of Φ in $\mathfrak{m}/\mathfrak{m}^2 \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\mathscr{Y}'} = \mathscr{O}_{\mathscr{Y}'} \hat{\otimes} \Gamma_{F_0}$.

For $V \in \mathscr{V}^\circ$ a coherent or incoherent pair of definite hermitian spaces as in §2.1.3, and v a finite place of F_0, we let:

- $\mathscr{C}(G')^{\mathrm{her,ord},V} \subset \mathscr{C}(G')^{\mathrm{her,ord}}$ be the subset of those isomorphism classes of representations Π such that for each finite place u of F_0, the space V_u is the one attached to Π_u by the local Gan–Gross–Prasad conjecture (Proposition 2.4.1).

- $\mathscr{H}(G'(\mathbf{A}^p), L)^{\circ, V}_{K_p, \mathrm{rs}, \mathrm{qc}} \subset \mathscr{H}(G'(\mathbf{A}^p), L)^\circ_{K_p, \mathrm{rs}, \mathrm{qc}}$ be the subspace of those f'^p that match (spectrally and geometrically, see Proposition 3.5.3) a function on $\mathscr{H}(G^V(\mathbf{A}^p), L)^\circ := \mathscr{H}(G^V(\mathbf{A}^p), L) \otimes_L Lf^\circ_\infty$;

- $\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)_V := \prod'_{u \nmid p} \mathrm{B}'_{\mathrm{rs}, u, V_u} \subset \mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)$;

- $V(v) \in \mathscr{V}^\circ$ be the pair such that $V(v)_u \cong V_u$ exactly for $u \neq v$; it is coherent if and only if V is incoherent.

Proposition 6.3.1. *Consider the situation of Theorem 6.1.3, and let $V \in \mathscr{V}^{\circ,-}$ be an incoherent pair. For all $f'^p \in \mathscr{H}(G'(\mathbf{A}^p), L)^{\circ, V}_{K_p, \mathrm{rs}, \mathrm{qc}}$, the following hold.*

(1) For all $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}$ and all $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)$,

$$\mathscr{I}_{K_p}(f'^p, \mathbf{1}) = \mathscr{I}_{\Pi}(f'^p, \mathbf{1}) = \mathscr{I}^p_{\gamma}(f'^p, \mathbf{1}) = 0.$$

(2) There is a spectral expansion

$$\partial \mathscr{I}_{K_p}(f'^p) = \sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V}_{K_p}} \partial \mathscr{I}_{\Pi,K_p}(f'^p)$$

where

$$\partial \mathscr{I}_{\Pi,K_p}(f'^p) = \frac{1}{4} \partial \mathscr{L}_p(\mathrm{M}_{\Pi}) \cdot (\otimes_{v \nmid p} \mathscr{I}_{\Pi_v})(f'^p, \mathbf{1}).$$

(3) The function $\partial \mathscr{I}^p_{(-)}(f'^p)$ is integrable for the Radon measure $I^{\mathrm{ord}}_{\gamma,p,K_p} := I^{\mathrm{ord}}_{\gamma,p,K_p}(\mathbf{1})$, and there is a geometric expansion

$$
\begin{aligned}
\partial \mathscr{I}_{K_p}(f'^p) &= \int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} \partial \mathscr{I}^p_{\gamma}(f'^p) \, dI^{\mathrm{ord}}_{\gamma,p,K_p} \\
&= \int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} \sum_{v \nmid p\infty \text{ nonsplit}} \mathbf{1}_{V(v)}(\gamma) \mathscr{I}^{vp}_{\gamma}(f'^{vp}, \mathbf{1}) \cdot \partial \mathscr{I}_{\gamma,v}(f'_v) \, dI^{\mathrm{ord}}_{\gamma,p,K_p},
\end{aligned}
\tag{6.3.1}
$$

where $\mathscr{I}^{vp}_{\gamma} := \kappa(\mathbf{1}_{\infty})^{-1} \cdot \otimes_{u \nmid vp} \mathscr{I}_{\gamma,u}$, and $\mathbf{1}_{V(v)}$ denotes the characteristic function of $\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)_{V(v)}$.

Proof. Consider the geometric terms $\mathscr{I}_{\gamma}(f'^p, \mathbf{1})$. For $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0) \cap B'^{\circ}_{\infty}$, let $V_{\gamma} \in \mathscr{V}^{\circ,+}$ be the unique coherent pair such that γ matches an orbit in $B_{\mathrm{rs}}(F_0)_{V_{\gamma}}$ as in (3.5.4); let $\Sigma(\gamma, V)$ be the non-empty finite set of non-archimedean (and necessarily nonsplit) places of F_0 such that $V_{\gamma,v} \not\cong V_v$. If $v \in \Sigma(\gamma, V)$, then by the assumption on f'^p we have $I_{\gamma,v}(f'_v, \mathbf{1}) = 0$; hence $\mathscr{I}^p_{\gamma}(f'^p)$ vanishes at $\mathbf{1}$ to order at least $|\Sigma(\gamma, V)| \geq 1$. Moreover, if $v \in \Sigma(\gamma, V)$ then

$$\partial \mathscr{I}_{\gamma}(f'^p) = \mathscr{I}^{vp}_{\gamma}(f'^{vp}, \mathbf{1}) \cdot \partial \mathscr{I}_{\gamma,v}(f'_v), \tag{6.3.2}$$

which can be nonzero only if $\Sigma(\gamma, V) = \{v\}$, equivalently $V_{\gamma} = V(v)$.

Consider now a representation $\Pi = \Pi_n \boxtimes \Pi_{n+1} \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}$. Let $V_{\Pi} \in \mathscr{V}^{\circ,\epsilon(\Pi)}$ be the pair such that $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V_{\Pi}}$ (cf. Remark 2.5.6). If $\epsilon(\Pi) = -1$, then $\mathscr{L}(\mathrm{M}_{\Pi}, \mathbf{1}) = 0$ by the functional equation of Rankin–Selberg L-functions; this implies $\mathscr{I}_{\Pi}(f'^{p\infty}, \mathbf{1}) = 0$. If $\epsilon(\Pi) = +1$, then for any finite place v such that $V_{\Pi,v} \not\cong V_v$, we have $I_{\Pi_v}(f'_v, \mathbf{1}) = 0$ by the assumption on f'^p. This completes the proof of part (1). More generally, we note that the last argument shows that

$$\Pi \notin \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V} \implies I_{\Pi_v}(f'_v, \mathbf{1}) = 0 \text{ for some } v \nmid p\infty. \tag{6.3.3}$$

This shows that the sum in part (2) indeed runs over $\mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V}$; as above, this implies that $\epsilon(\Pi) = -1$ and $\mathscr{L}_p(\mathrm{M}_{\Pi}, \mathbf{1}) = 0$, which implies the second equality in (2).

We now consider part (3). By the definition of the measure $I^{\mathrm{ord}}_{\gamma,p,K_p}$, the second equality follows from (6.3.2), whose right-hand side can be nonzero only if $V_{\gamma} = V(v)$. We consider the expansion in the first equality; it will hold without the condition that f'^p is quasicuspidal, and by linearity we may thus assume that $f'^p = \otimes_v f'_v$ is a pure tensor. Suppose first that K_p is a CSDI. Since $\int dI^{\mathrm{ord}}_{\gamma,p,K_p}$ is a bounded functional, we simply differentiate under the integral sign in Theorem 6.1.3 (5). We now consider the general case. Viewing $\mathscr{I}_{K_p}$ as in (6.2.5) we have

$$\partial \mathscr{I}_{K_p}(f'^p) = \mathscr{I}_{K_p}(f'^p, \ell)$$

(see for instance, [DL, Lemma 4.38]), where the 'logarithm' $\ell\colon \Gamma_{F_0} \to \Gamma_{F_0}^{\mathrm{fr}} \subset \Gamma_{F_0}\hat{\otimes}\mathbf{Q}_p$ is the projection onto the maximal $\mathbf{Z}_p$-free quotient $\Gamma_{F_0}^{\mathrm{fr}}$ of Γ_{F_0}.

For $s \geq 1$, let $\ell_s\colon \Gamma_{F_0} \to \Gamma_{F_0}^{\mathrm{fr}}/p^s$ be the reduction map, and let $\widetilde{\ell}_s\colon \Gamma_{F_0} \to \Gamma_{F_0}^{\mathrm{fr}}/p^s \to \Gamma_{F_0}^{\mathrm{fr}}$ be any lift of ℓ_s, which is a linear combination of characters whose conductors at places $v|p$ do not exceed s. By the definition of $\mathscr{I}_{K_p}$, the expansion (6.2.3), and linearity, we have

$$\mathscr{I}_{K_p}(\widetilde{\ell}_s) = \lim_{N\to\infty} I_{K_p}^{\dagger}(f'^p U_{t_p}^{N!}, \widetilde{\ell}_s)$$

Then by Proposition 4.2.2 (5), we have

$$\mathscr{I}_{K_p}(\widetilde{\ell}_s) = \lim_{N\to\infty} \sum_{\gamma\in \mathrm{B}'_{\mathrm{rs}}(F_0)} I_{\gamma}^{\dagger}(f'^p U_{t_p}^{N!}, \widetilde{\ell}_s). \tag{6.3.4}$$

By Lemma 6.2.1, up to multiplying f'^p by a power of p independent of s we have that all terms in (6.3.4) are p-integral; hence it makes sense to consider the reduction of that identity modulo p^s,

$$\mathscr{I}_{K_p}(\ell_s) = \lim_{N\to\infty} \sum_{\gamma\in \mathrm{B}'_{\mathrm{rs}}(F_0)} I_{\gamma}^{\dagger}(f'^p U_{t_p}^{N!}, \ell_s)$$

in $\Gamma_{F_0}^{\mathrm{fr}}/p^s$. Now $\ell_s = \sum_{v\nmid\infty} \ell_{s,v}$, where $\ell_{s,v} := \ell_{s|F_{0,v}^{\times}}$, so from Remark 4.5.1, the γ-summand equals

$$\sum_{v\nmid\infty} \frac{I_{\gamma}(f'_{\infty})}{\kappa(\mathbf{1}_{\infty})\kappa_{\infty}(\gamma', \mathbf{1})} \int_{\mathrm{H}_1(\mathbf{A}^{\infty})} \int_{\mathrm{H}_2(\mathbf{A}^{\infty})} f'^{\infty}(h_1^{-1}\gamma' h_2)\ell_{s,v}(h_{1,v})\eta(h_2) \frac{d^{\natural}h_1 d^{\natural}h_2}{d^{\natural}g} \tag{6.3.5}$$

in $\Gamma_{F_0}^{\mathrm{fr}}/p^s$; here $\gamma' \in \mathrm{G}'_{\mathrm{rs}}(F_0)$ is any preimage of γ. (Note that only finitely many v-summands are nonzero, hence it is trivial to interchange sum and integration.) For $v \nmid p$, the v-summand is

$$\frac{I_{\gamma}(f'_{\infty})}{\kappa(\mathbf{1}_{\infty})\kappa_{\infty}(\gamma', \mathbf{1})} I_{\gamma,p}^{\dagger}(U_{t_p}^{N!}, \mathbf{1})\mathscr{I}_{\gamma}^{vp\infty}(f'^{vp\infty}, \mathbf{1}) \cdot \mathscr{I}_{\gamma,v}(f'_v, \ell_{s,v})$$

$$\equiv \frac{1}{\kappa(\mathbf{1}_{\infty})\kappa_{\infty}(\gamma', \mathbf{1})} I_{\gamma,p}^{\dagger}(U_{t_p}^{N!}, \mathbf{1})\mathscr{I}_{\gamma}^{vp}(f'^{vp}, \mathbf{1}) \cdot \partial\mathscr{I}_{\gamma,v}(f'_v).$$

For $v|p$, the v-summand in (6.3.5) is a multiple of $\mathscr{I}_{\gamma}^p(f'^p, \mathbf{1})$, which is zero by part (1). Therefore $\partial\mathscr{I}_{K_p}(f'^p)$ is congruent to

$$\mathscr{I}_{K_p}(f'^p, \ell) \equiv \lim_{N\to\infty} \sum_{\gamma\in \mathrm{B}'_{\mathrm{rs}}(F_0)} I_{\gamma,p}^{\dagger}(U_{t_p}^{N!}, \mathbf{1}) \cdot \partial\mathscr{I}_{\gamma}^p(f'^p)$$

in $\Gamma_{F_0}^{\mathrm{fr}}/p^s$ for all s. We conclude that the above congruences amount to an equality in $\Gamma_{F_0}^{\mathrm{fr}}$; by definition, the right-hand side is

$$\int_{\mathrm{B}'_{\mathrm{rs}}(\mathbf{A}^p)} \partial\mathscr{I}_{\gamma}^p(f'^p) \, dI_{\gamma,p,K_p}^{\mathrm{ord}},$$

as desired. $\qquad\square$

Remark 6.3.2. The above discussion is valid when f'^p has regular semisimple support. In the more general case of regular support, we need to use the result mentioned in Remark 4.2.4. Analogously to Lemma 3.3.4, if f'^p has regular support at *two* split places, then the contributions to $\partial\mathscr{I}(f'^p)$ of those orbits that are not semisimple vanish.

7. Evaluation of p-adic orbital integrals

The goal of this section is to prove Proposition 5.3.2, of which we retain all the assumptions.

We also keep using the notation of § 5.1; however, at various steps of our descent into the argument, we will lighten (and sometimes recycle) the notation for the sake of readability. We start by dropping all apices from the notation, writing for instance f and G in place of f' and G'.

We define involutions w and ι on $\mathbf{Z}^\nu$ by $(\lambda^w)_i := \lambda_{\nu+1-i}$ and $\lambda^\iota := -\lambda^w$, and a notion of positivity by declaring $\lambda \in \mathbf{Z}^{\nu,+}$ if $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq \nu - 1$; thus ι preserves $\mathbf{Z}^{\nu,+}$. We also write $\lambda \succeq \lambda'$ if $\lambda - \lambda' \in \mathbf{Z}^{\nu,+}$. Then $\varpi^\lambda \in \underline{T}^+_{\nu,*}$ if and only if $\lambda \in \mathbf{Z}^{\nu,+}$, and $(\varpi^\lambda)^\iota = \varpi^{\lambda^\iota}$.

7.1. Reduction to p-adic linear algebra.

Extending the notation from (3.3.15), let $\mathrm{p}_\nu \colon G_\nu \to G_{\nu,0} \times G_{\nu,0}/F_0^\times$ be the projection, and let $\mathrm{p}_{\nu,*} \colon \mathscr{H}(G_\nu) \to \mathscr{H}(G_{\nu,0} \times G_\nu/F_0^\times)$ be the pushforward map. Thus $\mathrm{p} := \mathrm{p}_n \times \mathrm{p}_{n+1} \colon \widetilde{G} \to G$ and $\mathrm{p}_* = \mathrm{p}_{n,*} \otimes \mathrm{p}_{n+1,*}$.

Let $c \geq 1$ be such that K is of level $\leq c$. By the positivity condition on $f^\dagger$ and linearity, we may assume that

$$f_\nu^\dagger = U_{t',K} = \mathrm{p}_{\nu,*}(f_{\nu,1}^\dagger \otimes f_{\nu,2}^\dagger), \qquad f_{\nu,1}^\dagger = K_{\nu,0}\varpi^{\lambda_{\nu,1}}K_{\nu,0}, \quad f_{\nu,2}^\dagger = K_{\nu,0}^{w_c}\varpi^{\lambda_{\nu,2}^w}K_{\nu,0}^{w_c} \qquad (7.1.1)$$

for some $\lambda_{\nu,i} \in \mathbf{Z}^\nu$ with $\lambda_{\nu,i}, \lambda_{\nu,i}^\iota \succeq (s + c)\rho_\nu$.

We decompose the function

$$f = f_{K,s} = (5.2.10) = q_0^{d(n)(2s-c)} \cdot m_s U_t^{-s} U_{[1;t_0]}^c f^\dagger e_K[1; w_{0,c}^{-1}] = f_n \otimes f_{n+1}$$

where each f_ν is a Hecke measure on $G_\nu/F_0^\times$, and further decompose

$$f_\nu = q^{c(\nu)s} m_{\nu,s} f_\nu^\dagger U_{t_\nu}^{-s} = \mathrm{p}_{\nu,*}(f_{\nu,1} \otimes f_{\nu,2}),$$

where $f_{\nu,i} \in \mathscr{H}(G_{\nu,0})$ are defined up to some scalar ambiguities that we do not need to resolve. We also denote $f_i = f_{n,i} \otimes f_{n+1,i}$ for $i = 1, 2$.[15]

Fix a representative $\gamma = [\gamma_0; 1] = [(\gamma_{n,0}, \gamma_{n+1,0}); (1_n, 1_{n+1})] \in G$ under the decomposition $G = (G_{n,0} \times G_{n+1,0})^2/F_0^{\times,2}$.

Decompose $H_1 = H_{1,0}^2$, and write $h_1 \in H_1$ as $h_1 = (h_{1,0}, h'_{1,0})$. In the orbital integral (3.3.4), we first integrate over H_2 (noting that $\eta = 1$), to find

$$\chi(\gamma_n)I_\gamma(f^\dagger, \chi) = \int_{H_1} \int_{H_2} f([h_{1,0}^{-1}\gamma_0 h_2; h'^{-1}_{1,0} h_2])\, dh_2\, \chi(h_1)\, dh_1$$

$$= \int_{(H_{1,0})^2} f^*(h_{1,0}^{-1}\gamma_0 h'_{1,0})\, \chi((h_{1,0}, h'_{1,0}))\, dh_{1,0} dh'_{1,0} \qquad (7.1.2)$$

where

$$f^* = f_1 * f_2^\vee \in \mathscr{H}(G_{n,0} \times G_{n+1,0}).$$

Lemma 7.1.1. *Assume that* $f^\dagger = (7.1.1)$, *and let* $\lambda_\nu := \lambda_{\nu,1} + \lambda_{\nu,2}^\iota \succeq 2(s + c)\rho_\nu$. *Then* $f^* = f_n^* \otimes f_{n+1}^*$ *for*

$$f_\nu^* := q_0^{c(\nu)(2s-c)} m_s K \varpi^{\lambda_\nu - 2s\rho_\nu} w K m_s^{-1} \in \mathscr{H}_{G_\nu}$$

---

[15]The context should prevent any possible confusion from the clash of notation with $f_n \in \mathscr{H}(G_n/F_0^\times)$, since the integer in this f_n will never be specialized.

Proof. Let $\sigma_\nu := (\nu-1,\ldots,0) \in \mathbf{Z}^\nu$, so that $t = \varpi^{\sigma_\nu}$ and $\sigma_\nu + \sigma_\nu^\iota = 2\rho_\nu$. Abbreviate $w_{(c)} = w_{\nu,0,(c)}$; $m_s = m_{\nu,0,s}$; $t = t_{\nu,0}$; $K = K_{\nu,0}$; $K' = K^{w_c}$; $K'' = K_{\nu,0}^{\langle c\rangle}$. Then

$$
\begin{aligned}
f_\nu^* = f_{\nu,1} * f_{\nu,2}^\vee &= q_0^{c(\nu)(2s-c)} m_s U_t^{-s} f_{\nu,1}^\dagger K w_c K' (U_t^{c-s} f_{\nu,2}^\dagger)^\vee m_s^{-1} \\
&= q_0^{c(\nu)(2s-c)} m_s K'' \varpi^{\lambda_{\nu,1} - s\sigma_\nu} K w_c K' \varpi^{-\lambda_{\nu,2} + (s-c)\sigma_\nu} K'' m_s^{-1} \\
&= q_0^{c(\nu)(2s-c)} m_s K'' \varpi^{\lambda_{\nu,1} - s\sigma_\nu} K \varpi^{-\lambda_{\nu,2}^w + s\sigma_\nu^w} w K'' m_s^{-1} \\
&= q_0^{c(\nu)(2s-c)} m_s K \varpi^{\lambda_{\nu,1} - s\sigma_\nu} K \varpi^{-\lambda_{\nu,2}^w + s\sigma_\nu^w} w K m_s^{-1} \\
&= q_0^{c(\nu)(2s-c)} m_s K \varpi^{\lambda_\nu - 2s\rho_\nu} w K m_s^{-1},
\end{aligned}
$$

where we have used the symmetry of K'' and the algebra rules of Lemma 5.1.2. $\qquad\square$

Let

$$
X_\nu^\circ := \varpi^{\lambda_\nu - 2s\rho_\nu} w_{\nu,0} \in G_{\nu,0}. \tag{7.1.3}
$$

By Lemma 7.1.1, the integrand in (7.1.2) is non-vanishing at h_1 if and only if

$$
h_{1,0}^{-1} \gamma_{\nu,0} h_{1,0}' \in m_{\nu,0,s} K_{\nu,0} X_\nu^\circ K_{\nu,0} m_{\nu,0,s}^{-1}. \tag{7.1.4}
$$

Therefore, if the orbital integral $I_\gamma(f^\dagger, \chi)$ is non-vanishing, up to changing the representative γ_0 in its $H_{1,0}$-orbit we may and will assume that

$$
\gamma_{\nu,0} \in m_{\nu,0,s} K_{\nu,0} X_\nu^\circ K_{\nu,0} m_{\nu,0,s}^{-1}. \tag{7.1.5}
$$

We introduce the convenient variables

$$
X_\nu := m_{\nu,0,s}^{-1} \gamma_{\nu,0} m_{\nu,0,s}. \tag{7.1.6}
$$

Then (7.1.5) is equivalent to

$$
X_\nu \in K_{\nu,0} X_\nu^\circ K_{\nu,0} \tag{7.1.7}
$$

and (7.1.4) is equivalent to

$$
m_{\nu,0,s}^{-1} h_{1,0}^{-1} m_{\nu,0,s} \cdot X_\nu \cdot m_{\nu,0,s}^{-1} h_{1,0}' m_{\nu,0,s} \in K_{\nu,0} X_\nu^\circ K_{\nu,0}. \tag{7.1.8}
$$

We reduce Proposition 5.3.2 to the following.

Proposition 7.1.2. *Let* $X_\nu^\circ := \varpi^{\lambda_\nu'} w_{\nu,0}$ *for some* $\lambda_\nu' \in \mathbf{Z}^{\nu,+}$, *and let* $(X_n, X_{n+1}, h_1) \in G_{n,0} \times G_{n+1,0} \times H_1$ *satisfy* (7.1.7), (7.1.8) *for* $\nu = n, n+1$. *Then* $h_1 = (h_{1,0}, h_{1,0}') \in K_H^{(s)}$.

Corollary 7.1.3. *The function* $f = (5.2.10)$ *(where K is a CSDI) has regular support.*

Proof. Let $\gamma \in G$ be in the support of f. The set $K_{\gamma,f} \subset H_1$ of those h_1 satisfying (7.1.3) for $\nu = n, n+1$ is stable under the stabilizer $H_\gamma \subset H_1$. Assume that γ is not regular, so that H_γ is nontrivial. Then H_γ is isomorphic to the F_0-points of a non-trivial closed subgroup $\mathrm{H}_\gamma \subset \mathrm{H}_1$. Using the classification of stabilizer subgroups for the H_1-action on G by Rallis and Schiffmann [RS] (note that the question can be reduced to the action of GL_n on GL_{n+1} considered in [RS]), we see that, if H_γ is not trivial, then it has positive dimension and H_γ is non-compact. In particular, the set cannot be contained in a compact subset. This contradicts Proposition 7.1.2. $\qquad\square$

Lemma 7.1.4. *Proposition 7.1.2 implies Proposition 5.3.2.*

Proof. By linearity we may assume that $f^\dagger$ is of the form (7.1.1). Let $X^\circ := (X_n^\circ, X_{n+1}^\circ) \in G_{n,0} \times G_{n+1,0}$ (which depends on $f^\dagger$) be as in (7.1.3), and let $B_K^\dagger = B_K^\dagger(f^\dagger) \subset B'$ be the image of

$$m_{0,s}^{-1} K_0 X^\circ K_0 m_{0,s}^{-1} \times \{1\} \subset G.$$

We have already noted that if $\gamma \notin B_K^\dagger$, then $I_\gamma(f^\dagger) = 0$. Assume thus that $\gamma \in B_K^\dagger$, and pick a representative of the form $[\gamma_0; 1]$. Proposition 7.1.2 and the discussion preceding it, applied to $X_\nu = (7.1.6)$ and $\lambda_\nu' = \lambda_\nu - 2s\rho_\nu$, show that the integrand

$$f_{H,\gamma_0,\chi}^* : h_1 \longmapsto \chi(h_1) f^*(h_{1,0}^{-1} \gamma_0 h_{1,0}')$$

in (7.1.2) has support contained in $K_H^{(s)}$. Thus in order to prove Proposition 5.3.2 we need to show

$$f_{H,\gamma_0,\chi|K_H^{(s)}}^* = q^{d(n)s}. \tag{7.1.9}$$

Recall the observation from (5.1.8) that if $h_{1,0} \in K_{H,0}^{(s)}$, then $m_{0,s}^{-1} h_{1,0}^{-1} m_{0,s} \in K_0^{\langle s+1 \rangle} \subset K$, and similarly for $h_{1,0}'^{-1}$. Therefore the equivalent form (7.1.8) of (7.1.4) and the fact that $\chi_{|K_H^{(s)}} = 1$ imply (7.1.9). $\square$

In §7.2 we reduce Proposition 7.1.2 to a simpler statement, to be proved in the remainder of this section.

7.2. Contraction.

From now until the end of the section, we lighten the notation by: dropping all subscripts '0'; writing h in place of $h_{1,0}$, and h' in place of $h_{1,0}'$; and writing $m_s \in \mathrm{GL}_{n+1}(F)$ in place of $m_{n+1,s}$, whereas we recall that $m_{n,s} = t_n^s$.

We extract, from the pair of conditions on h, h' in (7.1.8), a single condition on h.

Let $e_{n+1,n} = \binom{1_n}{0} \in M_{n+1,n}(F)$ be the matrix with rows $(e_1, \ldots, e_n, 0)$. Denote $\underline{s} := (s, \ldots, s) \in \mathbf{Z}^n$ and $\varpi_n := \varpi^{\underline{1}} = \varpi 1_n \in \mathrm{GL}_n(F)$, and define the $(n+1) \times n$ matrices

$$X := X_{n+1} m_s^{-1} e_{n+1,n} t_n^s X_n^{-1},$$

$$X^\circ := X_{n+1}^\circ m_s^{-1} e_{n+1,n} t_n^s X_n^{\circ,-1} = \varpi^{\lambda_{n+1}'} w_{n+1} \begin{pmatrix} \varpi^{-\lambda_n' - 2s\rho_n - \underline{s}} \\ 0 \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & & \varpi^{\lambda_n} & \\ & 0 & \cdots & \\ 0 & \varpi^{\lambda_2} & & \\ \varpi^{\lambda_1} & & & \end{pmatrix},$$

where in the second-last matrix $0 \in (F^n)^{\mathrm{t}}$, and $\lambda_i := (\lambda_{n+1}')_{n+2-i} - (\lambda_n')_i - (n+2-2i)s$. Then

$$\lambda_{i+1} - \lambda_i \geq 2s$$

for all $1 \leq i \leq n-1$.

Let

$$\bar{h}_s := m_s^{-1} h^{-1} m_s = \begin{pmatrix} t_n^{-s} w_n^{-1} h^{-1} w_n t_n^s & \varpi_n^{-s} t_n^{-1} (w^{-1} h^{-1} - 1_n) u \\ & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(F),$$

$$h_s := t_n^{-s} h t_n^s \in \mathrm{GL}_n(F). \tag{7.2.1}$$

Denote by Y_ν the left-hand side of (7.1.8). Then those equations imply that

$$Y_{n+1} m_s^{-1} e_{n+1,n} t_n^s Y_n^{-1} = \overline{h}_s X h_s \in K_{n+1} X_{n+1}^\circ K_{n+1} m_s^{-1} \begin{pmatrix} t_n^s K_n X_n^{\circ,-1} K_n \\ 0 \end{pmatrix}. \tag{7.2.2}$$

We simplify the right-hand side. First, we have

$$K_n X_n^{\circ,-1} K_n = K_n w_n \varpi^{-\lambda_n'} K_n = K_n^{\langle 2s \rangle} w_n \varpi^{-\lambda_n'} K_n,$$

where the group $K_n^{\langle 2s \rangle}$ is as in § 5.1.4 (the second equality can be shown by observing that the quotient $K_n^{\langle 2s \rangle} \backslash K_n$ is represented by lower-triangular matrices). By the quasi-symmetry of $K_n^{\langle 2s \rangle}$, we have

$$K_{n+1} m_s^{-1} \begin{pmatrix} t_n^s K_n w_n \varpi^{-\lambda_n'} K_n \\ 0 \end{pmatrix} = K_{n+1} \begin{pmatrix} \varpi_n^{-s} t_n^{-s} w_n t_n^s K_n^{\langle 2s \rangle} w_n^{-1} \varpi^{-\lambda_n'} K_n \\ 0 \end{pmatrix}$$

$$= K_{n+1} \begin{pmatrix} \varpi_n^{-s} \cdot \varpi^{-2s\rho_n} w_n K_n^{\langle s \rangle} w_n \varpi^{2s\rho_n} \varpi^{-\lambda_n' - 2s\rho_n} K_n \\ 0 \end{pmatrix} = K_{n+1} \begin{pmatrix} K_n^{\langle s \rangle} \varpi^{-\lambda_n' - 2s\rho_n - \underline{s}} K_n \\ 0 \end{pmatrix}$$

Therefore (7.2.2) is equivalent to

$$\overline{h}_s X h_s \in K_{n+1} \varpi^{\lambda_{n+1}'} w_{n+1} K_{n+1} \begin{pmatrix} K_n^{\langle 2s \rangle} \varpi^{-\lambda_n' - 2s\rho_n - \underline{s}} K_n \\ 0 \end{pmatrix} K_n = K_{n+1} X^\circ K_n,$$

where the identity follows from writing

$$K_{n+1} \begin{pmatrix} K_n^{\langle 2s \rangle} \varpi^{-\lambda_n' - 2s\rho_n - \underline{s}} K_n \\ 0 \end{pmatrix} = \lim_{\varepsilon \to 0} K_{n+1} \begin{pmatrix} \varpi^{-\lambda_n' - 2s\rho_n - \underline{s}} & \\ & \varepsilon \end{pmatrix} K_{n+1} e_{n+1,n}$$

and applying the multiplication rules of Lemma 5.1.2. We conclude that we have

$$X \in K_{n+1} X^\circ K_n$$
$$\overline{h}_s X h_s \in K_{n+1} X^\circ K_n, \tag{7.2.3}$$

where the first containment follows from the above calculation and (7.1.7).

We show that the following solution to the contracted problem (7.2.3) implies Proposition 7.1.2.

Proposition 7.2.1. *Let K_ν be a deeper Iwahori of level $\leq s$. Let*

$$X^\circ = \begin{pmatrix} & & & & 0 \\ & & & \varpi^{\lambda_n} & \\ & & 0 & \cdots & \\ & 0 & \varpi^{\lambda_2} & & \\ \varpi^{\lambda_1} & & & & \end{pmatrix} \in M_{(n+1) \times n}(F) \tag{7.2.4}$$

with $\lambda_{i+1} \geq \lambda_i + 2s$ for all $1 \leq i \leq n - 1$, and let $X \in K_{n+1} X^\circ K_n$.
If $h \in \mathrm{GL}_n(F)$ satisfies

$$\overline{h}_s X h_s \in K_{n+1} X^\circ K_n$$

with the notation (7.2.1), then $h \in K_H^{(s)}$.

Lemma 7.2.2. *Proposition 7.2.1 implies Proposition 7.1.2.*

Proof. We revert for a moment to the notation of Proposition 7.1.2. The discussion preceding Proposition 7.2.1 shows that this proposition implies the conclusion that $h_{1,0} \in K_{H,0}^{(s)}$. Observe now that $(X_n^{\circ,-1}, X_{n+1}^{\circ,-1}; X_n^{-1}, X_{n+1}^{-1}; h_{1,0}', h_{1,0})$ also satisfies the hypothesis of Proposition 7.1.2. Then the previous argument applied to these data shows that $h_{1,0}' \in K_{H,0}^{(s)}$ as well. $\qquad\square$

The proof of Proposition 7.2.1 will occupy the rest of this section.

7.2.1. *Iwahori-invariants from minors.* We say that a size-r minor M of a matrix $X \in M_{m \times n}(F_0)$ is

— *Southwest principal* if it is obtained by deleting all but the last r rows and all but the first r columns of X;

— *quasi-SW-principal* if $r \geq 2$ and M contains the Southwest principal minor of size $r - 1$;

— *loose* if $r = 1$ and M is one of the entries of the last row of X.

Definition 7.2.3. Fix integers $\lambda_1 < \cdots < \lambda_n$. We say that $X \in M_{(n+1) \times n}(F)$ satisfies the *Minor Condition* if for every $1 \leq r \leq n$, every $r \times r$-minor $M_X^{(r)}$ of X, and the Southwest-principal $r \times r$-minor $P_X^{(r)}$, we have

$$v(\det M_X^{(r)}) \geq \sum_{i=1}^{r} \lambda_i, \qquad v(\det P_X^{(r)}) = \sum_{i=1}^{r} \lambda_i \qquad (7.2.5)$$

We say that $X \in M_{(n+1) \times n}(F)$ satisfies the *Weak Minor Condition* if (7.2.5) holds for all non-loose minors.

The first example of a matrix satisfying the Minor Condition is $X^\circ = (7.2.4)$.

Lemma 7.2.4. *Let $X, X' \in M_{(n+1) \times n}(F)$.*

(1) If

$$X' \in \mathrm{Iw}_{n+1} X \, \mathrm{Iw}_n,$$

then X satisfies the Minor Condition if and only if X' does;

(2) if

$$X' \in \begin{pmatrix} \mathrm{Iw}_n & \\ & 1 \end{pmatrix} X \, \mathrm{Iw}_n,$$

then X satisfies the Weak Minor Condition if and only if X' does.

Proof. This follows from the Cauchy–Binet formula for minors of products. $\qquad\square$

The next two subsections contain auxiliary lemmas to be deployed in the proof of our proposition in § 7.5 – the reader may wish to glance at § 7.5 before continuing.

7.3. **First auxiliary lemma.** We define some variants of the condition $h \in K_H^{(s)}$.

Definition 7.3.1. For $s \geq 1$, we say that a matrix $h \in \mathrm{GL}_n(F)$ is

— *s-small* if for all $1 \leq i, j \leq n$,

$$v(h_{ii}) = 0 \quad \text{and} \quad v(h_{ij}) \geq |j - i|s; \qquad (7.3.1)$$

— *upper-s-small up to row i* if there is a decomposition

$$h = h_-^{(i)} h_+^{(i)}$$

where $h_+^{(i)}$ is s-small, and $h_-^{(i)}$ admits a block decomposition

$$h_-^{(i)} = \begin{pmatrix} \alpha & \\ * & * \end{pmatrix} \tag{7.3.2}$$

such that $\alpha \in M_i(F)$ is lower-triangular with units on the diagonal.

— *extremely s-small* if it is s-small and $(wh - 1_n)u = 0$.

Remark 7.3.2. The set of extremely s-small matrices is a subgroup of $K_H^{(s)}$, which in turn is a subgroup of the group of s-small matrices. If h is of the form $h_-^{(i-1)}$ and it satisfies (7.3.1) for all $j \leq i$, then h is upper-s-small up to row i. (In fact, there is a decomposition $h = h_-^{(i)} h_+^{(i)}$ with $h_+^{(i)}$ differing from the identity only in row i.)

From now until the rest of this section, we write t in place of t_n. We denote $h^{-w} := w_n h^{-1} w_n^{-1}$ for $h \in \mathrm{GL}_n(F)$, and we simply denote by 0 the zero row vector of length n. The following remark will often be used in conjunction with Lemma 7.2.4.

Remark 7.3.3. If h is s-small, then $t^{-s} h t$ and $t^{-s} h^{-w} t$ belong to Iw_n.

Lemma 7.3.4. *Let $1 \leq i \leq n$, and consider the equation*

$$X._s h := \begin{pmatrix} t^{-s} h^{-w} t^s & \\ & 1 \end{pmatrix} X t^{-s} h t^s = X', \tag{7.3.3}$$

subject to:

— *X, $X' \in M_{(n+1)\times n}(F)$ satisfy the Weak Minor Condition of Definition 7.2.3;*

— *the entries of the last i rows of X below the lower antidiagonal are zero, that is*

$$v(X_{n+2-i',i'}) = \lambda_{i'}, \qquad X_{n+2-i',j} = 0 \text{ for all } j > i' \leq i \tag{7.3.4}$$

(where the first equation is a consequence of the second one and (7.2.5));

— *$h \in \mathrm{GL}_n(F)$.*

We have:

(1) for given X, every solution (h, X') has h upper-s-small up to row i;

(2) if h is of the form $h_-^{(i)}$ as in (7.3.2), then X' also satisfies (7.3.4).

(3) for given X', there exists a solution (h, X) with h extremely s-small.

Proof. We proceed by induction on i. Write

$$X = \begin{pmatrix} A \\ c \end{pmatrix}, \quad X' = \begin{pmatrix} A' \\ c' \end{pmatrix}$$

with $A, A' \in M_{n\times n}(F_{0,p})$

Consider first $i = 1$. The last row of (7.3.3) reads

$$c'_j = c_1 h_{1j}/\varpi^{(j-1)} \tag{7.3.5}$$

for $j \leq n$. Thus if X, X' satisfy (7.2.5), then $v(h_{11}) = 0$ and $v(h_{1j}) \geq (j-1)s$, hence the first statement is proved and the second one is immediate. On the other hand, substituting $h_{11} = 1 - \sum_{k=2}^{n} h_{ik}$, $c_1 = c_1' h_{11}^{-1}$ in (7.3.5) gives the integral linear system

$$\sum_{k=2}^{n} (c_1' \delta_{jk} + \varpi^{(k-1)s} c_j') \varpi^{(1-k)s} h_{1k} = c_j'$$

in the variables $\varpi^{(1-k)s} h_{1k}$. As the system is invertible, the third statement is proved too.

Now let $i \geq 2$ and suppose the first two statements known for $i-1$. By Remark 7.3.3 and Lemma 7.2.4, acting on the right by $h_+^{(i-1)}$ preserves the Weak Minor Condition on X'; hence we may and do replace h by $h_-^{(i-1)}$ in a decomposition $h = h_-^{(i-1)} h_+^{(i-1)}$. In other words, we may assume that for $j > i' \leq i-1$,

$$v(h_{i'i'}) = 0, \qquad h_{i',j} = 0.$$

The same conditions are then satisfied by h^{-1}.

For $j \geq i$, let

$$M^{n+2-i,j}$$

be the quasi-SW-principal minor of X' of size i whose upper-right corner is $X'_{n+2-i,j}$; thus by the induction hypothesis $M^{n+2-i,j}$ has zero entries below the antidiagonal, and its antidiagonal entries have valuations (in order, starting from the SW corner)

$$\lambda_1, \ldots, \lambda_{i-1}, v(X'_{n+2-i,j}).$$

In particular,

$$v(\det M^{n+2-i,j}) = \sum_{i'=1}^{i} \lambda_{i'} - \lambda_i + v(X'_{n+2-i,j}).$$

Hence the Minor Condition (7.2.5) implies

$$-\lambda_i + v(X'_{n+2-i,i}) = 0, \qquad -\lambda_i + v(X'_{n+2-i,j}) \geq 0 \text{ for all } j > i. \tag{7.3.6}$$

As $A' = t^{-s} h^{-w} t^s A t^{-s} h t^s$, we have for all $1 \leq j \leq n$:

$$\varpi^{-\lambda_i} X'_{n+2-i,j} = \varpi^{-\lambda_i} \sum_{k=1}^{n-1} (h^{-w})_{n+2-i,n+1-k} \varpi^{(k+1-i)s} X_{n+1-k,k-1} h_{k+1,j} \varpi^{(k+1-j)s}$$

$$= \sum_{k=1}^{i-1} h_{i-1,k}^{-1} \varpi^{(k+1-i)s} \varpi^{-\lambda_i} X_{n+1-k,k+1} h_{k+1,j} \varpi^{(k+1-j)s}, \tag{7.3.7}$$

by our assumptions on h. Moreover, for $j \geq i$ by induction hypothesis $h_{k+1,j} = 0$ for all $k < i-1$, hence

$$\varpi^{-\lambda_i} X'_{n+2-i,j} = h_{i-1,i-1}^{-1} h_{i,j} \varpi^{(i-j)s} \varpi^{-\lambda_i} X_{n+2-i,i}. \tag{7.3.8}$$

Since $h_{i-1,i-1}^{-1}$ and $\varpi^{-\lambda_i} X_{n+2-i,i}$ are units, condition (7.3.6) is equivalent to

$$v(h_{i,i}) = 0, \qquad v(h_{i,j}) \geq (j-i)s$$

for all $j > i$, establishing the first statement. If h is of the form $h_-^{(i)}$, the second statement is immediate from (7.3.8).

Consider now the third statement. After replacing X' by $X'._s(h')^{-1}$ where h' is as given by this statement for $i-1$, we may and do assume that X' satisfies (7.3.4) for $i' < i$. We seek h extremely s-small, upper-triangular and differing from the identity only in row i; hence h takes the form $h_-^{(i-1)}$, and by the second statement for $i-1$, we only need to find a solution to (7.3.8) in h (with the further simplification $h_{i-1,i-1} = 1$).

We set $h_{i,i} = 1 - \sum_{k \neq i} h_{ik}$, necessary for h to be extremely s-small, and substitute in $X_{n+2-i,i} = h_{i,i}^{-1} X'_{n+2-i,i}$. We find the linear system

$$\sum_{k=i+1}^{n} (\varpi^{-\lambda_i} X'_{n+2-i,i} \delta_{jk} + \varpi^{(k-i)s} \varpi^{-\lambda_i} X'_{n+2-i,j}) h_{ik} \varpi^{(i-k)s} = \varpi^{-\lambda_i} X'_{n+2-i,j}$$

in the variables $\varpi^{(i-k)s} h_{ik}$ for $k \geq i+1$. By our reductions, $-\lambda_i + v(X'_{n+2-i,i}) = 0$ and $-\lambda_i + v(X'_{n+2-i,j}) \geq 0$, hence the system is integral and invertible; its solvability implies the third statement. $\qquad\square$

7.4. Second auxiliary lemma.

Definition 7.4.1. We say that a lower-triangular matrix $h \in \mathrm{GL}_n(F)$ with units on the diagonal is *lower-s-small from column j* if

$$v(h_{ij'}) \geq (i - j')s \text{ for all } i > j' \geq j.$$

This is equivalent to the existence of a decomposition

$$h = {}^{(j)}h_- \cdot {}^{(j)}h_{--}$$

where ${}^{(j)}h_{--}$ is lower-triangular and s-small, and

$$^{(j)}h_- = \begin{pmatrix} \alpha_- & \\ * & 1_{n+1-j} \end{pmatrix} \tag{7.4.1}$$

with $\alpha_- \in M_{j-1}(F_{0,p})$ lower-triangular with units on the diagonal.

Remark 7.4.2. For a lower-triangular matrix h with units on the diagonal:

− h is lower-s-small from column j if and only if h^{-1} is;

− h is lower-s-small from column 1 if and only if it is s-small.

Lemma 7.4.3. *Let $1 \leq j \leq n$, and consider the equation*

$$X._s h = \begin{pmatrix} t^{-s} h^{-w} t^s & \\ & 1 \end{pmatrix} X t^{-s} h t^s = X', \tag{7.4.2}$$

subject to:

− *X, $X' \in M_{(n+1) \times n}(F)$ satisfy the Weak Minor Condition of Definition 7.2.3;*

− *the entries of X, X' below the lower antidiagonal are zero, that is*

$$v(X_{n+2-i,i}) = \lambda_i, \qquad X_{n+2-i,j'} = 0 \text{ for all } j' > i$$

(where the first equation is a consequence of the second one and (7.2.5)), and similarly for X';

– *the entries of the last $n - j$ columns of X above the lower antidiagonal are zero, that is,*

$$X_{n+2-i,j'} = 0 \text{ for all } i > j' \geq j + 1; \tag{7.4.3}$$

– *$h \in \mathrm{GL}_n(F)$ is lower-triangular with units on the diagonal.*

We have:

(1) for given X, every solution (h, X') has h lower-s-small from column j;

(2) if h is of the form $^{(j)}h_-$ as in (7.4.1), then X' also satisfies (7.4.3);

(3) for given X', there exists a solution (h, X) with h extremely s-small.

Proof. We prove this by decreasing induction on j, the case $j = n$ being trivial. Thus let $j \leq n-1$ and assume the statements proved for $j + 1$.

After replacing h by $^{(j-1)}h_-$ as in the decomposition (7.4.1), that is acting by $._s{}^{(j+1)}h_{--}$ on both sides of (7.4.2), by the induction hypothesis we are led to a situation that is equivalent for the purposes of the first two statements. Hence we may and do assume that h has the form $^{(j+1)}h_-$. For $i \geq j$, let

$$N^{n+1-i,j+1}$$

be the quasi-SW-principal minor of X' of size $j + 1$ whose upper-right corner is $X'_{n+1-i,j+1}$; thus the matrix $N^{n+1-i,j+1}$ has vanishing entries below the antidiagonal, and its antidiagonal entries (in order, starting from the SW corner) have valuations

$$\lambda_1, \ldots, \lambda_j, v(X'_{n+1-i,j+1}).$$

In particular,

$$v(\det N^{n+1-i,j+1}) = \sum_{j'=0}^{j+1} \lambda_{j'} - \lambda_{j+1} + v(X'_{n+1-i,j+1}).$$

Hence (7.2.5) implies

$$-\lambda_{j+1} + v(X'_{n+1-i,j+1}) \geq 0. \tag{7.4.4}$$

The same condition holds for X by assumption.

We have

$$\lambda_{j+1}^{-1} X'_{n+1-i,j+1} = \varpi^{-\lambda_{j+1}} \sum_{1 \leq k,l \leq n} (h^{-w})_{n+1-i,n+1-k} \varpi^{(k-i)s} X_{n+1-k,l+1} h_{l+1,j+1} \varpi^{(l-j)s}$$

$$= \sum_{k=j}^{i} h_{i,k}^{-1} \varpi^{(k-i)s} \varpi^{-\lambda_{j+1}} X_{n+1-k,j+1} \tag{7.4.5}$$

where we have used our assumptions on h and X. All terms are integral except possibly the last one, whose valuation is $v(h_{i,j}^{-1}) - (i - j)s$. That this should be non-negative, for all $i > j$, is equivalent to h being lower-s-small from column j, proving the first statement.

If moreover h has the form $^{(j)}h_-$, then in (7.4.5) all terms are zero unless $i = j$, in which case we only have the term corresponding to $i = j = k$, giving $X'_{n+1-j,j+1} = X_{n+1-j,j+1} = 0$. This proves the second statement.

For the third statement, we seek an extremely s-small matrix h that differs from the identity only in column j. Then h^{-1} satisfies the same conditions, h is of the form $^{(j-1)}h_-$, and we need it

to satisfy (7.4.5) (for some X), in whose right-hand side only the terms $k = j, i$ may be nonzero. Substituting

$$h_{jj}^{-1} := 1 - \sum_{i>j} h_{ij}^{-1}, \qquad X_{n+1-j,j+1} = (1 - \sum_{i>j} h_{ij})^{-1} X'_{n+1-j,j+1},$$

and observing that for $i \geq j + 1$ only the term $k = i$ may be nonzero in (7.4.5), we find

$$h_{ij}^{-1} \varpi^{(j-i)s} \varpi^{-\lambda_{j+1}} X_{n+1-j,j+1} = \varpi^{-\lambda_{j+1}} X'_{n+1-i,j+1}.$$

This is an invertible integral linear system

$$\sum_{k=j+1}^{i} (\varpi^{-\lambda_{j+1}} X'_{n+1-j,j+1} \delta_{kj} + \varpi^{k-j} \varpi^{-\lambda_{j+1}} X'_{n+1-i,j+1}) \varpi^{(j-k)s} h_{kj}^{-1} = \varpi^{-\lambda_{j+1}} X'_{n+1-i,j+1}.$$

in the variables $\varpi^{(j-k)s} h_{kj}^{-1}$. The solvability of the system implies our third statement. $\qquad\square$

7.5. **Proof of Propositions 5.3.2, 7.1.2, and 7.2.1.** By Lemmas 7.1.4, 7.2.2, it suffices to prove Proposition 7.2.1. Thus we need to show that for $X, Y \in K_{n+1} X^\circ K_n$, all the solutions in h to the equation

$$Y = \overline{h}_s X h_s \qquad\qquad (7.5.1)$$

have $h \in K_H^{(s)}$. By Lemma 7.2.4, both X and Y satisfy the Minor Condition.

Write $X = \binom{A}{c} \in M_{(n+1) \times n}(F)$, with $c \in F^{n,\mathrm{t}}$. Then

$$Y = \overline{h}_s X h_s = X' + X'',$$
$$X' := X_{\cdot s} h = \begin{pmatrix} t^{-s} h^{-w} t^s A t^{-s} h t^s \\ c t^{-s} h t^s \end{pmatrix}, \qquad\qquad (7.5.2)$$
$$X'' := X_{\cdot\cdot s} h := \begin{pmatrix} \varpi^{-s} t^{-s} (wh^{-1} - 1_n) u \\ 0 \end{pmatrix} c t^{-s} h t^s,$$

where the notation $X_{\cdot s} h$ is as in Lemmas 7.3.4, 7.4.3.

Note that $X'' = Y - X'$ is a rank-1 matrix whose rows are all multiples of row $n+1$ of X' (and whose last row is zero), so that the determinants of any pair of corresponding non-loose minors of Y, X' are equal. In particular, X' also satisfies the Weak Minor Condition of Definition 7.2.3.

We proceed in several steps to show that $h \in K_H^{(s)}$.

(1) By applying first Lemma 7.3.4 (3) for $i = n$, then Lemma 7.4.3 (3) for $j = 1$, we find an extremely s-small h' such that, first, $X_{\cdot\cdot s} h' = 0$ (which is automatic by the extreme smallness of h') and, second, $X_{\cdot s} h' = \overline{h}'_s X h_s$ has zero entries outside of the lower antidiagonal and of column 1. Hence, up to changing variables by such an h', we may assume X satisfies these vanishing conditions.

(2) Apply Lemma 7.3.4 (1) to the equation

$$X' = X_{\cdot s} h, \qquad\qquad (7.5.3)$$

to deduce that h is upper-s-small, $h = h_- h_+$ with h_- lower-triangular with units on the diagonal and h_+ s-small.

(3) Act on (7.5.3) by $._s h_+^{-1}$; by Remark 7.3.3 and Lemma 7.2.4 (2) this preserves the Weak Minor Condition. We can then apply Lemma 7.4.3 (1) (with $j = 1$) to the resulting equation, to conclude that h_- and h are s-small.

(4) By Remark 7.3.3 and Lemma 7.2.4 (1), we deduce that $X' = X._s h$ satisfies the full Minor Condition; in particular, all entries of X' have valuation no less than $v(\lambda_1)$. Since this also holds for the entries of Y, it must hold for the entires of X'' too. As $\lambda_1^{-1} c t^{-s} h t^s$ is integral with first entry a unit, the condition on X'' is satisfied if and only if $\varpi^{-s} t^{-s} (w h^{-1} - 1_n) u \in \mathcal{O}^n$; that is, $h \in K_H^{(s)}$.

The proof of Propositions 7.2.1, 7.1.2 and 5.3.2 is now complete.

Part 2. p-adic heights and the arithmetic relative-trace formula

We now study the p-adic heights of Gan–Gross–Prasad cycles. In §8, we recall the relevant Shimura varieties, the arithmetic diagonal cycles, and their moduli interpretations over the reflex fields. In §9, we study various integral models and prove some vanishing results for their cohomologies. In §10, we define the arithmetic relative-trace distribution encoding the heights of GGP cycles, and prove the corresponding RTF. In §11, we compare the arithmetic RTF with the derivative of the analytic RTF in order to prove our main theorems. Appendix A collects the necessary definitions and results on cycles and p-adic heights, mostly used in §10.

In §§8-9, we use slightly different notation on unitary groups from the rest of the paper.

8. Unitary Shimura varieties and arithmetic diagonals

For this section and the next one, we largely follow [RSZ20, RSZ21].

8.1. **Unitary Shimura varieties.** We keep denoting by F a CM number field with maximal totally real subfield F_0 and nontrivial F/F_0-automorphism c: $a \mapsto a^c$. For an algebraic group G over F_0, we denote its restriction of scalars to $\mathbf{Q}$ by

$$G^\flat := \mathrm{Res}_{F_0/\mathbf{Q}} G.$$

8.1.1. *Unitary Shimura data and the associated varieties.* We denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. Let ν be a positive integer. Recall from [RSZ21, §2.2] that a *generalized CM type* (or a *signature type*) of rank ν is a function $r \colon \mathrm{Hom}_{\mathbf{Q}}(F, \overline{\mathbf{Q}}) \to \mathbf{Z}_{\geq 0}$, denoted $\varphi \mapsto r_\varphi$, such that

$$r_\varphi + r_{\varphi^c} = \nu \quad \text{for all} \quad \varphi \in \mathrm{Hom}_{\mathbf{Q}}(F, \overline{\mathbf{Q}}); \tag{8.1.1}$$

here $\varphi^c := \varphi \circ c$. When $\nu = 1$, a generalized CM type is "the same" as a usual CM type, via

$$\Phi = \left\{ \varphi \in \mathrm{Hom}_{\mathbf{Q}}(F, \overline{\mathbf{Q}}) \mid r_\varphi = 1 \right\}.$$

Fix a CM type Φ of F, and let $(W, (\ ,\ ))$ be an F/F_0-hermitian vector space of dimension ν. The signatures of W at the archimedean places determine a generalized CM type r of rank n, by writing

$$\mathrm{sig}\, W_\varphi = (r_\varphi, r_{\varphi^c}), \quad \varphi \in \Phi, \quad W_\varphi := W \otimes_{F,\varphi} \mathbf{C}.$$

Consider the unitary group

$$G := U(W). \tag{8.1.2}$$

For each $\varphi \in \Phi$, choose a $\mathbf{C}$-basis of W_φ with respect to which the matrix of the hermitian form $(\ ,\ )$ is given by $\mathrm{diag}(1_{r_\varphi}, -1_{r_{\varphi^c}})$ We now define a Shimura datum $(G^\flat, \{h_G^\flat\})$, where $\{h_G^\flat\}$ is a $G^\flat(\mathbf{R})$-conjugacy class of homomorphisms $\mathbb{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_{m,\mathbf{R}} \to G_{\mathbf{R}}^\flat$. With respect to the inclusion and the identification induced by the fixed CM type Φ,

$$G^\flat(\mathbf{R}) \subset \mathrm{GL}_{F \otimes \mathbf{R}}(W \otimes \mathbf{R}) \xrightarrow[\sim]{\Phi} \prod_{\varphi \in \Phi} \mathrm{GL}_{\mathbf{C}}(W_\varphi), \tag{8.1.3}$$

we define $h_{\mathrm{G}^\flat}$ as $(h_{\mathrm{G}^\flat,\varphi})_{\varphi\in\Phi}$ where the φ-component is defined on $\mathbf{C}^\times$ by

$$h_{\mathrm{G}^\flat,\varphi}\colon z\longmapsto \mathrm{diag}\big(1_{r_\varphi},(z^c/z)1_{r_{\varphi^c}}\big).$$

The reflex field $E(\mathrm{G}^\flat,h_{\mathrm{G}^\flat})$ of this Shimura datum is the reflex field $E_{r^\natural}$ of the function $r^\natural$, characterized by

$$\mathrm{Gal}(\overline{\mathbf{Q}}/E_{r^\natural})=\big\{\,\sigma\in\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})\mid\sigma^*(r^\natural)=r^\natural\,\big\},\tag{8.1.4}$$

where we define a modified function

$$r^\natural\colon\ \mathrm{Hom}_{\mathbf{Q}}(F,\overline{\mathbf{Q}})\longrightarrow\mathbf{Z}_{\geq0}\tag{8.1.5}$$

$$\varphi\longmapsto\begin{cases}0,&\varphi\in\Phi;\\ r_\varphi,&\varphi\in\Phi^c.\end{cases}$$

We then obtain a tower of Shimura varieties $(\mathrm{Sh}_K(\mathrm{G}^\flat,\{h_{\mathrm{G}^\flat}\}))_{K\subset\mathrm{G}(\mathbf{A}^\infty)}$ over $E_{r^\natural}$.

Remark 8.1.1. The Shimura variety $\mathrm{Sh}_K(\mathrm{G}^\flat,\{h_{\mathrm{G}^\flat}\})$ is not of PEL type, i.e., it is not related to a moduli problem of abelian varieties (this can be seen already from the fact that the restriction of $\{h_{\mathrm{G}^\flat}\}$ to $\mathbf{G}_m\subset\mathbb{S}$ is not mapped via the identity map to the center of $\mathrm{G}^\flat$). However, this Shimura variety is of abelian type.

8.1.2. *A special signature type.* If the generalized CM type r satisfies

$$r_\varphi=\begin{cases}\nu-1,&\varphi=\varphi_0,\\ \nu,&\varphi\in\Phi\smallsetminus\{\varphi_0\},\end{cases}\tag{8.1.6}$$

for some CM type Φ and some $\varphi_0\in\Phi$, we say that the signature type of r is *strictly fake Drinfeld* (with respect to (Φ,φ_0)). In this case, we have $\varphi_0\colon F\xrightarrow{\sim}E_{r^\natural}$ for all $\nu\geq1$; in other words the reflex field is F via the embedding $\varphi_0\colon F\to\mathbf{C}$ (cf. [RSZ21, Example 2.3 (ii)]). In this paper, we will only consider data of strict fake Drinfeld type. We will abbreviate $\mathrm{Sh}_K(\mathrm{G})\coloneqq\mathrm{Sh}_K(\mathrm{G}^\flat,\{h_{\mathrm{G}^\flat}\})$, omitting the superscript $\flat$ and suppressing the datum $\{h_{\mathrm{G}^\flat}\}$.

8.1.3. *Hecke correspondences.* For each $K\subset\mathrm{G}(\mathbf{A}^\infty)$, and each characteristic-zero field L, we have an L-algebra homomorphism (see Definition A.3.3 for ÉtCorr)

$$T\colon\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty),L)_K\longrightarrow\text{ÉtCorr}(\mathrm{Sh}_K(\mathrm{G}))$$

$$KgK\longmapsto\left[\begin{array}{c}\mathrm{Sh}_{K'}(\mathrm{G})\\ \overset{\mathrm{nat}_1}{\swarrow}\qquad\overset{\mathrm{nat}_g}{\searrow}\\ \mathrm{Sh}_K(\mathrm{G})\qquad\qquad\mathrm{Sh}_K(\mathrm{G})\end{array}\right]\tag{8.1.7}$$

where $KgK\coloneqq\mathrm{vol}(K)^{-1}\mathbf{1}_{KgK}\,dg$, $K'\coloneqq K\cap gKg^{-1}$, and the map nat_1 is the natural map induced by the embedding $K'\subset K$ while nat_g is induced by the composition

$$\mathrm{Sh}_{K'}(\mathrm{G})\xrightarrow{\ \cdot g\ }\mathrm{Sh}_{g^{-1}K'g}(\mathrm{G})\longrightarrow\mathrm{Sh}_K(\mathrm{G})\ .$$

For the other Shimura varieties in this section, we also have Hecke correspondences defined in an entirely analogous way.

8.1.4. *Product Shimura varieties and the arithmetic diagonal.* Let Φ be a CM type, let W_n be a hermitian space of dimension $n \geq 1$, and assume that the associated generalized CM type r_n is of strict fake Drinfeld type. Let $W_{n+1} = W \oplus^{\perp} Fe$ where e has norm 1. Let $\mathrm{G}_{\nu} = \mathrm{U}(W_{\nu})$ for $\nu = n, n+1$, and let $(\mathrm{Sh}_{K_{\nu}}(\mathrm{G}_{\nu}))_{K_{\nu}}$ be the corresponding tower of Shimura varieties. We also have a product Shimura variety $\mathrm{Sh}_K(\mathrm{G}) = \mathrm{Sh}_K(\mathrm{G}) = \mathrm{Sh}_K(\mathrm{G}^{\flat}, h_{\mathrm{G}^{\flat}})$ associated with $\mathrm{G} = \mathrm{G}_n \times \mathrm{G}_{n+1}$ and $h_{\mathrm{G}^{\flat}} = h_{\mathrm{G}_n^{\flat}} \times h_{\mathrm{G}_{n+1}^{\flat}}$. Denote $\mathrm{H} := \mathrm{G}_n$. The map

$$\jmath \colon \mathrm{H} \longrightarrow \mathrm{G}$$

that is the graph of the natural embedding $\mathrm{G}_n \to \mathrm{G}_{n+1}$ induces a corresponding map of Shimura varieties

$$\jmath \colon \mathrm{Sh}_{K_{\mathrm{H}}}(\mathrm{H}) \longrightarrow \mathrm{Sh}_{K_{\mathrm{G}}}(\mathrm{G}) \tag{8.1.8}$$

whenever $K_{\mathrm{H}} \subset \jmath^{-1}(K_{\mathrm{G}}) \subset \mathrm{H}(\mathbf{A}^{\infty})$.

The target Shimura variety has dimension $2n - 1$, and the image of $\jmath$ has codimension n, in the arithmetic middle dimension (i.e., the codimension is just more than half the dimension of the ambient variety). We thus call the map (8.1.8) the arithmetic diagonal, and the image cycle (defined in more detail in § 10.2.1) the arithmetic diagonal cycle in $\mathrm{Sh}(\mathrm{G})$.

8.2. Incoherent Shimura varieties.
For our specific signature type, we may present the above Shimura varieties more symmetrically using incoherent hermitian spaces.

8.2.1. *Shimura varieties for incoherent unitary groups.* Let V be a totally positive definite incoherent F/F_0-hermitian space of dimension ν. The theory of conjugates of Shimura varieties ([MS82]; see also [Gro, ST]) shows that there exists a unique-up-to-isomorphism tower

$$(\mathrm{Sh}_K(\mathrm{G}))_{K \subset \mathrm{G}(\mathbf{A}^{\infty})}$$

over $\mathrm{Spec}\, F$ with the following property. For any CM type Φ of F and any archimedean place v_0 of F_0, let $\varphi_0 \in \Phi$ be the unique embedding above v_0, let $\mathrm{G}^{(v_0)} = \mathrm{U}(V(v_0))$ be the unitary group associated to the nearby hermitian space $V(v_0)$, and let $(\mathrm{Sh}_K(\mathrm{G}^{(v_0)}))_K$ be the tower of Shimura varieties associated with the data $(\mathrm{G}^{(v_0)}, \Phi, \varphi_0)$ as in § 8.1.2. Then

$$\mathrm{Sh}_K(\mathrm{G}) \times_{\mathrm{Spec}\, F, \varphi_0} \mathrm{Spec}\, \varphi_0(F) \xrightarrow{\;\sim\;} \mathrm{Sh}_K(\mathrm{G}^{(v_0)}) \;,$$

where we have an isomorphism $\mathrm{G}(\mathbf{A}^{\infty}) \simeq \mathrm{G}^{(v_0)}(\mathbf{A}^{\infty})$ induced from a fixed isometry $V(v_0)_v \simeq V_v$ for all $v \nmid \infty$. We will call $\mathrm{Sh}_K(\mathrm{G})$ the Shimura varieties attached to the incoherent hermitian space V (even though strictly speaking they are not Shimura varieties defined by Deligne).

From now on we will also make the assumption that all our unitary groups G are *anisotropic*; in the incoherent case this means that $\mathrm{G}^{(v_0)}$ is anisotropic for any (hence every) archimedean place $v_0 \in \mathrm{Hom}(F_0, \mathbf{R})$. Then $\mathrm{Sh}_K(\mathrm{G})$ is proper for any compact open subgroup $K \subset \mathrm{G}(\mathbf{A}^{\infty})$; this is guaranteed if $F_0 \neq \mathbf{Q}$.

8.2.2. *The arithmetic diagonal for incoherent Shimura varieties.* Fix now an incoherent pair $V = (V_n, V_{n+1}) \in \mathscr{V}^{\circ, -}$. We denote $\mathrm{G} = \mathrm{G}^V = \mathrm{G}_n^V \times \mathrm{G}_{n+1}^V := \mathrm{U}(V_n) \times \mathrm{U}(V_{n+1})$ (an incoherent unitary group as in § 1.3.1), and let $\mathrm{Sh}_{K_{\mathrm{G}}}(\mathrm{G})$ denote the product of Shimura varieties constructed in §8.1.4. Then for every place $v_0 \in \mathrm{Hom}(F_0, \mathbf{R})$, let $\mathrm{G}^{(v_0)} := \mathrm{G}_n^{(v_0)} \times \mathrm{G}_{n+1}^{(v_0)} := \mathrm{U}(V(v_0)_n) \times \mathrm{U}(V(v_0)_{n+1})$ be

the unitary group associated to the nearby hermitian space $V(v_0)$. Then there exists a projective system of varieties $(\mathrm{Sh}_K(\mathrm{G}))_{K \subset \mathrm{G}(\mathbf{A}^\infty)}$ over $\mathrm{Spec}\, F$ such that, for every embedding $\varphi_0 \colon F \to \mathbf{C}$ extending v_0 and every choice of CM type Φ such that $\varphi_0 \in \Phi$ we have

$$\mathrm{Sh}_K(\mathrm{G}) \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \varphi_0(F) \xrightarrow{\ \sim\ } \mathrm{Sh}_K(\mathrm{G}^{(v_0)}) \tag{8.2.1}$$

where $\mathrm{Sh}_K(\mathrm{G}^{(v_0)}) = \mathrm{Sh}_K(G^{(v_0),\flat}, h_{\mathrm{G}^{(v_0),\flat}})$ with $h_{\mathrm{G}^{(v_0),\flat}} = h_{\mathrm{G}_n^{(v_0),\flat}} \times h_{\mathrm{G}_{n+1}^{(v_0),\flat}}$ (the latter defined in § 8.1.1). Similarly, we have incoherent Shimura varieties $\mathrm{Sh}_{K_\mathrm{H}}(\mathrm{H})$ for the group $\mathrm{H} = \mathrm{H}^V = \mathrm{U}(V_n)$.

As in § 8.1.4, we have (finite) maps

$$\jmath \colon \mathrm{Sh}_{K_\mathrm{H}}(\mathrm{H}) \longrightarrow \mathrm{Sh}_{K_\mathrm{G}}(\mathrm{G}), \tag{8.2.2}$$

which are the pullbacks of (8.1.8) via (8.2.1).

8.3. **RSZ Shimura varieties.**

The unitary Shimura varieties above do not admit natural moduli descriptions. Hence we will relate them to RSZ Shimura varieties, which admit a PEL type moduli definition. They will play an auxiliary role when computing local heigts. We will follow [RSZ21].

8.3.1. *Shimura varieties for unitary similitude groups.*

We resume the notation from §8.1. Thus let Φ be a CM type, and let W be a hermitian space of dimension $\nu \geq 1$ whose associated generalized CM type r_ν is of strict fake Drinfeld type in the sense of §8.1.2. Recall also that $\mathbf{G}_m$ denotes the multiplicative group over $\mathbf{Q}$.

We first consider the group (over $\mathbf{Q}$)

$$\mathrm{G}^{\mathbf{Q}} := \mathrm{Res}_{F_0/\mathbf{Q}} \mathrm{GU}(W) \times_{\mathrm{Res}_{F_0/\mathbf{Q}} \mathbf{G}_{m,F_0}} \mathbf{G}_m$$

of unitary similitudes of $(W, (\ ,\ ))$ with similitude factor in $\mathbf{G}_m$.

Let $\{h_{\mathrm{G}^{\mathbf{Q}}}\}$ be the $\mathrm{G}^{\mathbf{Q}}(\mathbf{R})$-conjugacy class of the homomorphism $h_{\mathrm{G}^{\mathbf{Q}}} = (h_{\mathrm{G}^{\mathbf{Q}},\varphi})_{\varphi \in \Phi}$, where the components $h_{\mathrm{G}^{\mathbf{Q}},\varphi}$ are defined with respect to the inclusion

$$G^{\mathbf{Q}}(\mathbf{R}) \subset \mathrm{GL}_{F \otimes \mathbf{R}}(W \otimes \mathbf{R}) \xrightarrow[\sim]{\Phi} \prod_{\varphi \in \Phi} \mathrm{GL}_{\mathbf{C}}(W_\varphi), \tag{8.3.1}$$

and where each component is defined on $\mathbf{C}^\times$ by

$$h_{G^{\mathbf{Q}},\varphi} \colon z \longmapsto \mathrm{diag}(z \cdot 1_{r_\varphi}, z^c \cdot 1_{r_{\varphi^c}}).$$

We single out the special case $\nu = 1$. We let $W = W_0$ be totally definite and we write $\mathrm{Z}^{\mathbf{Q}} := \mathrm{G}^{\mathbf{Q}}$ (a torus over $\mathbf{Q}$) and $h_{\mathrm{Z}^{\mathbf{Q}}} := h_{\mathrm{G}^{\mathbf{Q}}}$. Explicitly,

$$\mathrm{Z}^{\mathbf{Q}} = \{\, z \in \mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m \mid \mathrm{Nm}_{F/F_0}(z) \in \mathbf{G}_m \,\}.$$

The reflex field of $(\mathrm{Z}^{\mathbf{Q}}, \{h_{\mathrm{Z}^{\mathbf{Q}}}\})$ is E_Φ, the reflex field of the CM type Φ.

8.3.2. *RSZ Shimura varieties.*

The Shimura varieties of [RSZ20] are attached to the group

$$\widetilde{\mathrm{G}} := \mathrm{Z}^{\mathbf{Q}} \times_{\mathbf{G}_m} \mathrm{G}^{\mathbf{Q}}, \tag{8.3.2}$$

where the maps from the factors on the right-hand side to $\mathbf{G}_m$ are respectively given by Nm_{F/F_0} and the similitude character. In terms of the Shimura data already defined, we obtain a Shimura

datum for $\widetilde{G}$ by defining the Shimura homomorphism to be

$$h_{\widetilde{G}} \colon \mathbf{C}^{\times} \xrightarrow{(h_{Z\mathbf{Q}}, h_{G\mathbf{Q}})} \widetilde{G}(\mathbf{R}).$$

Then $(\widetilde{G}, \{h_{\widetilde{G}}\})$ has reflex field $E \subset \overline{\mathbf{Q}}$ characterized by

$$\begin{aligned}
\mathrm{Gal}(\overline{\mathbf{Q}}/E) &= \big\{\, \sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \;\big|\; \sigma \circ \Phi = \Phi \text{ and } \sigma^*(r) = r \,\big\} \\
&= \big\{\, \sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \;\big|\; \sigma \circ \Phi = \Phi \text{ and } \sigma^*(r^{\natural}) = r^{\natural} \,\big\}.
\end{aligned}$$
(8.3.3)

In other words, the reflex field is the common composite $E = E_{\Phi} E_r = E_{\Phi} E_{r^{\natural}} = E_{\Phi}$ for our signature type (8.1.6).

Remark 8.3.1. The RSZ Shimura varieties are related to the unitary Shimura varieties as follows. The torus $Z^{\mathbf{Q}}$ embeds naturally as a central subgroup of $G^{\mathbf{Q}}$, which gives rise to a product decomposition

$$\begin{aligned}
\widetilde{G} &\xrightarrow{\;\sim\;} Z^{\mathbf{Q}} \times G^{\flat} \\
(z, g) &\longmapsto (z, z^{-1}g),
\end{aligned}$$
(8.3.4)

where $G^{\flat} \subset G^{\mathbf{Q}}$ is the restriction of scalars of the unitary group (8.1.2). The isomorphism (8.3.4) extends to a product decomposition of Shimura data,

$$\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) \cong \big(Z^{\mathbf{Q}}, \{h_{Z\mathbf{Q}}\}\big) \times \big(G^{\flat}, \{h_{G}^{\flat}\}\big).$$
(8.3.5)

Hence, for a decomposable compact open subgroup $K_{\widetilde{G}} = K_{Z\mathbf{Q}} \times K_{G^{\flat}}$, there is a product decomposition

$$\mathrm{Sh}_{K_{\widetilde{G}}}\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) \cong \mathrm{Sh}_{K_{Z\mathbf{Q}}}\big(Z^{\mathbf{Q}}, \{h_{Z\mathbf{Q}}\}\big) \times \mathrm{Sh}_{K_G}\big(G, \{h_G\}\big),$$

of Shimura varieties over E.

8.3.3. *Product Shimura varieties and the arithmetic diagonal.* Let now $W = (W_n, W_{n+1}) \in \mathscr{V}$ and $G := G_n \times G_{n+1} := U(W_n) \times U(W_{n+1})$ be as in § 8.1.4. Similar to (8.3.2) we set

$$\widetilde{G} := Z^{\mathbf{Q}} \times_{\mathbf{G}_m} G_n^{\mathbf{Q}} \times_{\mathbf{G}_m} G_{n+1}^{\mathbf{Q}}.$$
(8.3.6)

where $G_{\nu}^{\mathbf{Q}}$ is the similitude unitary group attached to W_{ν} as in (8.3.6). We have an analogous Shimura datum with the reflex field $E = E_{\Phi}$, and an isomorphism induced by (8.3.4)

$$\widetilde{G} \xrightarrow{\;\sim\;} Z^{\mathbf{Q}} \times G^{\flat}.$$
(8.3.7)

In this situation, we will always assume that the open compact $K_{\widetilde{G}}$ is *decomposable* of the form $K_{\widetilde{G}} = K_{Z\mathbf{Q}} \times K_G = K_{Z\mathbf{Q}} \times K_n \times K_{n+1}$. In particular, we have a finite étale morphism $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}) \to \mathrm{Sh}_{K_G}(G)_E$ over $\mathrm{Spec}\, E$.

Moreover, let $\widetilde{H} := \widetilde{G}_n$. Then we have a map $\jmath \colon \widetilde{H} \to \widetilde{G}$ and corresponding maps

$$\mathrm{Sh}_{K_{\widetilde{H}}}(\widetilde{H}) \longrightarrow \mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G})$$
(8.3.8)

that are the pullbacks of (8.1.8) along the projection $\mathrm{Sh}_{K_{Z\mathbf{Q}} \times K_G}(\widetilde{G}) \to \mathrm{Sh}_{K_G}(G)$ given by (8.3.7).

8.4. **Moduli functors over E.** We formulate the PEL type moduli functor for RSZ Shimura varieties, following [RSZ21, §3]. Denote by $(\mathrm{LNSch})_{/R}$ the category of locally noetherian schemes over a ring R, and by Sets the category of sets.

8.4.1. *The torus case.* First we consider the torus $Z^{\mathbf{Q}}$. The construction of [RSZ21, §2.2], specialized to $n = 1$, gives a Kottwitz PEL moduli functor $(\mathrm{LNSch})_{/E} \to$ Sets, which is represented by a finite étale stack $M_{0,K_{Z^{\mathbf{Q}}}}$ over $E = E_\Phi$. Since the precise definition of this functor plays only a minor auxiliary role in this paper, we omit it and refer the interested readers to *loc. cit.*; it suffices to recall that (among other data) one needs to fix a certain F/F_0-traceless element $\sqrt{\Delta} \in F^\times$ adapted to the CM type Φ. The stack $M_{0,K_{Z^{\mathbf{Q}}}}$ is isomorphic, over E, to finitely many copies of the Shimura variety $\mathrm{Sh}_{K_{Z^{\mathbf{Q}}}}(Z^{\mathbf{Q}})$. For our purposes, it suffices to work with a fixed copy, which we denote by $M^\tau_{0,K_{Z^{\mathbf{Q}}}}$.

8.4.2. *Definition of the moduli funtor.* Let now W be of dimension ν as in §§ 8.1.1, 8.3.1, and set

$$V = \mathrm{Hom}_F(W_0, W).$$

We now present the moduli functor $M_{K_{\widetilde{G}}}$ represented by the Shimura variety $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G})$. For simplicity, we will always assume

$$K_{\widetilde{G}} = K_{Z^{\mathbf{Q}}} \times K_{G}$$

where $K_G \subset G(\mathbf{A}^\infty)$ is a compact open subgroup. For each scheme S in $(\mathrm{LNSch})_{/E}$, $M_{K_{\widetilde{G}}}(S)$ is by definition the groupoid of tuples $(A_0, \iota_0, \lambda_0, \overline{\eta}_0, A, \iota, \lambda, \overline{\eta})$, where

- $(A_0, \iota_0, \lambda_0, \overline{\eta}_0)$ is an object of $M^\tau_{0,K_{Z^{\mathbf{Q}}}}(S)$;

- A is an abelian scheme over S;

- $\iota \colon F \to \mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbf{Q}$ is an action of F on A up to isogeny satisfying the Kottwitz condition of signature type r given by (8.1.6),

$$\mathrm{char}\big(\iota(a) \mid \mathrm{Lie}\, A\big) = \prod_{\varphi \in \mathrm{Hom}(F,\overline{\mathbf{Q}})} \big(T - \varphi(a)\big)^{r_\varphi} \quad \text{for all} \quad a \in F. \tag{8.4.1}$$

- λ is a quasi-polarization on A whose Rosati involution satisfies condition

$$\mathrm{Ros}_\lambda\big(\iota(a)\big) = \iota(\overline{a}) \quad \text{for all} \quad a \in F, \tag{8.4.2}$$

and

- $\overline{\eta}$ is a K_G-orbit (equivalently, a $K_{\widetilde{G}}$-orbit, where $K_{\widetilde{G}}$ acts through its projection $K_{\widetilde{G}} \to K_G$) of isometries of $\mathbf{A}_F^\infty/\mathbf{A}^\infty$-hermitian modules

$$\eta \colon \widehat{V}(A_0, A) \xrightarrow{\sim} V \otimes_F \mathbf{A}_F^\infty. \tag{8.4.3}$$

Here, denoting by $\widehat{V}(A')$ the adelic Tate module of an abelian variety A',

$$\widehat{V}(A_0, A) := \mathrm{Hom}_{\mathbf{A}_F^\infty}\big(\widehat{V}(A_0), \widehat{V}(A)\big), \tag{8.4.4}$$

endowed with its natural $\mathbf{A}_F^\infty$-valued hermitian form h,

$$h(x, y) := \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}_{\mathbf{A}_F^\infty}\big(\widehat{V}(A_0)\big) = \mathbf{A}_F^\infty, \quad x, y \in \widehat{V}(A_0, A). \tag{8.4.5}$$

Finally, there are natural functors interpreting Hecke correspondences $T(KgK)$ for $g \in G(\mathbf{A}^\infty)$.

Proposition 8.4.1 ([RSZ21]). *The functor $M_{K_{\widetilde{G}}}$ is represented by $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G})$.*

9. Integral models

We define and study various integral models of the RSZ unitary Shimura varieties introduced in the last section.

9.1. Integral models with parahoric levels.
We follow [RSZ21, § 4] with slightly different formulation. We continue with the notation of § 8, we we fix a rational prime ℓ, and we denote by $\mathcal{V}_\ell$ the set of places of F_0 over ℓ. If $\ell = 2$, then we assume that every $v \in \mathcal{V}_\ell$ is *split* in F.

We will assume that $K_{\mathrm{Z^Q},\ell} \subset \mathrm{Z^Q}(\mathbf{Q}_\ell)$ is maximal. Then the auxiliary moduli stack $M_{0,K_{\mathrm{Z^Q}}}$ (respectively its substack $M_{0,K_{\mathrm{Z^Q}}}$) has a natural integral model $\mathcal{M}_{0,K_{\mathrm{Z^Q}}}$ (respectively $\mathcal{M}^\tau_{0,K_{\mathrm{Z^Q}}}$), which is finite étale over $\mathrm{Spec}\,\mathcal{O}_{E,(\ell)}$. For each $v \in \mathcal{V}_\ell$, we endow the $F_v/F_{0,v}$-hermitian space $W_v := W \otimes_F F_v$ with the $\mathbf{Q}_\ell$-valued alternating form $\mathrm{tr}_{F_v/\mathbf{Q}_\ell} \sqrt{\Delta}^{-1}(\ ,\ )$, and we fix a *vertex lattice* $\Lambda_v \subset W_v$ with respect to this form, i.e., Λ_v is an $\mathcal{O}_{F,v}$-lattice such that

$$\Lambda_v \subset \Lambda_v^\vee \subset \pi_v^{-1}\Lambda_v.$$

Here π_v denotes a uniformizer in F_v (if v splits in F, this means the image in F_v of a uniformizer for $F_{0,v}$), and $\Lambda_v^\vee \subset W_v$ denotes the dual lattice with respect to $\mathrm{tr}_{F_v/\mathbf{Q}_\ell} \sqrt{\Delta}^{-1}(\ ,\ )$.

We assume that $K_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}^\infty_{F_0})$ is of the form $K_{\mathrm{G}} = K^\ell_{\mathrm{G}} \times K_{\mathrm{G},\ell}$, where $K^\ell_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}^{\ell\infty})$ is arbitrary and where

$$K_{\mathrm{G},\ell} = \prod_{v \in \mathcal{V}_\ell} K_{\mathrm{G},v} \subset \mathrm{G}(F_{0,\ell}) = \prod_{v \in \mathcal{V}_\ell} G_v,$$

with

$$K_{\mathrm{G},v} := \mathrm{Stab}_{G_v}(\Lambda_v). \tag{9.1.1}$$

We note that if v is unramified in F, then $K_{\mathrm{G},v}$ is a maximal parahoric subgroup of $\mathrm{U}(W)(F_{0,v})$.

We then define $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ as the functor that associates to each scheme S in $(\mathrm{LNSch})_{/\mathcal{O}_{E,(\ell)}}$ the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^\ell)$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}^\tau_0(S)$;

- A is an abelian scheme over S;

- $\iota \colon \mathcal{O}_{F,(\ell)} \to \mathrm{End}_{(\ell)}(A)$ is an action up to prime-to-ℓ isogeny satisfying the Kottwitz condition (8.4.1) on $\mathcal{O}_{F,(\ell)}$;

- $\lambda \in \mathrm{Hom}(A, A^\vee)_{\mathbf{Z}_{(\ell)}}$ is a quasi-polarization on A whose Rosati involution satisfies condition (8.4.2) on $\mathcal{O}_{F,(\ell)}$; and

- $\overline{\eta}^\ell$ is a K^ℓ_{G}-orbit of isometries of $\mathbf{A}^{\ell\infty}_F/\mathbf{A}^{\ell\infty}_{F_0}$-hermitian modules

$$\eta^\ell \colon \widehat{\mathrm{V}}^\ell(A_0, A) \overset{\sim}{\longrightarrow} V \otimes_F \mathbf{A}^{\ell\infty}_F, \tag{9.1.2}$$

where

$$\widehat{\mathrm{V}}^\ell(A_0, A) := \mathrm{Hom}_{\mathbf{A}^{\ell\infty}_F}\big(\widehat{\mathrm{V}}^\ell(A_0), \widehat{\mathrm{V}}^\ell(A)\big), \tag{9.1.3}$$

and where the hermitian form on $\widehat{\mathrm{V}}^\ell(A_0, A)$ is the obvious prime-to-ℓ analog of (8.4.5).

We impose the following further conditions on the above tuples.

(i) Consider the decomposition of ℓ-divisible groups

$$A[\ell^\infty] = \prod_{v \in \mathcal{V}_p} A[v^\infty] \tag{9.1.4}$$

induced by the action of $\mathcal{O}_{F_0} \otimes \mathbb{Z}_\ell \cong \prod_{v \in \mathcal{V}_\ell} \mathcal{O}_{F_0,v}$. Since Ros_λ is trivial on $\mathcal{O}_{F_0}$, λ induces a polarization $\lambda_v \colon A[v^\infty] \to A^\vee[v^\infty] \cong A[v^\infty]^\vee$ of ℓ-divisible groups for each v. The condition we impose is that $\ker \lambda_v$ is contained in $A[\iota(\pi_v)]$ of rank $\#(\Lambda_v^\vee/\Lambda_v)$ for each $v \in \mathcal{V}_\ell$.

(ii) We require that the *sign condition*, the *Eisenstein condition* hold; we omit the definitions and refer to [RSZ21, §5].

The morphisms in the groupoid $\mathcal{M}_{K_{\widetilde{G}}}(S)$ are the obvious ones.

 We have the following result from [RSZ20, RSZ21].

Theorem 9.1.1. *The stack $\mathcal{M}_{K_{\widetilde{G}}}$ is Deligne-Mumford, and regular with strictly semistable reduction at all places u of E above ℓ, provided that u is unramified over F. It is smooth over $\mathrm{Spec}\, \mathcal{O}_{E,(\ell)}$ if the lattices Λ_v have type 0 or n for every $v \mid \ell$. The generic fibre of $\mathcal{M}_{K_{\widetilde{G}}}$ is $M_{K_{\widetilde{G}}}$.*

 Finally, there are natural functors interpreting Hecke correspondences $T(f^\ell)$ for all $f^\ell \in \mathscr{H}(\mathrm{G}(\mathbf{A}^{\ell\infty}))_{K_{\mathrm{G}}}$. The correspondences $T(f^\ell)$ are all étale.

9.2. More integral models at split places.
We need to have regular integral models for deeper levels at split places. We will consider two cases: the Iwahori case and the principal congruence subgroup case.

9.2.1. *Setup.* Fix a place $v \in \mathcal{V}_\ell$ that splits in F, say $v = w\overline{w}$. Let $u \colon E \to \overline{\mathbf{Q}}_\ell$ be a place of E above v; we will assume that E_u is unramified over $F_{0,v}$. Let $\widetilde{u} \colon \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$ be an embedding extending u. Then $\widetilde{u}$ induces a bijection $\mathrm{Hom}(F, \overline{\mathbf{Q}}) \simeq \mathrm{Hom}(F, \overline{\mathbf{Q}}_\ell)$. Let $\mathrm{Hom}_w(F, \overline{\mathbf{Q}})$ be the subset of $\mathrm{Hom}(F, \overline{\mathbf{Q}})$ consisting of $\varphi \in \mathrm{Hom}(F, \overline{\mathbf{Q}})$ such that $\widetilde{u} \circ \varphi$ induces w. The set $\mathrm{Hom}_w(F, \overline{\mathbf{Q}})$ depends only on u but not on the choice of $\widetilde{u}$. Note that the distinguished element φ_0 belongs to $\mathrm{Hom}_w(F, \overline{\mathbf{Q}})$. We will assume that the *matching condition* between the CM type Φ and the chosen place u of E is satisfied:

$$\mathrm{Hom}_w(F, \overline{\mathbf{Q}}) \subset \Phi, \tag{9.2.1}$$

cf. [RSZ20, §4.3]. Note that, for our signature type (8.1.6), this is equivalent to the condition that the restriction $r|_{\mathrm{Hom}_w(F,\overline{\mathbf{Q}})}$ of the signature function is of the form

$$r_\varphi = \begin{cases} n-1, & \varphi = \varphi_0 \in \mathrm{Hom}_w(F, \overline{\mathbf{Q}}); \\ n, & \varphi \in \mathrm{Hom}_w(F, \overline{\mathbf{Q}}) \smallsetminus \{\varphi_0\}. \end{cases} \tag{9.2.2}$$

9.2.2. *Principal congruence subgroups.* We now recall from [RSZ20, §4.3] the moduli problem in the case of principal congruence subgroups. Let m be a nonnegative integer, and define $K_{\mathrm{G},v}^m$ to be the principal congruence subgroup mod $\mathfrak{p}_v^m$ inside $K_{\mathrm{G},v}$, where $\mathfrak{p}_v$ denotes the prime ideal in $\mathcal{O}_{F_0}$ determined by v. Let

$$K_{\widetilde{G}}^m := K_{\mathbb{Z}\mathbb{Q}} \times K_{\mathrm{G}}^\ell \times K_{\mathrm{G},v}^m \times \prod_{v' \in \mathcal{V}_\ell \smallsetminus \{v\}} K_{\mathrm{G},v'} \subset K_{\widetilde{G}}.$$

Then one can extend the definition of $\mathcal{M}_{K_{\widetilde{G}},\mathscr{O}_{E,u}}$ to the case of the level subgroup $K_{\widetilde{G}}^m$ by adding a Drinfeld level-m structure at v. More precisely, consider the factors occurring in the decomposition (9.1.4) of the ℓ-divisible group $A[\ell^\infty]$,

$$A[v^\infty] = A[w^\infty] \times A[\overline{w}^\infty]. \tag{9.2.3}$$

The condition (9.2.2) implies that $A[\overline{w}^\infty]$ is a one-dimensional formal $\mathscr{O}_{F,w_0}$-module. We introduce $T_{\overline{w}}(A_0, A)[w_0^m] := \underline{\mathrm{Hom}}_{\mathscr{O}_{F,\overline{w}}}(A_0[\overline{w}^m], A[\overline{w}^m])$ and $T_{\overline{w}}(A_0, A) := \varinjlim_m T_{\overline{w}}(A_0, A)[w_0^m]$. Note that $T_{\overline{w}}(A_0, A)$ is a 1-dimensional formal $\mathscr{O}_{F,w_0}$-module. The datum we add to the moduli problem is an $\mathscr{O}_{F,\overline{w}}$-linear homomorphism of finite flat group schemes,

$$\phi \colon \pi_{\overline{w}}^{-m}\Lambda_{\overline{w}}/\Lambda_{\overline{w}} \longrightarrow T_{\overline{w}}(A_0, A)[\overline{w}^m], \tag{9.2.4}$$

which is a Drinfeld $\overline{w}^m$-structure on the target. Here $\Lambda_{\overline{w}}$ is the summand attached to $\overline{w}$ in the natural decomposition

$$\Lambda_v = \Lambda_w \oplus \Lambda_{\overline{w}} \tag{9.2.5}$$

with Λ_v the vertex lattice at v chosen in §9.1. See [RSZ20, §4.3] (which we note interchanges the roles of w and $\overline{w}$) for more details.

Then by [RSZ20, Theorem 4.7], the moduli problem $\mathcal{M}_{K_{\widetilde{G}}^m}$ is relatively representable by a finite flat morphism to $\mathcal{M}_{K_{\widetilde{G}}}$ and consequently it coincides with the normalization of $\mathcal{M}_{K_{\widetilde{G}}}$ in the generic fiber of $\mathcal{M}_{K_{\widetilde{G}}^m}$. It is regular and flat over $\mathrm{Spec}\,\mathscr{O}_{E,(u)}$. Furthermore, the generic fiber $\mathcal{M}_{K_{\widetilde{G}}^m} \times_{\mathrm{Spec}\,\mathscr{O}_{E,(u)}} \mathrm{Spec}\,E$ is canonically isomorphic to $M_{K_{\widetilde{G}}^m}$.

9.2.3. *Iwahori subgroups.* We will also need the Iwahori case. For simplicity we assume that the vertex lattice Λ_v in (9.2.5) is self-dual. We now choose a chain of $\mathscr{O}_{F,w}$-lattices

$$\Lambda_{\overline{w}} = \Lambda_{\overline{w}}^{(0)} \subset \Lambda_{\overline{w}}^{(1)} \subset \cdots \subset \Lambda_{\overline{w}}^{(n)} = \pi_w^{-1}\Lambda_{\overline{w}},$$

where each inclusion has colength one. Equivalently, we choose a full flag in the k_v-vector space $\Lambda_w/\pi_w\Lambda_w$. This chain determines uniquely a chain of vertex $\mathscr{O}_{F,v} = \mathscr{O}_{F,w} \times \mathscr{O}_{F,\overline{w}}$-lattices $\Lambda_v^{(i)} := \Lambda_w \oplus \Lambda_{\overline{w}}^{(i)}, 0 \leq i \leq n$. The stabilizer of the chain $\Lambda_v^{(i)}$ is an Iwahori subgroup Iw_v of $\mathrm{Stab}(\Lambda_v)$. To the moduli problem $\mathcal{M}_{K_{\widetilde{G}},\mathscr{O}_{E,u}}$, we add the datum of a chain of isogenies of $\mathscr{O}_{F,\overline{w}}$-divisible modules

$$\mathcal{G}_0 = T_{\overline{w}}(A_0, A) \longrightarrow \mathcal{G}_1 \longrightarrow \cdots \longrightarrow \mathcal{G}_n = \mathcal{G}_0/\mathcal{G}_0[\overline{w}] \tag{9.2.6}$$

with equal heights $\#k_v$. An equivalent datum is an Iw_v-orbit of the Drinfeld level structure

$$\phi \colon \pi_{\overline{w}}^{-1}\Lambda_{\overline{w}}/\Lambda_{\overline{w}} \longrightarrow T_{\overline{w}}(A_0, A)[\overline{w}].$$

The resulting moduli functor is then denoted by $\mathcal{M}_{K_{\widetilde{G}}^{\mathrm{Iw}_v}}$, where $K_{\widetilde{G}}^{\mathrm{Iw}_v}$ denotes the compact subgroup of $K_{\widetilde{G}}$ with the Iwahori factor at v. Then the moduli problem $\mathcal{M}_{K_{\widetilde{G}}^{\mathrm{Iw}_v}}$ is relatively representable by a finite flat morphism to $\mathcal{M}_{K_{\widetilde{G}}}$ and consequently it coincides with the normalization of $\mathcal{M}_{K_{\widetilde{G}}}$ in the generic fiber of $\mathcal{M}_{K_{\widetilde{G}}^{\mathrm{Iw}_v}}$. It is regular, proper and flat over $\mathrm{Spec}\,\mathscr{O}_{E,(u)}$. Moreover, by the theory of local models, the scheme $\mathcal{M}_{K_{\widetilde{G}}^{\mathrm{Iw}_v}}$ has strictly semistable reduction over $\mathrm{Spec}\,\mathscr{O}_{E,(u)}$ (namely, its generic fiber is smooth and every closed point of the special fiber admits an open neighborhood which is smooth over the scheme $\mathrm{Spec}\,\mathscr{O}_{E,(u)}[x_1, \cdots, x_m]/(\prod_{i=1}^m x_i - \varpi)$ for some

$m \geq 1$, cf. [Har01, Prop. 1.3]). Moreover, there is a natural morphism from $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{m=1}}$ to $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iwv}}}$, which is finite flat. There is a stratification of the special fiber $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iwv}}} \otimes k_u$, where k_u denotes the residue field of $\mathscr{O}_{E,(u)}$:

$$\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iwv}}} \otimes k_u = \bigcup_{1 \leq i \leq n} \mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iwv}}, k_u, i}, \tag{9.2.7}$$

where $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iwv}}, k_u, i}$ is the closed subscheme where the kernel of the isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ in (9.2.6) is connected, cf. [TY07, §3] for a similar case. By [TY07, Prop. 3.4] (or rather its proof), each of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iwv}}, k_u, i}$ is smooth over $\mathrm{Spec}\, k_u$.

9.2.4. *Hecke correspondences.* We recall from [RSZ20, §4.3] that, in each of the above two cases (principal and Iwahori level), there are natural functors interpreting Hecke correspondences attached to functions $\mathbf{1}_{KgK}$ for any $g \in \mathrm{G}(\mathbf{A}^\infty)$, where we simply denote $K = K_{\mathrm{G}}$:

$$\begin{array}{ccc} & \mathcal{M}_{K'_{\widetilde{\mathrm{G}}}} & \\ {}^{\mathrm{nat}_1}\swarrow & & \searrow^{\mathrm{nat}_g} \\ \mathcal{M}_{K_{\widetilde{\mathrm{G}}}} & & \mathcal{M}_{K_{\widetilde{\mathrm{G}}}} \end{array} \tag{9.2.8}$$

where $K'_{\widetilde{\mathrm{G}}} = K_{\mathbf{Z}^{\mathbf{Q}}} \times K'_{\mathrm{G}}$, and K'_{G} is a subgroup of $K_{\widetilde{\mathrm{G}}} \cap g K_{\widetilde{\mathrm{G}}} g^{-1}$. We refer to [RSZ20, §4.3] for the unexplained notation. (Note that in *loc. cit.*, the authors only consider the case of a principal congruence subgroup $K_{\mathrm{G}} = K_{\mathrm{G}}^m$. The Iwahori case is similar and may be reduced to the case $K_{\mathrm{G}} = K_{\mathrm{G}}^m$ as follows. We can factorize [IwgIw] as $e_{\mathrm{Iw}} * [KgK] * [\mathrm{Iw}g\mathrm{Iw}]$ for some $K = K_{\mathrm{G}}^m \subset \mathrm{Iw}$, and accordingly we define the correspondence for [IwgIw] as the composition of the three factors: the middle one is as in *loc. cit.*, and the other two are given by the natural map from the principal level to the Iwahori level.)

Both maps nat_1 and nat_g are finite flat, and étale if $g_\ell = 1$. The Hecke correspondence (9.2.8) induces an endomorphism (by the usual pull-back and then push-forward maps) on the group of cycles (with L-coefficients), rather than merely cycles modulo rational equivalence. This endomorphism is independent of the choice of $K'_{\widetilde{\mathrm{G}}}$ in the diagram above. The resulting map

$$T \colon \mathscr{H}(\mathrm{G}(\mathbf{A}^{\ell\infty}), L)_K \longrightarrow \text{ÉtCorr}(\mathcal{M}_K)_L$$

is a ring homomorphism. However, we do not know if the assertion remains true for the full Hecke algebra $\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty), L)_K$. When $m = 0$, the recent work of Li–Mihatsch [LM, Proposition 3.4] shows that the assertion holds. Moreover, in the Iwahori case, the away-from-ℓ Hecke correspondences preserve the stratification (9.2.7).

9.3. **Moduli functors for the product Shimura varieties.** It is now easy to extend the construction in §8.3 to the product unitary group $\widetilde{\mathrm{G}}$ defined in § 8.3.3. There are analogous moduli functors over E and over $\mathscr{O}_{E,(\ell)}$. For example, the ℓ-integral model may be succinctly defined as

$$\mathcal{M}_{K_{\widetilde{\mathrm{G}}}} = \mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}} \times_{\mathcal{M}_0^\tau} \mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_{n+1})}}, \tag{9.3.1}$$

where $K_{\widetilde{\mathrm{G}}(V_\nu)} = K_{\mathbf{Z}^{\mathbf{Q}}} \times K_\nu$ for $\nu \in \{n, n+1\}$.

The product $\mathcal{M}_{K_{\widetilde{G}}}$ may no longer be regular even if both factors are regular, and we may need to resolve the product singularity. We will need to study two cases: the vertex parahoric case at an inert place, and the Iwahori case at a split place.

9.3.1. *Vertex parahoric level at an inert place.* We first consider the vertex parahoric case from §9.1. Fix a place $v \in \mathcal{V}_\ell$ that is inert in F and we let w denote the unique place of F above v. We fix a vertex lattice $\Lambda_v^\flat \subset V_{n,v}$ of type $0 \le t \le n$ and let $\Lambda_v = \Lambda_v^\flat \oplus \langle e \rangle_{\mathcal{O}_{F,v}} \subset V_{n+1,v}$ where the hermitian norm of the special vector e has valuation $\epsilon \in \{0,1\}$.[16]Then Λ_v is a vertex lattice of type $t + \epsilon$. We let $u : E \to \overline{\mathbf{Q}}_p$ be a place of E above v and we further assume that E_u is unramified over $F_{0,v}$. We let $K_{n,v}$ and $K_{n+1,v}$ be the stabilizer of $\Lambda_v^\flat$ and Λ_v respectively. We then call $K_v = K_{n,v} \times K_{n+1,v}$ a vertex parahoric subgroup of type $(t, t + \epsilon)$. The (self-dual) hyperspecial case corresponds to type $(0,0)$.

In this case, the integral models $\mathcal{M}_{K_{\widetilde{G}(V_n)}}$ and $\mathcal{M}_{K_{\widetilde{G}(V_{n+1})}}$ have strictly semistable reduction over $\operatorname{Spec} \mathcal{O}_{E,u}$; and $\mathcal{M}_{K_{\widetilde{G}(V_n)}}$ (resp. $\mathcal{M}_{K_{\widetilde{G}(V_{n+1})}}$) is smooth over $\operatorname{Spec} \mathcal{O}_{E,u}$ only when $t \in \{0, n\}$ (resp. $t + \epsilon \in \{0, n+1\}$); see Theorem 9.1.1. When $\mathcal{M}_{K_{\widetilde{G}(V_n)}}$ or $\mathcal{M}_{K_{\widetilde{G}(V_{n+1})}}$ is non-smooth over $\operatorname{Spec} \mathcal{O}_{E,u}$, its special fiber admits a "balloon–ground" stratification ([LTX$^+$22] for $t = 1$ and [ZZh] for general t): the special fiber is a union of two Weil divisors

$$\mathcal{M}_{K_{\widetilde{G}(V_n)},k_u} = \mathcal{M}^\circ_{K_{\widetilde{G}(V_n)},k_u} \cup \mathcal{M}^\bullet_{K_{\widetilde{G}(V_n)},k_u} \tag{9.3.2}$$

where the first one $\mathcal{M}^\circ_{K_{\widetilde{G}(V_n)},k_u}$ is called the balloon stratum and the second one $\mathcal{M}^\bullet_{K_{\widetilde{G}(V_n)},k_u}$ is called the ground stratum. (When $t \in \{0, n\}$ we understand that the balloon stratum is empty.) When $\mathcal{M}_{K_{\widetilde{G}}}$ is not regular, we let $\widetilde{\mathcal{M}}_{K_{\widetilde{G}}}$ be the blow up along the product of the balloon strata of the two factors, and denote the blow-up morphism

$$\pi : \widetilde{\mathcal{M}}_{K_{\widetilde{G}}} \longrightarrow \mathcal{M}_{K_{\widetilde{G}}}. \tag{9.3.3}$$

For $(?_n, ?_{n+1}) \in \{\circ, \bullet\}^2$, we denote by $\widetilde{\mathcal{M}}^{(?_n,?_{n+1})}_{K_{\widetilde{G}},k_v}$ the strict transform of $\mathcal{M}^{?_n}_{K_{\widetilde{G}(V_n)},k_u} \times \mathcal{M}^{?_{n+1}}_{K_{\widetilde{G}(V_{n+1})},k_u}$. For later reference we record the following result from [LTX$^+$22] for $t = 1$ and [ZZh] for general t.

Proposition 9.3.1. *The scheme $\widetilde{\mathcal{M}}_{K_{\widetilde{G}}}$ is regular with strictly semistable reduction*

$$\widetilde{\mathcal{M}}_{K_{\widetilde{G}}} \otimes k_u = \bigcup_{(?_n,?_{n+1}) \in \{\circ,\bullet\}^2} \widetilde{\mathcal{M}}^{(?_n,?_{n+1})}_{K_{\widetilde{G}},k_v}, \tag{9.3.4}$$

where $\widetilde{\mathcal{M}}^{(?_n,?_{n+1})}_{K_{\widetilde{G}},k_v}$ are smooth of pure dimension $2n - 1$.

The map π is small, i.e., a proper birational morphism with the property that

$$\operatorname{codim}\{z \in \mathcal{M}_{K_{\widetilde{G}}} \mid \dim \pi^{-1}(z) \ge i\} \ge 2i + 1,$$

for all $i \ge 0$.

9.3.2. *Iwahori level at a split place.* Fix as in §9.2 a place $v \in \mathcal{V}_\ell$ that splits in F into $v = w\overline{w}$ and we let $u : E \to \overline{\mathbf{Q}}_\ell$ be a place of E above v. We further assume that E_u is unramified over

---

[16]Note that in §2.1.3 we have assumed the special vector has norm 1. For the general discussion of the geometry of Shimura varieties with parahoric levels, it is more convenient to relax this condition.

$F_{0,v}$. Then the integral model $\mathcal{M}_{K_{\widetilde{G}}}$ over $\operatorname{Spec} \mathscr{O}_{E,(u)}$ is smooth if one of the two compact open subgroups $K_{n,v}$ and $K_{n+1,v}$ is hyperspecial. When both $K_{n,v}$ and $K_{n+1,v}$ are Iwahori, $\mathcal{M}_{K_{\widetilde{G}}}$ is no longer regular and we need to resolve the product singularity. More precisely, we consider the fiber product of the stratifications from (9.2.7)

$$\mathcal{M}_{K_{\widetilde{G}},k_u,(i,j)} := \mathcal{M}_{K_{\widetilde{G}(V_n)},k_u,i} \times_{\mathcal{M}_0^\tau} \mathcal{M}_{K_{\widetilde{G}(V_{n+1})},k_u,j}. \tag{9.3.5}$$

We choose an ordering of the set $\{(i,j) \mid 1 \le i \le n, 1 \le j \le n+1\}$, and rename the component $\mathcal{M}_{K_{\widetilde{G}},k_u,(i,j)}$ as $\mathcal{M}_{K_{\widetilde{G}},k_u,r}$ where $1 \le r \le n(n+1)$. Let $\mathcal{M}_{K_{\widetilde{G}}}^{(0)} := \mathcal{M}_{K_{\widetilde{G}}}$ and for $1 \le r \le n(n+1)$ we let $\mathcal{M}_{K_{\widetilde{G}}}^{(r)}$ blow-up $\mathcal{M}_{K_{\widetilde{G}}}^{(r-1)}$ along (the strict transforms of) $\mathcal{M}_{K_{\widetilde{G}},k_u,r}$. We denote $\widetilde{\mathcal{M}}_{K_{\widetilde{G}}}$ for $\mathcal{M}_{K_{\widetilde{G}}}^{(n(n+1))}$, and denote $\widetilde{\mathcal{M}}_{K_{\widetilde{G}},k_u,(i,j)}$ for the strict transform of $\mathcal{M}_{K_{\widetilde{G}},k_u,(i,j)}$. The composition of the natural blow-up maps is denoted as

$$\pi \colon \widetilde{\mathcal{M}}_{K_{\widetilde{G}}} \longrightarrow \mathcal{M}_{K_{\widetilde{G}}}. \tag{9.3.6}$$

We also note that the resolution in the inert case earlier can also be view a special case of the current procedure: one simply orders the components such that the first one is the product of the balloon strata.

Proposition 9.3.2. *The scheme $\widetilde{\mathcal{M}}_{K_{\widetilde{G}}}$ is regular with strictly semistable reduction*

$$\widetilde{\mathcal{M}}_{K_{\widetilde{G}},k_u} = \bigcup_{\substack{1 \le i \le n, \\ 1 \le j \le n+1}} \widetilde{\mathcal{M}}_{K_{\widetilde{G}},k_u,(i,j)}, \tag{9.3.7}$$

where $\widetilde{\mathcal{M}}_{K_{\widetilde{G}},k_u,(i,j)}$ are smooth of pure dimension $2n-1$. The map π is a small map.

Proof. The first part is well-known, for example see [Har01, Prop. 2.1] or [GS95]. For the smallness, we use the explicit description as in the proof of [Har01, Prop. 2.1]. Consider a point $P = (a,b)$ on the special fiber $\widetilde{\mathcal{M}}_{K_{\widetilde{G}},k_u}$ with an open neighborhood that is smooth over

$$\operatorname{Spec} \mathscr{O}_{E,(u)}[x_1, \cdots, x_r, y_1, \cdots y_s] / (\prod_{i=1}^r x_i - \varpi, \prod_{j=1}^s y_j - \varpi)$$

for some (uniquely-determined) integers $r, s \ge 1$, such that P lies over the point defined by $x_i = 0, y_j = 0, 1 \le i \le r, 1 \le j \le s$. Then keeping track of the steps of the blow-ups in *loc. cit.* shows that the dimension of the fiber of P is $\min\{r-1, s-1\}$. Note that the locus of P with fixed $r, s \ge 1$ is contained in the union of

$$(\mathcal{M}_{K_{\widetilde{G}(V_n)},k_u,i_1} \cap \cdots \cap \mathcal{M}_{K_{\widetilde{G}(V_n)},k_u,i_r}) \times (\mathcal{M}_{K_{\widetilde{G}(V_{n+1})},k_u,j_1} \cap \cdots \cap \mathcal{M}_{K_{\widetilde{G}(V_{n+1})},k_u,j_s})$$

for all possible $1 \le i_1 \le \cdots \le i_r \le n, 1 \le j_1 \le \cdots \le j_s \le n+1$. The codimension of such locus in $\mathcal{M}_{K_{\widetilde{G}}}$ is $r+s-1 \ge 2\min\{r-1, s-1\} + 1$, which proves the smallness of the map π. $\square$

This procedure depends on the choice of an ordering and therefore it is not canonical. Nevertheless the smallness of π shows that the resolution has the property that $\pi_* \mathbf{Q}_\ell \simeq \mathrm{IC}$, the latter being the intersection complex of the $\mathbf{Q}_p$-sheaf (for $p \ne \ell$). Moreover, the resulting $\widetilde{\mathcal{M}}_{K_{\widetilde{G}}}$ and each of $\widetilde{\mathcal{M}}_{K_{\widetilde{G}},k_u,(i,j)}$ still has an action of $\mathscr{H}(\mathrm{G}(\mathbf{A}^{\ell\infty}))_{K_\mathrm{G}}$ by correspondences.

9.3.3. *Integral arithmetic diagonal.* We have an integral model

$$\jmath \colon \; \mathcal{M}_{K_{\widetilde{\mathrm{H}}}} \longrightarrow \mathcal{M}_{K_{\widetilde{\mathrm{G}}}} \tag{9.3.8}$$

of the morphism (8.3.8). In the two cases discussed above, over a place u of E, we have the small resolution $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ and we denote by

$$\widetilde{\jmath} \colon \; \widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{H}}}} \longrightarrow \widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}} \tag{9.3.9}$$

the strict transform of $\mathcal{M}_{K_{\widetilde{\mathrm{H}}}}$ along the resolution morphism. For uniformity of notation, we will put $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{H}}}} \coloneqq \mathcal{M}_{K_{\widetilde{\mathrm{H}}}}$, $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}} \coloneqq \mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$, $\widetilde{\jmath} \coloneqq \jmath$ in the cases where those schemes are already regular.

9.4. **Vanishing of absolute cohomology.**
We continue to consider the scheme $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ over $\operatorname{Spec} \mathscr{O}_{E,u}$ and we aim to prove $H^{2n}(\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}, L(n))_{\mathfrak{m}} = 0$ in the following three cases:

(1) The split-(Drinfeld-level, hyperspecial) case: the place v is split in F and in the product (9.3.1) one of the two factors has Drinfeld-level for some integer m and the other has hyperspecial level.

(2) The split-(Iwahori, Iwahori) case: the place v is split in F and in the product (9.3.1) both factors have Iwahori level; in this case $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ is the small resolution in Proposition 9.3.2.

(3) The inert-vertex-parahoric case: the place v is inert in F and in the product (9.3.1) both factors have vertex-parahoric levels (of type $(t, t+\epsilon)$); in this case $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ is the small resolution in Proposition 9.3.1.

Theorem 9.4.1. *Let $\mathfrak{m}$ be the maximal ideal of the away-from-ℓ Hecke algebra $\mathbb{T}$ attached to a representation $\pi \in \mathscr{C}(\mathrm{G})(L)$. Suppose we are in one of the above three cases, and suppose moreover that the following hold:*

(1) In the split-(Iwahori, Iwahori) case, the representation π_v is a (tempered) principal series.

(2) In the inert-vertex-parahoric case, the type (t_n, t_{n+1}) satisfies $t_n \in \{0, 1, n-1, n\}$ and $t_{n+1} \in \{0, 1, n, n+1\}$.

Then we have

$$H^{2n}(\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}, L(n))_{\mathfrak{m}} = 0. \tag{9.4.1}$$

Proof. We wish to apply the vanishing Theorem A.3.5 of Li–Liu. For this, we need to specify a stratification of the reduced special fiber of $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$. In the split-(Drinfeld-level, hyperspecial) case, for simplicity we consider the case the Drinfeld level takes place on the first factor $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}}$. Then the special fiber, denoted by Y_{n+1}, of the second factor in the product (9.3.1) is smooth. In [LL22, §4.3] the authors have defined a stratification of the reduced special fiber, denoted by Y_n, of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}}$, essentially a refinement of the Newton stratification

$$Y_n = \coprod_{i=0}^{n-1} \coprod_{M \in \mathfrak{S}_i} Y_n^{(M)},$$

where $\mathfrak{S}_i$ denotes the $\mathfrak{S}_m^i$ in *loc. cit.*. Here we simply take the stratification of $Y = Y_n \times Y_{n+1}$ as the product of the stratification of Y_n with Y_{n+1}

$$Y = \coprod_{i=0}^{n-1} \coprod_{M \in \mathfrak{S}_i} Y_n^{(M)} \times Y_{n+1}.$$

By [LL22, §4.3] this stratification of Y_n verifies the two conditions stated before Theorem A.3.5 (for $\mathscr{X} = \mathcal{M}_{K_{\widetilde{G}(V_n)}}$). It follows easily that the above stratification of Y verifies the two conditions stated before Theorem A.3.5 (for $\mathscr{X} = \mathcal{M}_{K_{\widetilde{G}}}$).

In the split-(Iwahori, Iwahori) case and the inert-vertex-parahoric case, the scheme $\mathscr{X} = \mathcal{M}_{K_{\widetilde{G}}}$ has strictly semistable reduction. The special fiber Y is already reduced and we define the stratification induced by the union (9.3.7) and (9.3.4) respectively, as follows. Let $\mathcal{J}$ denote the set of indices in (9.3.7) and (9.3.4), and denote $Y = \cup_{j \in \mathcal{J}} Y_j$. Then we define $\mathfrak{S}^i$ to be the set of subsets M of I with $\#M = \#I - i$ such that $Y^{[M]} := \cap_{j \in M} Y_j$ is non-empty (then it has codimension $\#M + 1$ in $\mathscr{X}$). Set $Y^{[i]} = \cup_{M \in \mathfrak{S}^i} Y^{[M]}$ and $Y^{(M)} = Y^{[M]} \setminus Y^{[\#M+1]}$. Then we have the resulting stratification

$$Y = \coprod_{i=0}^{\#\mathcal{J}} Y^{[i]} = \coprod_{i=0}^{\#\mathcal{J}} \coprod_{M \in \mathfrak{S}_i} Y^{(M)}. \tag{9.4.2}$$

The strict semistability of $\mathscr{X}$ implies that the stratification verifies the two conditions stated before Theorem A.3.5. (Note that the scheme $Y^{[i]}$ is empty once $i > \dim Y$.)

We write $\mathbb{T} = \mathbb{T}_n \otimes \mathbb{T}_{n+1}$ and $\mathfrak{m}$ corresponding to $(\mathfrak{m}_n, \mathfrak{m}_{n+1})$ for maximal ideals $\mathfrak{m}_\nu$ of $\mathbb{T}_\nu, \nu \in \{n, n+1\}$. We will distinguish the three cases.

Split-(Drinfeld-level, hyperspecial) case. By Theorem A.3.5 (1), it suffices to verify that, for every $M \in \mathfrak{S}$, we have $H^j(Y^{[M]} \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0$ whenever $j \neq \dim Y^{[M]}$. This follows from the Kunneth formula, [LL22, Prop. 4.25] for $H^j(Y_n^{[M]} \otimes_k \overline{k}, L)_{\mathfrak{m}_n} = 0, j \neq \dim Y_n^{[M]}$ and the similar vanishing result for $H^j(Y_{n+1} \otimes_k \overline{k}, L)_{\mathfrak{m}_{n+1}} = 0, j \neq \dim Y_{n+1}$.

Split-(Iwahori, Iwahori) case. We first define a stratification of the special fiber Z of $\mathcal{M}_{K_{\widetilde{G}}}$ prior to the resolution, similar to (9.4.3)

$$Z = \coprod_{i=0}^{\#\mathcal{J}} Z^{[i]} = \coprod_{i=0}^{\#\mathcal{J}} \coprod_{M \in \mathfrak{S}_i} Z^{(M)}. \tag{9.4.3}$$

Then, under the condition (1), it follows from [LL21, (3), p. 859] that $H_c^i(Z^{(M)})_{\mathfrak{m}} = 0$ for all i and $M \in \mathfrak{S}$, unless $Z^{(M)}$ are maximal dimensional, in which case $H_c^i(Z^{(M)})_{\mathfrak{m}} = 0$ unless $i = \dim Z^{(M)}$. In *loc. cit.* the authors only treated the case of Drinfeld levels; but the proof applies vebatim to the Iwahori case. Now we return to the stratum $Y^{(M)}$ in (9.4.3). It is easy to see that the natural map $\pi_M \colon Y^M \to Z^{(M)}$ is smooth and the direct images $R\pi_{M,!}^j L$ are constant on $Z^{(M)}$. It follows that $H_c^i(Y^{(M)})_{\mathfrak{m}} = 0$ for all i and $M \in \mathfrak{S}$, unless $Y^{(M)}$ are maximal dimensional hence equal to $Z^{(M)}$, in which case $H_c^i(Y^{(M)})_{\mathfrak{m}} = 0$ unless $i = \dim Z^{(M)}$. It follows from the cohomological exact sequence associated to $Y^{[M]} = Y^{(M)} \cup (Y^{[M]} \setminus Y^{(M)})$ (see for example (9.4.5) below) and an induction that $H^i(Y^{(M)})_{\mathfrak{m}} = 0$ for all i and $M \in \mathfrak{S}$, unless $Y^{(M)}$ are maximal dimensional, in which case $H^i(Y^{[M]})_{\mathfrak{m}} = 0$ for $i > \dim Y^{[M]}$ and by Poincaré duality $H^i(Y^{[M]})_{\mathfrak{m}} = 0$ for

$i \neq \dim Y^{[M]}$. We have thus verified the condition in case (1) of Theorem A.3.5 and therefore we have proved $H^{2n}(\mathscr{X})_{\mathfrak{m}} = 0$ in this case.

Inert-vertex-parahoric case. We note that the moduli space $\mathcal{M}_{K_{\widetilde{G}(V_\nu)}}$ for type t_ν (at v) is isomorphic to another similarly defined moduli space of type $\nu - t$. Therefore it suffices to consider the cases when $t_n, t_{n+1} \in \{0, 1\}$. We first recall from [LL21, §9, p. 868] that, when the type is $t_n = 1$, we have the cohomology of the balloon and the ground strata

$$H^i(Z \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0, \quad i \neq \dim Z \tag{9.4.4}$$

for $Z = Y_n^\circ, Y_n^\bullet, Y_n^\dagger$ respectively, where we simplify the notation $Y_n^? = \mathcal{M}_{K_{\widetilde{G}(V_n)}, k}^?$ in (9.3.2) for $? \in \{\circ, \bullet\}$, and define $Y_n^\dagger = Y_n^\circ \cap Y_n^\bullet$. If one of t_n, t_{n+1} is 0, the proof is now similar to the split-(Drinfeld-level, hyperspecial) case, using case (1) Theorem A.3.5. It remains to consider the type $(1,1)$ case. For $(?_n, ?_{n+1}) \in \{\circ, \bullet\}^2$, we will write $Y^{?_n, ?_{n+1}}$ for the strict transform of $Y_n^{?_n} \times Y_{n+1}^{?_{n+1}}$. Then by the formula for cohomology of blow-up, $H^i(Y^{?_n, ?_{n+1}} \otimes_k \overline{k}, L)$ is isomorphic to

$$H^i((Y_n^{?_n} \times Y_{n+1}^{?_{n+1}}) \otimes_k \overline{k}, L) \oplus \begin{cases} 0, & (?_n, ?_{n+1}) = (\circ, \bullet) \text{ or } (\bullet, \circ) \\ H^{i-2}((Y_n^\dagger \times Y_{n+1}^\dagger) \otimes_k \overline{k}, L), & (?_n, ?_{n+1}) = (\circ, \circ) \text{ or } (\bullet, \bullet). \end{cases}$$

Similarly we can compute the cohomology of all of the closed strata $Y^{[M]}$ in terms of the notation in (9.4.3) using (9.4.4)

$$H^i(Y^{[M]} \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0, \quad i \neq \dim Y^{[M]},$$

for all $M \in \mathfrak{S}$ but one exception: the stratum $Y^{[M_0]} := Y^{\circ, \circ} \cap Y^{\bullet, \bullet}$, which is a $\mathbb{P}^1$-bundle over $Y^{\dagger, \dagger}$. Nonetheless the exceptional case has vanishing (localized at $\mathfrak{m}$) cohomology at all degree outside $i = \dim Y^{\dagger, \dagger}$ and $i = \dim Y^{\dagger, \dagger} + 2$. Using (9.4.4) (for both n and $n+1$) we can deduce that

$$H_c^i(Y^{(M)} \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0, \quad i > \dim Y^{[M]}$$

for $M \neq M_0 \in \mathfrak{S}$. To treat the exceptional case, we use the exact sequence

$$H^{i-1}(Y^{[M_0]} \setminus Y^{(M_0)}) \longrightarrow H_c^i(Y^{(M_0)}) \longrightarrow H^i(Y^{[M_0]}). \tag{9.4.5}$$

Note that the stratum $Y^{[M_0]}$ has codimension 2 in $\mathscr{X}$, and $Y^{[M_0]} \setminus Y^{(M_0)}$ is smooth of codimension 3 in $\mathscr{X}$. By $H^i(Y^{[M_0]})_{\mathfrak{m}} = 0$ when $i \geq \dim Y^{[M_0]} + 2$, and $H^i(Y^{[M_0]} \setminus Y^{(M_0)})_{\mathfrak{m}} = 0$ when $i \neq \dim Y^{[M_0]} \setminus Y^{(M_0)} = \dim Y^{[M_0]} - 1$, we conclude that $H_c^i(Y^{(M_0)})_{\mathfrak{m}} = 0$ when $i \geq \dim Y^{[M_0]} + 2$. By Poincaré duality we have $H^i(Y^{(M_0)})_{\mathfrak{m}} = 0$ when $i \leq \dim Y^{[M_0]} - 2$. Since $H^{2n}(X, L(n)) = 0$, we have verified the condition in case (2) of Theorem A.3.5 and therefore we have proved $H^{2n}(\mathscr{X})_{\mathfrak{m}} = 0$. $\qquad\square$

Remark 9.4.2. The condition (1) in Theorem 9.4.1 may be unnecessary if one makes a more careful study on the stratification of the special fiber of the small resolution.

10. THE ARITHMETIC RELATIVE-TRACE FORMULA

Let $V \in \mathscr{V}^{\circ, -}$ be an incoherent pair, and let $G = G^V$, $H := H^V$. In this section, we define our cycles of interest, and a distribution $\mathscr{J} = \mathscr{J}_{K_p}$ on (part of) the Hecke algebra for $G(\mathbf{A}^p)$

that encodes their p-adic heights. The main result of this section is the arithmetic RTF for $\mathscr{J}$ (Theorem 10.5.3).

We will denote $X_K := \mathrm{Sh}_K(\mathrm{G})$, $Y_{K_{\mathrm{H}}} := \mathrm{Sh}_{K_{\mathrm{H}}}(\mathrm{H})$. In § 10.1 we study the étale cohomology of X_K and define the Galois representation of interest. in § 10.2, we define and study the arithmetic diagonal cycles and Gan–Gross–Prasad cycles. In § 10.3 we define $\mathscr{J}$ by means of height pairings of those cycles, and give its spectral expansion. In § 10.4 we prove some vanishing results to decompose $\mathscr{J}$ as a sum indexed by the nonsplit places of F_0. Finally, in § 10.5 we state the geometric expansion of $\mathscr{J}$.

10.1. **Cohomology and automorphic Galois representations.** Let L be an algebraic extension of $\mathbf{Q}_p$.

10.1.1. *Ordinary representations of* $\mathrm{G}(\mathbf{A})$. We say that $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$ is *ordinary* if for every place $v|p$ of F_0, the base-change $\mathrm{BC}(\pi_v)$ satisfies the ordinariness conditions of § 1.1.2. If $K_p \subset \mathrm{G}(F_{0,p})$ is a compact open subgroup, we say that π_p is K_p-*ordinary* if it is ordinary and moreover $\pi^{K_p} \neq 0$. These conditions define ind-subschemes

$$\mathscr{C}(\mathrm{G})^{\mathrm{ord}}_{K_p} \subset \mathscr{C}(\mathrm{G})^{\mathrm{ord}} \subset \mathscr{C}(\mathrm{G})_{\mathbf{Q}_p}.$$

We also denote by $\mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}$ and $\mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}$ their ind-subschemes of Galois orbits of distinguished representations. Finally, for the above decorations '?', we define $\widetilde{\mathscr{C}}(\mathrm{G})^?(L)$ as the corresponding sets of isomorphism classes of representations such that $\mathscr{C}(\mathrm{G})^?(L) = \widetilde{\mathscr{C}}(\mathrm{G})(\overline{L})/G_L$ (cf. § 2.5.2).

10.1.2. *Duals and Hecke actions.* If S is a finite set of places of F_0 and M is an admissible (left) $L[\mathrm{G}(\mathbf{A}^S)]$-module, we denote

$$M^* := \varprojlim_{K^S \subset \mathrm{G}(\mathbf{A}^{S\infty})} M^{K^S,\vee}$$

the algebraic dual of M, whereas as usual we denote by $M^\vee = \varinjlim_{K^S} M^{K^S,\vee}$ the contragredient; for any compact subgroup $K' \subset \mathrm{G}(\mathbf{A}^{S\infty})$, we denote by $M^*_{K'}$ the K'-coinvariants (thus $M^*_{K'} = M^{\vee,K'}$ if K' is open). The left Hecke action on M induces a right action

$$M^* \times \mathscr{H}(\mathrm{G}(\mathbf{A}^S), L) \longrightarrow M^\vee.$$

10.1.3. *Hecke and Galois actions on the cohomology of unitary Shimura varieties.* For $i \in \mathbf{Z}$, we put

$$M^{i,K} := H^i(\mathrm{Sh}_K(\mathrm{G})_{\overline{F}}, \mathbf{Q}_p(n)), \qquad M^i := \varinjlim_K M^i_K. \tag{10.1.1}$$

where the limit is with respect to the pullback maps. For $? = \emptyset, K$, we also put $M^{\oplus,?} := \bigoplus M^{i,?}$; it has a natural (left) action by $\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty), \mathbf{Q}_p)_?$ and by the Galois group G_F.

Let $\diamond \in \mathbf{Z} \cup \{\oplus\}$. Then the $\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty), \mathbf{Q}_p)$-action on $M^\diamond$ makes it into an admissible $\mathrm{G}(\mathbf{A}^\infty)$-module, so that we may consider $M^{\diamond,*}$. It is helpful to think of $(M^\oplus)^*$ as the inverse limit of homology (whereas $M^\oplus$ is the direct limit of cohomology).

For $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$, let
$$\rho[\pi]^\diamond := \mathrm{Hom}_{\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty))}(\pi^\vee, M_L^{\diamond,\vee}(1)),$$
$$M^{\diamond,\pi} := \pi^* \boxtimes \rho[\pi] \subset M_L^{\diamond,*}(1),$$

so that we have a Hecke-equivariant map
$$\pi \longrightarrow \mathrm{Hom}_{G_F}(M^{\diamond,*}(1), \rho[\pi]) \tag{10.1.2}$$

factoring through $\mathrm{Hom}_{G_F}(M^{\diamond,\pi}, \rho[\pi])$. In fact, it is known (see [BW80, Theorem III.5.1]) that the temperedness implies
$$M^{\oplus,\pi} = M^{2n-1,\pi} \tag{10.1.3}$$

so that we will simply write $M^\pi := M^{2n-1,\pi}$, $\rho[\pi] := \rho[\pi]^{2n-1}$.

We put $M_{\pi^\vee}^K := (M_K^\pi)^\vee$ and $M_{\pi^\vee} := \varinjlim_K M_{\pi^\vee}$, so that $M^\pi = M_{\pi^\vee}^*(1)$. For $? \in \{\mathrm{temp}, \mathrm{t\text{-}ord}\}$, we put
$$M_{?,\overline{\mathbf{Q}}_p} := \bigoplus M_\pi \subset M_{\overline{\mathbf{Q}}_p}^\oplus, \qquad M_{\overline{\mathbf{Q}}_p}^? := \bigoplus M^\pi \subset M_{\overline{\mathbf{Q}}_p}^{\oplus,*}(1)$$

where the sums run over $\mathscr{C}(\mathrm{G})(\overline{\mathbf{Q}}_p)$ and $\mathscr{C}(\mathrm{G})_{\mathrm{ord}}(\overline{\mathbf{Q}}_p)$ respectively. These are base-changes of L-subspaces $M_? \subset M^{2n-1}$, $M_L^? \subset M^{2n-1,*}(1)$. Poincaré duality gives an isomorphism $M_K \cong M_K^*(1)$, which induces isomorphisms
$$M_?^K \cong M_K^? \tag{10.1.4}$$

for $? \in \{\mathrm{temp}, \mathrm{t\text{-}ord}\} \cup \widetilde{\mathscr{C}}(\mathrm{G})(\mathrm{L})$.

10.1.4. *Automorphic Galois representations and decomposition of the cohomology.* Assume from now on that the extension L of $\mathbf{Q}_p$ is finite, and denote by $\overline{\mathbf{Q}}_p$ an algebraic closure of L. Let $\pi = \pi_n \boxtimes \pi_{n+1} \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$.

Lemma 10.1.1. *For $\nu \in \{n, n+1\}$ there is a semisimple continuous representation*
$$\rho_{\pi_\nu, \overline{\mathbf{Q}}_p} : G_F \to \mathrm{GL}_\nu(\overline{\mathbf{Q}}_p)$$

characterized, up to isomorphism, by the property that for all but finitely many places w of F split over F_0, the restriction $\rho_{\pi_\nu, \overline{\mathbf{Q}}_p | \mathcal{G}_{F_w}}$ is unramified, and a geometric Frobenius at w acts with a characteristic polynomial equal to the Satake polynomial of π_w viewed as a representation of $\mathrm{GL}_\nu(E_w)$. If π_ν is stable, then
$$\rho_{\pi_\nu, \overline{\mathbf{Q}}_p} \cong \rho_{\mathrm{BC}(\pi_\nu), \overline{\mathbf{Q}}_p} \tag{10.1.5}$$

(where the latter is as in § 1.2).

Proof. The construction is as in [DL, Lemma 4.8], using [LTX$^+$22, Proposition 3.2.8] (due to Shin) instead of [Mok15]. Property (10.1.5) is immediate from the construction. □

Let
$$\rho_{\pi, \overline{\mathbf{Q}}_p} : G_F \to \mathrm{GL}_{n(n+1)}(\overline{\mathbf{Q}}_p)$$

be defined by $\rho_{\pi, \overline{\mathbf{Q}}_p}(-n) := \rho_{\pi_n, \overline{\mathbf{Q}}_p} \otimes \rho_{\pi_{n+1}, \overline{\mathbf{Q}}_p}$. If $\rho : G_F \to \mathrm{GL}_n(L)$ is a continuous representation, denote by $\rho_{\overline{\mathbf{Q}}_p} := \rho \otimes_L \overline{\mathbf{Q}}_p$ the base-change and by $\rho_{\overline{\mathbf{Q}}_p}^{\mathrm{ss}}$ its semisimplification.

The following key hypothesis gives an explicit description of $\rho[\pi]$ (at least in the stable case).

Hypothesis 10.1.2. *Let $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$, and let $K \subset \mathrm{G}(\mathbf{A}^\infty)$ be an open compact subgroup. Then $\rho[\pi]^{\mathrm{ss}}_{\overline{\mathbf{Q}}_p}$ is is isomorphic to a direct summand of $\rho_{\pi,\overline{\mathbf{Q}}_p}$. Moreover, if π is stable then $\rho[\pi]^{\mathrm{ss}}_{\overline{\mathbf{Q}}_p} \cong \rho_{\pi,\overline{\mathbf{Q}}_p}$.*

Remark 10.1.3. Let $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her},-}_{\mathbf{Q}_p}$, and let $L = \mathbf{Q}_p(\Pi)$. Let $\overline{\pi} \in \mathscr{C}(\mathrm{G}^V)(\overline{\mathbf{Q}}_p)$ be the preimage of Π under (2.5.1); a priori we know it is isomorphic to its G_L-conjugates but not that it arises from some $\pi \in \widetilde{\mathscr{C}}(\mathrm{G}^V)(L)$. Assume that Hypothesis 10.1.2 holds. By the definitions, the space

$$M^{\overline{\pi}}$$

is isomorphic to $\overline{\pi}^* \boxtimes \rho[\overline{\pi}]$ as a Hecke- and $\overline{\mathbf{Q}}_p[G_F]$-module, and it is a G_L-invariant subspace of $M^{2n-1,*}_{\overline{\mathbf{Q}}_p}(1)$. Let $M^\pi := (M^{\overline{\pi}})^{G_L} \subset M^{2n-1,*}_L(1)$, and define the $L[G_F]$-module

$$\rho_\Pi := (M^\pi)^{\mathrm{H}(\mathbf{A}^\infty)}$$

Then we have

$$\rho_\Pi \otimes_L \overline{\mathbf{Q}}_p \cong (\overline{\pi}^*)^{\mathrm{H}(\mathbf{A}^\infty)} \otimes_{\overline{\mathbf{Q}}_p} \rho[\overline{\pi}].$$

The first tensor factor is 1-dimensional, so that by Remark 2.5.6 and (10.1.5), the representation

$$\rho_\Pi : G_F \longrightarrow \mathrm{GL}_{n(n+1)}(L)$$

satisfies

$$(\rho_\Pi \otimes_L \overline{\mathbf{Q}}_p)^{\mathrm{ss}} \cong \rho_{\Pi_n,\overline{\mathbf{Q}}_p} \otimes \rho_{\Pi_{n+1},\overline{\mathbf{Q}}_p}(n).$$

(In fact, it is conjectured that $\rho_{\Pi_\nu,\overline{\mathbf{Q}}_p}$ is irreducible for $\nu = n, n+1$, so that the semisimplification should be superfluous.) This also implies that $\overline{\pi}$ has a model $\pi = \mathrm{Hom}_{L[G_F]}(M^\pi, \rho_\Pi)$ defined over L; in other words, for an incoherent $V \in \mathscr{V}^{\circ,-}$ we have $\widetilde{\mathscr{C}}(\mathrm{H}^V \backslash \mathrm{G}^V)^{\mathrm{st}}_{\mathbf{Q}_p} = \mathscr{C}(\mathrm{H}^V \backslash \mathrm{G}^V)^{\mathrm{st}}_{\mathbf{Q}_p}$.[17]

10.1.5. *Properties of automorphic Galois representations.*

Proposition 10.1.4. *Let $\pi \in \mathscr{C}(\mathrm{G})_{\mathbf{Q}_p}(L)$. The Galois representation $\rho := \rho_{\pi,\overline{\mathbf{Q}}_p}$ satisfies the following properties:*

(1) *For every nonarchimedean place w of F, the representation $\rho_{|G_{F_w}}$ is pure of weight -1 in the sense of* [DL, Definition A.8].

(2) *The representations ρ^{c} and $\rho^*(1)$ are isomorphic.*

(3) *For every place $v|p$ of F_0 and every place $w|v$ of F:*
    (a) *if π_v is unramified, then $\rho_{|G_{F_w}}$ is crystalline;*
    (b) *if moreover π_v is ordinary, then $\rho_{|G_{F_w}}$ is Panchishkin-ordinary.*

If Hypothesis 10.1.2 holds, then the conclusions (1)-(3) above also hold for $\rho = \rho[\pi]$ and $\rho = M^\pi_K$.

Proof. Part (1) is a fundamental result of Caraiani [Car12, Car14] (see also [TY07, Lemma 1.4 (3)]). Part (2) follows from the last statement in [DL, Lemma 4.8] for $\rho_{\pi,\overline{\mathbf{Q}}_p}$, and from the Galois-equivariance of the Poincaré pairing for $\rho[\pi]$, M_π. The proof of part (3) is as in [DL, Lemmas 4.14, 4.15]. (In fact the assumption on $\pi_{\nu,v}$ in (b) is stronger than the analogous assumption in

---

[17]It is plausible that this kind of equality holds more generally, but we do not explore this here.

loc. cit.; correspondingly each factor $\rho_{\pi_\nu | G_{F_w}}$ is also ordinary in the sense of [Nek93, Definition 1.29]; however, only Panchishkin-ordinariness is stable under tensor products.)   □

For the rest of the paper, we will assume Hypothesis 10.1.2 for every[18] representation $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(\overline{\mathbf{Q}}_p)$.

10.2. Gan–Gross–Prasad cycles.
From now on, we freely use the notation and results of Appendix A; see especially § A.1 for the present subsection.

10.2.1. *Arithmetic diagonal cycles.*
We have a fundamental cycle

$$[Y]^\circ = ([Y_{K_\mathrm{H}}]^\circ) \in \varprojlim_{K_\mathrm{H}} \mathrm{Z}^0(Y_{K_\mathrm{H}})_\mathbf{Q},$$

where the transition maps on the right are pushforwards and $[Y_{K_\mathrm{H}}]^\circ = \mathrm{vol}(K_\mathrm{H}, dh)[Y_{K_\mathrm{H}}]$. Let $\jmath$ be (system of) arithmetic diagonal maps (8.2.2). The *arithmetic diagonal cycle*

$$Z := \jmath_*[Y]^\circ \in \varprojlim_K \mathrm{Z}^n(X_{\mathrm{G},K})_\mathbf{Q} \tag{10.2.1}$$

is well-defined. We denote by Z_K its image in $\mathrm{Z}^n(X_{\mathrm{G},K})_\mathbf{Q}$.

10.2.2. *Limits of Selmer groups.*
Let L be a finite extension of $\mathbf{Q}_p$, and let $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$. For $? \in \{\mathrm{temp}, \mathrm{t\text{-}ord}, \pi\}$, define

$$H_f^1(F, M^?) := \varprojlim_K H_f^1(F, M_K^?), \qquad H_f^1(F, M_?) := \varinjlim_K H_f^1(F, M_{?,K}).$$

10.2.3. *GGP cycles and associated functionals.*
Let

$$Z_{\pi,K} \in H_f^1(F, M_{\pi,K})$$

be the Hecke-eigencomponent of $\widetilde{\mathrm{cl}}(Z_K)$. Here, by the discussion in § A.1, the fact that $Z_{\pi,K}$ belongs to the Bloch–Kato Selmer group is a consequence of the vanishing of $M_{\pi,K} \cap M_{2n}$ and Proposition 10.1.4.

Definition 10.2.1. The *Gan–Gross–Prasad cycle* of π is

$$Z_\pi := \varprojlim_K Z_{\pi,K} \quad \in H_f^1(F, M^\pi);$$

The $\mathrm{H}(\mathbf{A})$-invariant functional associated to it via (10.1.2) will still be denoted by

$$Z_\pi \colon \pi \longrightarrow H_f^1(E, \rho[\pi])$$

$$\phi \longmapsto Z_\pi \phi := \phi_* Z_\pi.$$

From the $\mathrm{H}(\mathbf{A})$-invariance it follows that Z_π vanishes unless $\pi \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})$.

Remark 10.2.2. The linear functional Z_π valued in the Selmer group can be viewed as an arithmetic analog of the automorphic period functional (3.4.4).

---

[18]In fact, it would be enough to assume it for the representation π in order to prove Theorem D for π, at the cost of some complication in the exposition.

10.2.4. *Ordinary cycles.* Suppose that every place $v|p$ of F_0 splits in F. For each v, we may fix a place $w|v$ of F and compatible bases of $V_{\nu,w}$, giving isomorphisms $\mathrm{G}_{F_{0,v}} \cong \mathrm{GL}_n \times \mathrm{GL}_{n+1}$, $\mathrm{H}_{F_{0,v}} \cong \mathrm{GL}_n$ of algebraic groups over $F_{0,v} = F_w$. Then we may and will use the notation, definitions and results of §§ 5.1-5.2; we generally also denote $\square_p := \prod_{v|p} \square_v$; for instance, $t_{0,p} = \prod_{v|p} t_{0,v}$, $N_{0,p}^{\circ} = \prod_{v|p} N_{0,v}^{\circ} \subset \mathrm{GL}_n(F_{0,p}) \times \mathrm{GL}_{n+1}(G_{F_{0,p}}) \cong \mathrm{G}(F_{0,p})$. We define an operator $e^{\mathrm{ord}} := \lim U_{t_0}^{N!}$; it acts on $M_{N_{0,p}^{\circ}}$ and on $\pi^{N_{0,p}^{\circ}}$ for any $\pi \in \mathscr{C}(\mathrm{G})_{\mathbf{Q}_p}$; the representation π is ordinary if and only if it is not annihilated by e^{ord}.

Let $K_p \subset \mathrm{G}(F_{0,p})$ be an open compact subgroup containing $N_{0,p}^{\circ}$, and let $c \geq 1$ be such that K_p contains $K_0^{\langle c+1 \rangle}$. For positive integers r, N with $N! \geq r \geq c$, set $m_{0,r} = \prod_{v|p} m_{0,r,v}$ (cf. (5.1.4) for the definition of twisting matrices), and define

$$Z_{K_p}^{\dagger,N} := \prod_{v|p} q_v^{rd(n)} \cdot Z.T(m_{0,r} U_{t_0,p}^{N!-r} e_{K_p})_{\mathbf{Q}} \ \in \mathrm{Z}^n(X_{\mathrm{G},K_p}),$$

which is independent of r by Corollary 5.1.5.

We define the *ordinary arithmetic diagonal cycle* by

$$Z_{K_p}^{\mathrm{ord}} := \lim_{N \to \infty} \widetilde{\mathrm{cl}}(Z_{K_p}^{\dagger,N}) \quad \in \varprojlim_{K^p}(H^{2n}(X_{K^p K_p}, \mathbf{Q}_p(n))/\mathrm{Fil}^2).e^{\mathrm{ord}}.$$

For any $\pi \in \mathscr{C}(\mathrm{H} \backslash \mathrm{G})^{\mathrm{ord}}$, we define the ordinary GGP cycle

$$Z_{\pi,K_p}^{\mathrm{ord}} \in H^1_f(F, M_{K_p}^{\pi})$$

to be the eigencomponent of $Z_{K_p}^{\mathrm{ord}}$. By the definitions, for any sufficiently large $r \leq N!$,

$$Z_{\pi,K_p}^{\mathrm{ord}} = \prod_{v|p} q_v^{rd(n)} \lim_{N \to \infty} Z_\pi.T(m_{0,r} U_{t_0,p}^{N!-r} e_{K_p}).$$

We have an induced $\mathrm{H}(\mathbf{A}^p)$-invariant functional still denoted by the same name

$$Z_{\pi,K_p}^{\mathrm{ord}} \colon \pi^{K_p} \longrightarrow H^1_f(E, \rho[\pi]).$$

It factors through $e^{\mathrm{ord}} e_{K_p}$.

10.2.5. *Norm relation.* Continue with the notation and assumptions of § 10.2.4. In order to study p-adic heights, it will be useful to know that $Z_{\pi,K_p}^{\dagger,N}$ is a norm from some ring class fields of F of conductors that are high powers of p.

Let S be the unitary group of the 1-dimensional hermitian space $(F, N_{F/F_0})$. We have a map

$$\mathrm{rec} \colon G_F \longrightarrow \overline{F^\times} \backslash \mathbf{A}_F^{\infty,\times} \longrightarrow \overline{\mathrm{S}(F_0)} \backslash \mathrm{S}(\mathbf{A}^\infty),$$

where the first map is the reciprocity law of class field theory and the second map is $x \mapsto x^{\mathrm{c}}/x$ (and the bars denote Zariski closures). For $v|p$ and $r \geq 0$, let $K_{S,v}^{(r)} := \mathrm{S}(\mathscr{O}_{F_{0,v}}) \cap 1 + v^r \mathscr{O}_{F_{0,v}}$, let $\Gamma_r = \Gamma_r^{(v)} := \mathrm{S}(F_0) \backslash \mathrm{S}(\mathbf{A}^\infty)/\mathrm{S}(\widehat{\mathscr{O}}_{F_0}^v) K_{S,v}^{(r)}$, and let

$$F_r = F_r^{(v)}/F$$

be the abelian extension such that $\mathrm{Gal}(F_r/F) \cong \Gamma_r$ under the reciprocity map. We have the norm map

$$\mathrm{N}_{F_r/F} \colon Z^n(X_{\mathrm{G},K,F_r}) \longrightarrow Z^n(X_{\mathrm{G},K}).$$

Lemma 10.2.3. *Fix a place $v|p$ of F_0. For any $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^{p\infty}), \mathscr{O}_L)_{K^p}$ and any integer r with $\max\{1, c(K_v) - 1\} \le r \le N!$, there exists a cycle $Z_r = Z_{K^p}^{\dagger,N}(f^p)_r^{(v)} \in Z^n(X_{\mathrm{G},K,F_r})_{\mathscr{O}_L}$ such that*

$$Z_{K^p}^{\dagger,N}(f^p) = \mathrm{N}_{F_r/F}(Z_r).$$

Proof. We may assume $f^p = e_{K^p}$, and abbreviate $Z_K^{\dagger,N} = Z_{K^p}^{\dagger,N}(f^p)$. Let $K_H^p := \mathrm{H}(\mathbf{A}^{p\infty}) \cap K^p$ and let $Y_r := Y_{K_H^p K_{H,0,p}^{(r)}}$. Then by (5.1.8), the map $Y \xrightarrow{\jmath} X \xrightarrow{m_{0,r}} X \to X_K$ factors through Y_r, and we can write

$$Z_K^{\dagger,N} = \prod_{v|p} q_v^{rd(n)} \cdot (\jmath_*[Y_r]^{\circ}).T(m_{0,r}U_{t_0,p}^{N!-r}e_K) = \mathrm{vol}^{\circ}(K_{H,0,p}) \cdot (\jmath_*[Y_r]).T(m_{0,r}U_{t_0,p}^{N!-r}e_K)$$

(see (5.1.7) for $\mathrm{vol}^{\circ}(K_{H,0,p})$).

Let $\det \colon \mathrm{H} \to \mathrm{S}$ be the determinant map. For a compact open subgroup $K \subset \mathrm{H}(\mathbf{A}^{\infty})$, by Shimura's reciprocity law we have an isomorphism of G_F-sets

$$\pi_0(Y_{K,\overline{F}}) \cong \mathrm{S}(F_0) \backslash \mathrm{S}(\mathbf{A}^{\infty}) / \det K.$$

Now we have $\det K_{H,0,v}^{(r)} = 1 + v^r \mathscr{O}_{F_{0,v}} = 1 + w^r \mathscr{O}_{F_{0,w}} \cong K_{S,v}^{(r)}$, where the last identification comes from the natural map $F_w^{\times} \subset F_v^{\times} \to S_v$. Thus we deduce a natural surjection $\mathrm{p} \colon \pi_0(Y_{r,\overline{F}}) \to \Gamma_r$. For each $\gamma \in \Gamma_r$, let $Y_{r,\gamma,\overline{F}} \subset Y_{r,\overline{F}}$ be the union of connected components in $\mathrm{p}^{-1}(\gamma)$; it arises as $Y_{r,\gamma} \times_{F_r} \overline{F}$ for an F_r-subvariety

$$Y_{r,\gamma} \subset Y_{r,F_r}.$$

Then for any $\gamma_0 \in \Gamma_r$, we have

$$Z_K^{\dagger,N} = \mathrm{vol}^{\circ}(K_{H,0,p}) \cdot \mathrm{N}_{F_r/F}(\jmath_*[Y_{r,\gamma_0}]).T(m_{0,r}U_{t_0,p}^{N!-r}e_K),$$

which belongs to $\mathrm{N}_{F_r/F}(Z^n(X_{\mathrm{G},K,F_r})_{\mathbf{Z}_p}$ since $\mathrm{vol}^{\circ}(K_{H,0,p})$ is a p-unit. $\qquad\square$

10.3. The distribution and its spectral expansion.
From now until the end of the paper, we suppose that every place $v|p$ of F_0 splits in F and that $K_p \subset \mathrm{G}(F_{0,p})$ is the hyperspecial subgroup $\mathrm{G}(\mathscr{O}_{F_{0,p}})$.

10.3.1. *Height pairings.* Considering the setup of § A.2.3, we denote by

$$h \colon H_f^1(F, M_{\text{t-ord}}^{K_p}) \times H_f^1(F, M_{\text{t-ord}}^{K_p}) \longrightarrow \Gamma_{F_0,L} \qquad\qquad (10.3.1)$$

the pairing induced by the family, $h_{X_K,\lambda} \colon H_f^1(F, M_{\text{t-ord}}^K)^{\otimes 2} \to \Gamma_{F_0,L}$ for $K = K^p K_p$, where

$$\lambda \colon \Gamma_{F,L} \longrightarrow \Gamma_{F_0,L}$$

is the natural surjection. It is well-defined by the projection formula (Lemma A.2.5). Note that the conditions of § A.2.1 for the definition of h (as well as for the definition of the pairing h_π from § 1.3.7) are satisfied by Proposition 10.1.4. We also denote by

$$h \colon H_f^1(F, M_{\text{t-ord}}^{K_p}) \times H_f^1(F, M_{K_p}^{\text{t-ord}}) \longrightarrow \Gamma_{F_0,L}$$

the pairing sending (Z^{K^p}, Z') to $h_{(10.3.1)}(Z^{K^p}, Z'.T(e_{K^p}))$ whenever $Z^{K^p} \in H^1_f(F, M^{K_p}_{\text{t-ord}})$ is in the image of $H^1_f(F, M^{K^p K_p}_{\text{t-ord}})$; the context should dispel any ambiguities caused by the conflated notation.

For a non-archimedean place w of F, we denote by h_w the corresponding local pairings (A.2.5) on pairs of (limits of) cycles with disjoint supports in $\mathrm{Z}^n_{\text{t-ord}}(X_{K,F_w})^0_L$ (if $w|p$) or $\mathrm{Z}^n_{\text{temp}}(X_{K,F_w})^0_L$ (if $w \nmid p$). For $w \nmid p$, this requires a projection formula for w-local heights, which is equivalent to Lemma A.2.4 (2).

10.3.2. *Definition of the distribution.* For S a finite set of non-archimedean places of F_0, denote by
$$\mathscr{H}(\mathrm{G}(\mathbf{A}^S), L)^\circ_{K_S\text{-temp}} \subset \mathscr{H}(\mathrm{G}(\mathbf{A}^S), L)_{K_S}$$
the subalgebra of measures $f^S = f^{S\infty} f_\infty$ such that $f_\infty \in L f^\circ_\infty$ (where $f^\circ_\infty = (4.1.3)$) and $M^\oplus.T(f^S e_{K_S}) \subset M^{K_S}_{\text{temp}}$.

Define first
$$\mathscr{J}_{K_p} \colon (\mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-temp}})^2 \longrightarrow \Gamma_{F_0, L}$$
$$(f^p_1, f^p_2) \longmapsto h(Z^{\text{ord}}_{K_p}.T(f^p_1), Z^{\text{ord}}_{K_p}.T(f^p_2)). \tag{10.3.2}$$

Definition 10.3.1. For any $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p}$ that can be written as
$$f^p = f^p_1 * f^{p,\vee}_2 \tag{10.3.3}$$
with $f^p_i \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-temp}}$, we define the *arithmetic relative-trace distribution* by[19]
$$\mathscr{J}_{K_p}(f^p) := \mathscr{J}_{K_p}(f^p_1, f^p_2).$$

Remark 10.3.2. The definition is independent of the decomposition (10.3.3). Indeed, let $K^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$ be such that $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^{p\infty}))_{K^p}$, and let S be a finite set of finite places of F_0, not above p, such that $K^S := K \cap \mathrm{G}(\mathbf{A}^{Sp\infty})$ is a maximal hyperspecial subgroup. Let $e^{\text{temp}}_K \in \mathscr{H}(\mathrm{G}(\mathbf{A}^{S\infty}))_{K^S}$ be an element acting as the idempotent projection $M^K \to M^K_{\text{temp}}$. Then by the projection formula (Lemma A.2.5),
$$\mathscr{J}_{K_p}(f^p) = h(Z^{\text{ord}}_{K_p}.T(f^p), Z^{\text{ord}}_{K_p}.T(e^{\text{temp}})_K).$$

Let now
$$f_{p, K_p, N} := \prod_{v|p} q^{rd(n)}_v \cdot m_{0,r,p} U^{N!-r}_{t_0,p} e_{K_p}, \tag{10.3.4}$$
where $1 \le r \le N!$. By the definitions, we have
$$\mathscr{J}_{K_p}(f^p_1, f^p_2) = \lim_{N \to \infty} h(Z.T(f^p_1 f_{p,K_p,N}), Z.T(f^p_2 f_{p,K_p,N})).$$
It is independent of the choice of $r \le N!$.

10.3.3. *Spectral expansion.* Let $\pi \in \widetilde{\mathscr{C}}(\mathrm{H}\backslash\mathrm{G})^{\text{ord}}_{K_p}(L)$. Denote by
$$h_{M_\pi} \colon H^1_f(F, M^{K_p}_\pi) \times H^1_f(F, M^{\pi^\vee}_{K_p}) \longrightarrow \Gamma_{F_0, L}$$

---

[19]The abuse of notation with respect to (10.3.2) should cause no confusion.

the restriction of h. For any $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ$, we define

$$\mathscr{J}_{\pi, K_p}(f^p) := h_{M_\pi}(Z^{\mathrm{ord}}_{\pi, K_p}.T(f^p), Z^{\mathrm{ord}}_{\pi^\vee, K_p}) = \mathrm{Tr}^{h_\pi \circ \left(Z^{\mathrm{ord}}_{\pi, K_p} \boxtimes Z^{\mathrm{ord}}_{\pi^\vee, K_p}\right)}_{(,)_{\pi^{K_p}}}(T(f^p)),$$

where the pairing $(,)_{\pi^{K_p}}$ is the restriction of $(,)_\pi = (1.3.4)$ to $\pi^{K_p} \times \pi^{\vee, K_p}$. Then it is clear that if f^p is as in Definition 10.3.1, we have

$$\mathscr{J}_{K_p}(f^p) = \sum_{\pi \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}} \mathscr{J}_{\pi, K_p}(f^p),$$

where for a Galois orbit $\pi = \{\pi^\sigma\} \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}$ of isomorphism classes of representations, we put $\mathscr{J}_{\pi, K_p} := \sum \mathscr{J}_{\pi^\sigma, K_p}$.

10.4. **Decomposition over nonsplit places.** We will complete the arithmetic relative-trace formula by finding a geometric expansion for the distribution $\mathscr{J}_{K_p}$. Each term in the expansion will be a sum over all nonsplit finite places of F_0. The goal of this subsection is to show the preliminary result that $\mathscr{J}_{K_p}$ has a decomposition as a sum over nonsplit places, by proving some vanishing results for local height pairings at split (p-adic and non-p-adic) places.

10.4.1. *Decomposition over all places.* Let v be a non-archimedean place of F_0. We define

$$\mathscr{J}^{(v),N}_{K_p}(f^p_1, f^p_2) := \sum_{w|v} h_w(Z^{\dagger,N}_{K_p}.T(f^p_1), Z^{\dagger,N}_{K_p}.T(f^p_2))$$

for any $f^p_1, f^p_2 \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-temp}}$ (respectively $\mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-t-ord}}$ if $v|p$) such that the two cycles involved have disjoint supports. Here, the sum ranges over the (one or two) places of F above v.

It is clear from the definitions that we have a decomposition

$$\mathscr{J}_{K_p}(f^p_1, f^p_2) = \lim_{N\to\infty} \sum_{v\nmid\infty} \mathscr{J}^{(v),N}_{K_p}(f^p_1, f^p_2). \tag{10.4.1}$$

whenever the terms in the right-hand side are defined.

In the rest of this subsection, we show the vanishing of the contribution at split (p-adic and non-p-adic) places.

Remark 10.4.1. If $v \nmid p\infty$, we can more generally define

$$\mathscr{J}^{(v)}(f_1, f_2) := \sum_{w|v} h_w(Z.T(f_1), Z.T(f_2))$$

for $f_1, f_2 \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^\circ_{\text{temp}}$ such that the two cycles involved have disjoint supports; then

$$\mathscr{J}^{(v),N}_{K_p}(f^p_1, f^p_2) = \mathscr{J}^{(v)}(f^p_1 f_{p,K_p,N}, f^p_2 f_{p,K_p,N}). \tag{10.4.2}$$

10.4.2. *Auxiliary Shimura varieties.* Let v be a place of F_0 and w a place of F above v. Choose an "admissible CM type Φ (relative to v)" in the sense of [LL21, p.851] and a place u of the reflex field E above the place w of F such that u is unramified over w. (Note that Φ depends on v.) Recall from §9.1 that, by our assumption, the compact open subgroup $K_{\mathbf{Z}\mathbf{Q}}$ is maximal at v. We

then have the auxiliary Shimura variety

$$X'_u \coloneqq X'_{K/E_u} \coloneqq \mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}(\widetilde{\mathrm{G}})_{E_u} \tag{10.4.3}$$

and its integral model $\mathscr{X}'_u \coloneqq \mathcal{M}_{K_{\widetilde{\mathrm{G}}}, \mathscr{O}_{E,u}}$ from §9.3. We observe that X'_u is of the form $X_{F_w} \times_{F_w} A$ for some finite étale F_w-algebra A.

We denote by

$$\mathscr{L}' = \mathscr{L}'_u \coloneqq \widetilde{j}_* \left(\mathrm{vol}(K_{\mathrm{H}}) [\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{H}}}}] \right)$$

the $\mathscr{O}_{E,u}$-integral model of the arithmetic diagonal cycle, where $\widetilde{j}$ is as in § 9.3.3.

10.4.3. *Local heights at split places.* The following lemma will be useful for considerations both at places above p and away from p. We first need a definition.

Definition 10.4.2. We say that a pair $(f_{1,v}, f_{2,v}) \in \mathscr{H}(G_v)^2_{K_v}$ has K_v-*regular support*, or simply that it is K_v-*regular*, if $f_{1,v}$ has regular support and $f_{2,v'} = e_{K_v}$.

If S is a finite set of finite places of F_0 and $v \notin S$ is another finite place of F_0, we say that a pair $(f_1^S, f_2^S) \in \mathscr{H}(\mathrm{G}(\mathbf{A}^S), L)^{\circ}_{K^S}$ is K_v-regular at v if we can write $K^S = K^{Sv} K_v$ and $f_i = f_{i,v} \otimes f_i^v$ with $(f_{1,v}, f_{2,v})$ K_v-regular.

Lemma 10.4.3. *Let v be a split place of F_0. Let $K = \prod_v K_v$ be an open subgroup of $\mathrm{G}(\mathbf{A}^{\infty})$. Suppose that $f_1, f_2 \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^{\circ}_K$ satisfy:*

- *(f_1, f_2) has $K_{v'}$-regular support at some finite place $v' \neq v$;*

- *the subgroup $K_v = K_{n,v} \times K_{n+1,v}$ satisfies either of the following conditions:*
  - *(a) for some labelling $\{\nu, \nu'\} = \{n, n+1\}$, the subgroup $K_{\nu,v}$ is maximal hyperspecial and $K_{\nu',v}$ is the principal congruence subgroup of level $m \in \mathbf{Z}_{\geq 0}$ (cf.§9.2);*

  - *(b) for both $\nu = n, n+1$, the subgroup $K_{\nu,v}$ is Iwahori (that is, $G_{\nu,v}$-conjugate to the standard Iwahori $\mathrm{Iw}_{\nu,v,0}$).*

Then the following statements hold.

(i) The cycles $Z.T(f_1)$ and $Z.T(f_2)$ have disjoint support (on the generic fiber).

(ii) Abusing notation, we still let $T(f_i)$ denote the (flat) correspondence on the integral model $\mathscr{X}'_u$. Then the cycles $\mathscr{L}'_u.T(f_1)$ and $\mathscr{L}'_u.T(f_2)$ have disjoint supports in $\mathscr{X}'_u$.

Proof. Part (i) follows from [RSZ20, Theorem 8.5 (i)]. (The result in *loc. cit* only treats the auxiliary Shimura variety attached to $\widetilde{\mathrm{G}}$; but it implies the desired result for G.)

For (ii) case (a), the integral model with Drinfeld m-level structure at one factor and with hyperspecial level ($m = 0$) at the other factor is regular. The proof of [RSZ20, Theorem 8.5 (ii)] (only the case $f_2 = e_K$ was considered there) still applies to show that the cycles $\mathscr{X}'_u.T(f_1)$ and $\mathscr{X}'_u.T(f_2)$ have disjoint supports in $\mathscr{X}'_u$.

In case (b), the integral model is the resolution given in §9.3 of the moduli scheme and the Hecke correspondences are obtained by base change and hence remain finite flat. The cycles are obtained by strict transforms. Hence it suffices to show the disjointness before the resolution, which again follows from [RSZ20, Theorem 8.5 (ii)]. (Strictly speaking, the result of *loc. cit.*

concerns the case of Drinfeld m-levels rather than Iwahori level. However, we may pull back the cycles to the moduli scheme with Drinfeld level for $m = 1$ and then apply that result.)

$\square$

Proposition 10.4.4. *Let $v \nmid p$ be a split place of F_0. Let f_1, $f_2 \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)_{\mathrm{temp}}^{\circ}$ be as in Lemma 10.4.3. Assume furthermore that either K_v is hyperspecial or that $T(f_1^v)$, $T(f_2^v)$ annihilate $H^{2n}(\mathscr{X}_u', L(n))$. Then*

$$\mathscr{J}^{(v)}(f_1, f_2) = 0.$$

Proof. We show that $h_w(Z.T(f_1), Z.T(f_2)) = 0$ for each of the two places $w|v$. By Lemma A.2.4 (2), it suffices to show the vanishing of the local height after pull-back to the auxiliary Shimura variety $\mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}(\widetilde{\mathrm{G}})$ over E_u. Finally, under our assumption, Proposition A.3.4 further reduces the question to the vanishing of the arithmetic intersection pairing on the integral model $\mathscr{X}_u'$ over $\mathscr{O}_{E,u}$. This last vanishing follows from Lemma 10.4.3 (ii). $\square$

10.4.4. *Vanishing at p-adic places.*

Proposition 10.4.5. *Let v be a place of F_0 above p (hence split in F). If $n > 1$, assume $p > 2n$. Let $f^p = f_1^p * f_2^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)_{K_p\text{-t-ord}}^{\circ}$, and assume that the pair (f_1^p, f_2^p) has regular support. Then*

$$\lim_{N \to \infty} \mathscr{J}_{K_p}^{(v),N}(f^p) = 0.$$

Proof. Write $Z_i := Z_{K_p}^{\dagger,N}.T(f_i^p)$, and let K^p be such that f_1, f_2 are right-K^p-invariant. For any finite extension E of F_w, denote by $\lambda_E' : E^\times \hat{\otimes} L \to E^\times \hat{\otimes} L$ the identity map, and by $h_{X_{K,E}} := h_{X_{K,E},\lambda_E'}$ the corresponding height pairing. We will show that

$$h_{X_{K,F_w}}(Z_1, Z_2) \quad \in p^{N!-C} \mathscr{O}_{F_w}^\times \hat{\otimes} \mathscr{O}_L$$

for some constant C; after taking limits, this implies the desired vanishing. Up to multiplying by a nonzero scalar, we may assume that $f_i^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), \mathscr{O}_L)^{\circ}$.

By Lemma 10.2.3, for some constant C' cancelling the denominators of f_i, and for any sufficiently large $r \leq N!$, we have $Z_i = \mathrm{N}_{F_r/F}(Z_{i,r})$ for some $Z_{i,r} \in p^{-C'}Z^n(X_{\mathrm{G},K,F_r})_{\mathscr{O}_L}$. Denote by $F_{w,r}$ the localization of F_r at its unique place above w. First, we show that

$$h_{X_{K,F_{w,r}}}(Z_1, Z_{2,r}) \quad \in \mathscr{O}_{F_{w,r}}^\times \hat{\otimes} L. \tag{10.4.4}$$

By Lemma A.2.4 (1) (which applies thanks to the observation made after (10.4.3)), it is enough to show the same result for the corresponding height pairing of arithmetic diagonal cycles on the auxiliary Shimura variety (10.4.3). This follows from Lemma 10.4.3 (ii), Proposition A.3.2, and Remark A.2.3.

By the integrality results of [Nek95, Proposition II.1.11], we have in fact

$$h_{X_{K,F_{w,r}}}(Z_1, Z_{2,r}) \quad \in p^{-C_r''-C'} \mathscr{O}_{F_{w,r}}^\times \hat{\otimes} \mathscr{O}_L, \tag{10.4.5}$$

for a constant C''_r that, similarly to [DL, Proof of Proposition 4.2.6], can be bounded as follows. Let $T := M_{\text{t-ord},K,L} \cap H^{2n-1}(X_{K,\overline{F}}, \mathscr{O}_L(n))/(\text{tors})$, and denote by

$$N_\infty H^1_f(F_{w,r}, T) := \bigcap_{s \geq r} \text{Im}\left[\text{Tr}_{F_{w,s}/F_{w,r}} : H^1_f(F_{w,s}, T) \longrightarrow H^1_f(F_{w,s}, T)\right].$$

Then $p^{C''_r} \leq c''_r := |H^1_f(F_{w,r}, T)/N_\infty H^1_f(F_{w,r}, T)|$. However c''_r is bounded independently of r: this follows by the same argument as for [DL, Lemma 4.28] from the fact that $M_{\text{t-ord},K,L}$, as a representation of G_{F_w}, is crystalline, Panchishkin-ordinary, and pure of weight -1 (Proposition 10.1.4). Thus in (10.4.5) we may replace $C''_r + C'$ by a constant C''.

Finally, by Lemma A.2.4 (1), we have

$$p^{C''} \cdot h_{X_{K,F_w}}(Z_1, Z_2) = p^{C''} N_{F_{w,r}/F_w}(h_{X_K, F_{w,r}}(Z_{1,F_{w,r}}, Z_{2,r})) \quad \in N_{F_{w,r}/F_w}(\mathscr{O}^\times_{F_{w,r}} \hat{\otimes} \mathscr{O}_L).$$

By the definition of $F_{w,r}$ and local class field theory, $N_{F_{w,r}/F_w}(\mathscr{O}^\times_{F_{w,r}} \hat{\otimes} \mathscr{O}_L) \subset p^{r-C'''}(\mathscr{O}^\times_{F_w} \hat{\otimes} \mathscr{O}_L)$ for some constant C'''. This completes the proof. $\qquad\square$

10.5. The arithmetic relative-trace formula.

The previous subsection shows that, for suitable f^p_1, f^p_2, we have a decomposition

$$\mathscr{J}_{K_p}(f^p_1, f^p_2) = \lim_{N\to\infty} \sum_{v \nmid \infty \text{ nonsplit}} \mathscr{J}^{(v),N}_{K_p}(f^p_1, f^p_2).$$

We state a geometric expansion of $\mathscr{J}^{(v),N}_{K_p}$ (in fact, $\mathscr{J}^{(v)}$) for inert places v. When F/F_0 is unramified, we then deduce a geometric expansion of $\mathscr{J}_{K_p}$, thus completing the corresponding RTF.

10.5.1. *Local arithmetic intersection numbers and geometric expansions at inert places.* Let $v \nmid 2p$ be an inert finite place of F_0 and let w be the unique place of F above v. We define for $\delta \in G^{V(v)}_{\text{rs}}(F_{0,v})$,

$$\mathscr{J}_{\delta,v}(e_{K_v}) := -(\delta \cdot \mathcal{N}_{n,v}, \mathcal{N}_{n,v})\, \lambda(\varpi_w), \tag{10.5.1}$$

where, in the right hand side, $(-,-)$ denotes the arithmetic intersection number on the unitary Rapoport–Zink space $\mathcal{N}_{n,v} \times_{\text{Spf } \mathscr{O}_{\breve{F}_v}} \mathcal{N}_{n+1,v}$ (resp. the small resolution in [ZZh]) if K_v is hyperspecial (resp. K_v is vertex parahoric), relative to the quadratic field extension $F_w/F_{0,v}$. Since $\mathscr{J}_{\delta,v}(e_{K_v})$ only plays an intermediate role, we refer to [MZ] (resp. [ZZh]) for the unexplained notation in the hyperspecial (resp. parahoric) case.

Proposition 10.5.1. *Let $v \nmid 2p$ be an inert finite place of F_0. Let f_1, $f_2 \in \mathscr{H}(G(\mathbf{A}), L)^\circ_{\text{temp}}$ and let $f = f_1 * f^\vee_2$. Suppose that:*

(1) (f_1, f_2) is regular at a place different from v;

(2) $f_{1,v} = f_{2,v} = e_{K_v}$ where K_v is a vertex parahoric subgroup of type (t,t) (cf. §9.3);

(3) K_v is hyperspecial or $T(f_1)$, $T(f_2)$ annihilate $H^{2n}(\mathscr{X}'_u, L(n))$.

Then

$$\mathscr{J}^{(v)}(f_1, f_2) = \sum_{\delta \in \mathrm{B}_{\mathrm{rs}}^{V(v)}(F_0)} J_\delta^v(f^v) \mathscr{J}_{\delta,v}(f_v)$$

$$= \sum_{\gamma \in \mathrm{B}_{\mathrm{rs}}'(F_0)} e_{V(v)}(\gamma) J_{\underline{\delta}(\gamma)}^v(f^v) \mathscr{J}_{\underline{\delta}(\gamma),v}(f_v),$$

where in the last expression, $\underline{\delta}$ is the matching of orbits of (3.5.3).

Proof. It suffices to show the first equality. Similar to the proof of Proposition 10.4.4, by the base change property of Lemma A.2.4(2) we have

$$\mathscr{J}^{(v)}(f_1, f_2) = \frac{1}{\deg(X_u'/X_w)} h_u(Z'.T(f_1), Z'.T(f_2)),$$

where h_u denotes the local height on X_u' over $\mathscr{O}_{E,u}$. Under our assumption, by Proposition A.3.4 we have

$$h_u(Z'.T(f_1), Z'.T(f_2)) = (\mathscr{Z}'.T(f_1), \mathscr{Z}'.T(f_1)) \lambda(\mathrm{Nm}_{E_u/F_w} \varpi_u).$$

Since $\lambda_{|F_w^\times}$ is necessarily unramified and E_u/F_w is an unramified extension, we have

$$\lambda(\mathrm{Nm}_{E_u/F_w} \varpi_u) = \deg(E_u/F_w)\lambda(\varpi_w).$$

In the hyperspecial case, by [RSZ20, Theorem 8.15] (the statement there is for the sum over all places of E above w, but the proof contains the formula for each place u), we obtain

$$(\mathscr{Z}_u'.T(f_1), \mathscr{Z}_u'.T(f_1)) = \deg(X_u'/X_w) \sum_{\delta \in \mathrm{B}_{\mathrm{rs}}^{V(v)}(F_0)} J_\delta^v(f^v) \mathscr{J}_{\delta,v}(f_v).$$

The vertex parahoric case is similar and we omit the details. $\square$

Remark 10.5.2. For the condition (2), we could relax it to allow vertex parahoric subgroup K_v of type $(t, t + \epsilon)$ with $\epsilon \in \{0, 1\}$. But this implicitly violates the convention in §2.1.3 and we will need to renormalize the matching of orbits that appears in the statement of the proposition.

10.5.2. *The arithmetic relative-trace formula.* We are ready to deduce the following relative-trace formula for $\mathscr{J}_{K_p}$. Recall the matching of global orbits $\underline{\delta}$ of (3.5.4).

Theorem 10.5.3 (Arithmetic relative-trace formula). *Suppose that:*

— *F/F_0 is unramified,*

— *$p > 2n$ if $n > 1$,*

— *all places $v|2p$ of F_0 are split in F.*

Suppose that there is a finite set S of places of F_0, not above p or ∞, and a compact open subgroup $K^p = \prod_{v \nmid p} K_v$ satisfying:

— *K_v is (self-dual) hyperspecial for $v \notin S$,*

— *for every split place $v \in S$, $K_v = K_{n,v} \times K_{n+1,v}$ where either at least one of the factors is maximal hyperspecial or both are Iwahori,*

— *for every inert $v \in S$, K_v is a vertex parahoric subgroup of type (t, t) (cf. §9.3),*

For $i = 1, 2$, let $f_i^p = f_i^p = f_i^{Sp} \otimes \otimes_{v|S} f_{i,v} \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)_{K^p}^\circ$ $f_i^p = f_i^{Sp} \otimes \otimes_{v|S} f_{i,v}$ satisy the following properties:

- *for every inert v, $f_{1,v} = f_{2,v} = e_{K_v}$,*

- *the pair $(f_{1,v}, f_{2,v})$ has regular support at two (necessarily split) places $v \in S$,*

- *for every finite place $v \in S$, $T(f_i^{Sp})$ annihilates $H^{2n}(\mathscr{X}_u', L(n))$ for some place u of E that is unramified over v.*

Let $f^p := f_1^p \star f_2^{p,\vee} \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)_{K^p}^\circ$. Then we have a spectral and a geometric expansion

$$\mathscr{J}_{K_p}(f^p) = \sum_{\pi \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})_{K_p}^{\mathrm{ord}}} \mathscr{J}_{\pi, K_p}(f^p)$$

$$= \int_{\mathrm{B}_{\mathrm{rs}}'(\mathbf{A}^p)} \sum_{\substack{v \nmid p\infty \\ \mathrm{nonsplit}}} \mathbf{1}_{V(v)}(\gamma) J_{\underline{\delta}(\gamma)}^{vp}(f^{vp}) \mathscr{J}_{\underline{\delta}(\gamma),v}(f_v) \, dI_{\gamma,p,K_p'}^{\mathrm{ord}},$$

where $dI_{\gamma,p,K_p'}^{\mathrm{ord}} = dI_{\gamma,p,K_p'}^{\mathrm{ord}}(\mathbf{1}_p)$ is as in (6.1.3) for $K_p' = \mathrm{G}'(\mathscr{O}_{F_{0,p}})$.

Proof. The spectral expansion was noted in § 10.3.3. We establish the geometric expansion. By (10.4.1), we have

$$\mathscr{J}(f^p) = \lim_{N \to \infty} \sum_{v \nmid \infty} \mathscr{J}_{K_p}^{(v),N}(f_1^p, f_2^p).$$

By Propositions 10.4.4, 10.4.5, only the terms corresponding to nonsplit places $v \nmid p$ contribute. (We use the 'second' place of regular support to apply Proposition 10.4.4 to the 'first' one.) By (10.4.2) and Proposition 10.5.1, we then have

$$\mathscr{J}(f^p) = \lim_{N \to \infty} \sum_{\substack{v \nmid p\infty \\ \mathrm{nonsplit}}} \sum_{\gamma \in \mathrm{B}_{\mathrm{rs}}'(F_0)} \mathbf{1}_{V(v)}(\gamma) J_{\underline{\delta}(\gamma)}^{vp}(f^{vp}) \mathscr{J}_{\underline{\delta}(\gamma),v}(f_v) \cdot J_{\underline{\delta}(\gamma)}(f_{p,K_p,N} * f_{p,K_p,N}^\vee).$$

The asserted form of the geometric expansion then follows, via Lemma 3.5.6 and Lemma 5.2.11, from the definition of $dI_{\gamma,p,K_p'}^{\mathrm{ord}}$. $\qquad\square$

Epilogue

11. Comparison of RTFs and proof of the main theorem

We compare the distributions $\mathscr{J}_{K_p}$ and $\partial\mathscr{I}_{K_p'}$ to deduce our main theorem. Throughout this section we assume:

- F/F_0 is unramified,

- $p > 2n$ if $n > 1$,

- all places $v|2p$ of F_0 are split in F.

11.1. Comparison of relative-trace formulas.

The comparison will be based on the following local result.

Theorem 11.1.1 ([Zha21, MZ, ZZh]). *Let v be an inert place of F_0 and assume that either of the following conditions on $K_v \subset G_v$, $K'_v \subset G'_v$ hold:*

(1) K_v is hyperspecial, and $K'_v = G(\mathscr{O}_{F_{0,v}})$;

(2) $K_v = K_{n,v} \times K_{n+1,v}$ is a vertex parahoric subgroup of type (t,t) (cf. §9.3), and $K'_v = K'_{n,v} \times K'_{n+1,v}$ where $K'_{\nu,v}$ is the stabilizer in $G'_\nu(F_{0,v})$ of both the vertex lattice defining $K_{\nu,v}$ and its dual lattice.

Suppose that $\gamma \in B'_{\mathrm{rs},v}$ matches an orbit $\delta = \underline{\delta}(\gamma) \in B_{\mathrm{rs},v,V_v}$ for the hermitian pair V_v with $\epsilon(V_v) = -1$ (cf. (1.3.1)). Then

$$\mathscr{J}_{\delta,v}(e_{K_v}) = \partial\mathscr{I}_{\gamma,v}(e_{K'_v}).$$

Proof. By the definitions, the identity is equivalent to

$$-(\delta \cdot \mathcal{N}_{n,v}, \mathcal{N}_{n,v}) = \partial\mathscr{I}_\gamma(e_{K'_v})/\lambda(\varpi_w) = \left.\frac{d}{ds}\right|_{s=0} I^{\mathbf{C}}_{\gamma,v}(e_{K'_v}, |\cdot|^{-s}_{F_v})/(-\log q^2_v) \qquad (11.1.1)$$

(where w is the place of F above v, and the 'division' in the second term has the obvious meaning).

In the hyperspecial case, the identity (11.1.1) is the Arithmetic Fundamental Lemma conjecture proved in [Zha21, MZ]. In the vertex parahoric case, (11.1.1) (an instance of Arithmetic Transfer conjecture) is recently proved by Z. Zhang [ZZh].

There are two points where the formulation in those works appears different. First, they consider a version with derivatives of 'inhomogenous' orbital integrals; this is verified to be equivalent to the above homogeonous version as in [RSZ18, Proposition 14.1 (ii)]. Second, their identity apparently differs from ours by a sign -1: the reason is that their orbital integral contains a transfer factor defined as in § 2.4 *ibid.*; under our assumptions on γ and v, that transfer factor (in its inhomogeneous version), evaluated at a preimage $\gamma' \in G'_{\mathrm{rs},v}$ of γ, differs from our $\kappa_v(\gamma', \mathbf{1})$ by -1. $\square$

We can now make the global comparison.

Theorem 11.1.2 (Comparison of RTFs). *Let S, $K^p = \prod_{v\nmid p} K_v$, and*

$$f^p \in \mathscr{H}(G(\mathbf{A}^p), L)^\circ_{K_p}$$

be as in Theorem 10.5.3. Write $S = S^{\mathrm{spl}} \sqcup S^{\mathrm{in}}$ as a union of sets of split and inert places.

Let $K'_p := G'(\mathscr{O}_{F_{0,p}})$ and let $K'^p = \prod_{v\nmid p} K'_v \subset G'(\mathbf{A}^{p\infty})$ be a compact open subgroup satisfying:

– *for every $v \notin S$, $K'_v = G'_\nu(\mathscr{O}_{F_{0,v}})$ is hyperspecial;*

– *for every inert $v \in S$, $K'_v = K'_{n,v} \times K'_{n+1,v}$ and $K'_{\nu,v}$ is the stabilizer in $G'_\nu(F_{0,v})$ of both the vertex lattice defining $K_{\nu,v}$ and its dual lattice.*

Let

$$f'^p = f'^{Sp} \otimes f'_{S^{\mathrm{spl}}\infty} \otimes f'_{S^{\mathrm{in}}} \in \mathscr{H}(G'(\mathbf{A}^p), L)^\circ_{K'^p\text{-rs, qc}}$$

be a quasicuspidal, regularly supported Gaussian whose factors satisfy the following properties:

– $f'^{S^{\mathrm{spl}}p} = \otimes_v f'_v$ *with $f'_v = e_{K'_v}$;*

– $f'_{S^{\mathrm{spl}}\infty}$ *matches $f_{S^{\mathrm{spl}}\infty}$;*

Then $\mathscr{I}_{K'_p}(f'^p, \mathbf{1}) = 0$ *and*

$$\mathscr{J}_{K_p}(f_p) = \partial \mathscr{I}_{K'_p}(f'_p).$$

Proof. We first show that f^p and f'^p match under the assumption. By our conditions and the Jacquet–Rallis Fundamental lemma (Proposition 3.5.4), $f^{S^{\mathrm{in}}p}$ and $f'^{S^{\mathrm{in}}p}$ match. The theorem of [ZZh] on transfer at vertex parahoric levels shows that f_v and f'_v match at the places in S_{in} as well.

It follows that the function f'^p is incoherent, hence $\mathscr{I}_{K'_p}(f'^p, \mathbf{1}) = 0$ by Proposition 6.3.1.

Next we compare the geometric expansions of both sides of the desired equality, given by Theorem 10.5.3 and Proposition 6.3.1 (3) respectively. By the identity of Theorem 11.1.1, these are equal term by term. The proof is complete. $\qquad\square$

Remark 11.1.3. The theorem is implicitly using the result mentioned in Remark 4.2.4 (cf. Remark 6.3.2).

11.2. **Test Hecke measures.** We find some $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ$, $f'^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ$ to which the comparison may be applied, and that isolate a given pair of representations over L.

We will from now admit the following local hypothesis, which is a special case of a result expected to appear in [Dan].

Hypothesis 11.2.1. *Let v be an inert place of F_0 and let $\pi_v = \pi_{n,v} \boxtimes \pi_{n+1,v}$ be a representation of G_v such that $\pi_{n,v}$ is either unramified or almost unramified, and $\pi_{n+1,v}$ is almost unramified. Let $f_v = e_{K_v}$ where $K_v \subset G_v$ is a vertex parahoric subgroup of type (t, t). Then*

$$J_{\pi_v}(f_v) \neq 0.$$

Lemma 11.2.2. *Let $\pi \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord,st}}_{K_p}(L)$ and let $\Pi = \mathrm{BC}(\pi)$. Assume that:*

– *for every place v of F_0 that is split in F/F_0, at least one of $\pi_{n,v}$ and $\pi_{n+1,v}$ is unramified;*

– *for every place v of F_0 that is inert in F/F_0, $\pi_{n,v}$ and $\pi_{n+1,v}$ are either unramified or almost unramified, and if $\pi_{n,v}$ is almost unramified then $\pi_{n+1,v}$ is also almost unramified.*

Then there exist:

– *a finite set S of places of F_0, not above p or ∞,*

– *open compact subgroups $K^p = \prod_{v\nmid p} K_v \subset \mathrm{G}(\mathbf{A}^{p\infty})$ and $K'^p = \prod_{v\nmid p} K'_v \subset \mathrm{G}'(\mathbf{A}^{p\infty})$,*

– *Hecke measures $f_1^p, f_2^p, f^p := f_1^p * f_2^{p,\vee} \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p}$ and $f'^p \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ_{K'_p\text{-rs, qc}}$,*

such that:

– *$(S, K^p, f_1^p, f_2^p, K'^p, f'^p)$ satisfy the conditions of Theorem 10.5.3 and of Theorem 11.1.2;*

– *$M^{\oplus,*}.T(f_i^p e_{K_p}) \subset M_\pi^{K_p}$;*

– *$\Pi'(f'^p e_{K_p}) = 0$ for every $\Pi \neq \Pi' \in \mathscr{C}$;*

– *$\otimes_{v\nmid p} J_{\pi_v}(f^p) = \otimes_{v\nmid p} I_{\Pi,v}(f'^p) \neq 0$.*

Proof. We construct f_1, f_2^p, f'^p as products whose various factors take care of the required conditions.

Regularity of the supports. Let S^{rs} be a set consisting of two split places of F_0 at which Π is an unramified regular principal series (cf. Lemma 4.3.3). For each $v \in S^{\mathrm{rs}}$, we take K'_v to be the standard Iwahori subgroup, and $f'_v \in \mathscr{H}(G'_v, L)_{K'_v}$ a regularly supported element with $I_{\Pi_v}(f'_v, \mathbf{1}) \neq 0$ as provided by Lemma 4.3.1 (3). Upon a choice of a basis of V_v, we have the matching $f_{v,1} \in \mathscr{H}(G_v, L)_{K_v}$ for an Iwahori subgroup $K_v \subset G_v$; we put $f_{v,2} \coloneqq e_{K_v}$. Thus $f_v = f_{v,1} * f_{v,2}^\vee = f_{v,1}$ still matches f'_v and is regularly supported. For $i = 1, 2$, we put

$$f_{S^{\mathrm{rs}},i} = \otimes_{v \in S^{\mathrm{rs}}} f_{v,i}, \qquad f'_{S^{\mathrm{rs}}} = \otimes_{v \in S^{\mathrm{rs}}} f'_v.$$

Choices at places of ramification. Let S^R be the finite set of places $v \notin S^{\mathrm{rs}} p \infty$ of F_0 where at least one of $\pi_{n,v}, \pi_{n+1,v}$ is ramified. Then for every *split* $v \in S^R$, we let $K_v = K_{n,v} \times K_{n+1,v}$ such that $\pi^{K_v} \neq 0$ and $K_{\nu,v}$ is hyperspecial if $\pi_{\nu,v}$ is unramified. Then we pick any $f_{v,1}, f_{v,2}, f_v \coloneqq f_{v,1} * f_{v,2}^\vee \in \mathscr{H}(G_v, L)_{K_v}$ such that $J_{\pi_v}(f_v) \neq 0$. For every *inert* $v \in S^R$, we let K_v be the vertex parahoric subgroup such that $\pi_v^{K_v} \neq 0$ and we let $f_v = e_{K_v}$. The non-vanishing $J_{\pi_v}(f_v) \neq 0$ is well-known in the hyperspecial case, and it is Hypothesis 11.2.1 in the remaining vertex parahoric case. We put $f_{S^R,i} = \otimes_{v \in S^R} f_{v,i}$ and we let $f'_{S^R} \in \mathscr{H}(G'_{S^R})$ be the element matching f_{S^R}.

Isolation of π and Π. Now take $S = S^R \cup S^{\mathrm{rs}}$. For $v \notin Sp$, we let K_v, K'_v be hyperspecial, and form $K = \prod_v K_v$, $K' = \prod_v K'_v$. Consider the split Hecke algebras

$$\mathbb{T} = \mathbb{T}^{\mathrm{spl}} \coloneqq \bigotimes_{\substack{v \nmid Sp \\ \mathrm{split}}} \mathscr{H}(G_v, L)_{K_v} \qquad\qquad \subset \mathscr{H}(\mathrm{G}(\mathbf{A}^{Sp}, L)^\circ_{K^S}$$

$$\mathbb{T}' = \mathbb{T}'^{\mathrm{spl}} \coloneqq \bigotimes_{\substack{v \nmid Sp \\ \mathrm{split}}} \mathscr{H}(G'_v, L)_{K'_v} \otimes_L \mathscr{H}(G'_\infty, L)^\circ \quad \subset \mathscr{H}(\mathrm{G}'(\mathbf{A}^{Sp}, L)^\circ_{K'Sp}.$$

Let $f_{\pi,1} = f_{\pi,2} \in \mathbb{T}$ be an element acting as the idempotent projection from $M_K^{\oplus,*}$ onto M_π^K, which exists by Lemma **??** (for Σ the finite set of representations occurring in $M_K^{\oplus,*}$). Let $f'_\pi \in \mathbb{T}'$ be an element supported at the finite places and matching $f_\pi \coloneqq f_{\pi,1} * f_{\pi,2}^\vee$.

Let $f'_\Pi \in \mathbb{T}'$ be an element such that $\Pi(f'_\Pi) = 1$ and for each $\iota \colon L \hookrightarrow \mathbf{C}$, $R(f'^\iota_\Pi)$ sends $\mathscr{A}(\mathrm{G}')^K$ into $\Pi^{\iota,K}$, which exists by Proposition 4.3.2; let $f_{\Pi,1} \in \mathbb{T}$ be a matching element and let $f_{\Pi,2}$ be the unit of $\mathbb{T}$.

Annihilation of absolute cohomology. For every place $v \in S$, by the vanishing Theorem 9.4.1 (1) (applied to the maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ corresponding to the eigensystem attached to π), there exists $f_{\{v\},1} = f_{\{v\},2} \in \mathbb{T}$ which annihilates $H^{2n}(\mathscr{X}'_u, L(n))$ (for $\mathscr{X}'_u$ as in § 10.4.2), and acts by a non-zero scalar on the line $\bigotimes_{\substack{v \nmid Sp \\ \mathrm{split}}} \pi_v^{K_v}$. Let $f'_{\{v\}} \in \mathbb{T}'$ be an element supported at the finite places and matching $f_{\{v\}} \coloneqq f_{\{v\},1} * f_{\{v\},2}^\vee$.

Assembly. For $i = 1, 2, \emptyset$, we define

$$f_i^{Sp} = f_{\pi,i} f_{\Pi,i} \otimes \otimes_{v \in S} f_{\{v\},i} \in \mathbb{T}, \qquad f'^{Sp} = f'_\pi f'_\Pi \otimes \otimes_{v \in S} f'_{\{v\}} \in \mathbb{T}',$$

viewed naturally as elements in $\mathscr{H}(\mathrm{G}(\mathbf{A}^{Sp}), L)^\circ_{K^{Sp}}$, $\mathscr{H}(\mathrm{G}'(\mathbf{A}^{Sp}), L)^\circ_{K'Sp}$. Then we define

$$f_i^p = f_{S,i} f_i^{Sp}, \qquad f'^p = f'_S f'^{Sp}.$$

Then it is easy to see that, by construction, f_i^p satisfies the required conditions. To check the condition on spherical characters, we use

$$\otimes_{v \nmid p} J_{\pi_v}(f^p) = \otimes_{v \notin Sp} J_{\pi_v}(f^{Sp}) \prod_{v \in S} J_{\pi_v}(f_v).$$

The product over $v \in S$ does not vanish by construction; the first factor is the product of $\otimes_{v \notin Sp} J_{\pi_v}(e_{K^{Sp}}) \neq 0$ and of the eigenvalue of f^{Sp} acting on the line $\pi^{K^{Sp}}$, which is a non-zero scalar. $\qquad\square$

11.3. **Proof of the main theorem.** We first reduce the identity

$$h_\pi(Z_\pi(\phi), Z_{\pi^\vee}(\phi')) = e_p(M_\Pi)^{-1} \cdot \frac{1}{4} \partial \mathscr{L}_p(M_\Pi) \cdot \alpha(\phi, \phi') \tag{11.3.1}$$

of Theorem D to the factorization

$$\mathscr{I}_{\pi, K_p}(f^p) = \frac{1}{4} \partial \mathscr{L}_p(M_\Pi) \cdot \otimes_{v \nmid p} J_{\pi_v}(f^p). \tag{11.3.2}$$

Lemma 11.3.1. *Let* $\pi \in \mathscr{C}(G)_{K_p}^{\mathrm{ord}}$*, and let* $\Pi = \mathrm{BC}(\pi)$*,* $L = \mathbf{Q}_p(\pi)$*. The following are equivalent:*

(1) For every $\phi \in \pi$*,* $\phi' \in \pi^\vee$*, the identity* (11.3.1) *holds.*

(2) For some $\phi \in \pi$*,* $\phi' \in \pi^\vee$ *such that* $\alpha(\phi, \phi') \neq 0$*, the identity* (11.3.1) *holds.*

(3) For every $f^p \in \mathscr{H}(G(\mathbf{A}), L)^\circ$*, the factorization* (11.3.2) *holds.*

(4) For some $f^p \in \mathscr{H}(G(\mathbf{A}), L)^\circ$ *such that* $\otimes_{v \nmid p} J_{\pi_v}(f^p) \neq 0$*, the factorization* (11.3.2) *holds.*

Proof. It is trivial that (1) implies (2), and (3) implies (4). The two converse implications follow from multiplicity one and the nonvanishing of α.

We prove that (3) is equivalent to (1). It is clear that (1) is equivalent to

$$\mathrm{Tr}_{(,)_\pi}^{h \circ Z_\pi \boxtimes Z_{\pi^\vee}}(\tau) = e_p(M_\Pi)^{-1} \cdot \frac{1}{4} \partial \mathscr{L}_p(M_\Pi) \cdot \mathrm{Tr}_{(,)_\pi}^\alpha(\tau) \tag{11.3.3}$$

for all $\tau \in \mathrm{End}(\pi)$, and equivalently for some τ such that $\mathrm{Tr}_{(,)_\pi}^\alpha(\tau) \neq 0$. Thus it is enough to show that (11.3.2) is equivalent to (11.3.3) for some such τ.

Choose a factorization $(,)_\pi = (,)_{\pi^p}(,)_{\pi_p}$. For any $N \geq 1$, let $f_{p, K_p, N} := (10.3.4) \in \mathscr{H}(G_p, L)$, let $f_{p, K_p, N}^* := f_{p, K_p, N} * f_{p, K_p, N}^\vee$, and for $? \in \{\emptyset, \vee\}$, let

$$\pi_p^?(f_{p, K_p}) := \lim_{N \to \infty} \pi_p^?(f_{p, K_p, N}) \in \mathrm{End}(\pi_p).$$

(This does not depend on the integer $1 \leq r \leq N!$ implicit in (10.3.4).) Let

$$\pi_p(f_{p, K_p}^*) := \pi_p(f_{p, K_p}) \circ (\pi_p^\vee(f_{0, p, K_p}))^\vee,$$

where $(-)^\vee$ denotes the transpose with respect to $(,)_{\pi_p}$. Then by the definition in § 10.3.3, we have

$$\mathrm{Tr}_{(,)_\pi}^{h \circ Z_\pi \boxtimes Z_{\pi^\vee}}(\pi^p(f^p) \pi_p(f_{p, K_p}^*)) = \mathscr{I}_{\pi, K_p}(f^p) \tag{11.3.4}$$

On the other hand, it is clear from the definitions that

$$\mathrm{Tr}_{(,)_\pi}^\alpha(\pi^p(f^p) \pi_p(f_{p, K_p}^*)) = \otimes_{v \nmid p} J_{\pi_v}(f^p) \cdot \lim_{N \to \infty} J_{\pi_p}(f_{p, K_p, N}^*) \tag{11.3.5}$$

Now by Lemma 3.5.6, $f^*_{p,K_p,N}$ matches the function $f'_{p,K'_p,N}$ attached to $U^{N!}_{t_p}$ as in Lemma 5.2.11. By the definitions and Corollary 5.2.10, we then have

$$\lim_{N\to\infty} J_{\pi_p}(f^*_{p,K_p,N}) = \lim_{N\to\infty} I_{\Pi_p}(f'_{p,K'_p,N}) = e_p(\mathrm{M}_\Pi). \tag{11.3.6}$$

(Recall that $e_p(\mathrm{M}_\Pi)$ is the product of the factors $e(\Pi_v, \mathbf{1}_v)$ of (5.2.9).) Thus by (11.3.4), (11.3.5), (11.3.6), the identity (11.3.2) for f^p is equivalent to (11.3.3) for $\tau = \pi^p(f^p)\pi_p(f^*_{p,K_p})$. This completes the proof. $\qquad\square$

We may now prove Theorem D based on the comparison of relative-trace formulas in Theorem 11.1.2. i have edited the proof using the lemma

Proof of Theorem D. By Lemma 11.3.1, it suffices to prove

$$\mathscr{J}_{\pi,K_p}(f^p) = \frac{1}{4}\partial\mathscr{L}_p(\mathrm{M}_\Pi) \cdot \otimes_{v\nmid p} J_{\pi_v}(f^p) \tag{11.3.7}$$

for any f^p such that $\otimes_{v\nmid p} J_{\pi_v}(f^p) \neq 0$.

Let S, K^p, f^p, f'^p be as in Lemma 11.2.2. By construction, $\otimes_{v\nmid p} J_{\pi_v}(f^p) \neq 0$, the elements f^p and f'^p match (geometrically), and Theorem 11.1.2 is applicable and it gives

$$\mathscr{J}_{K_p}(f_p) = \partial\mathscr{I}_{K'_p}(f'_p).$$

By Theorem 10.5.3 and Proposition 6.3.1 (2), we have an equality of spectral expansions

$$\sum_{\pi\in\mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}} \mathscr{J}_{\pi,K_p}(f^p) = \sum_{\Pi\in\mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V}_{K_p}} \partial\mathscr{I}_{\Pi,K_p}(f'^p),$$

but by construction only the terms corresponding to π and Π may be nonzero. We deduce that

$$\mathscr{J}_{\pi,K_p}(f^p) = \frac{1}{4}\partial\mathscr{L}_p(\mathrm{M}_\Pi) \cdot \otimes_{v\nmid p} I_{\Pi_v}(f'^p),$$

which is equivalent to the desired factorization (11.3.7) by the (spectral) matching of f^p and f'^p. $\qquad\square$

Proof of Theorem C. The main implication follows immediately from Theorem D, choosing the unique distinguished π such that $\Pi = \mathrm{BC}(\pi)$. The strengthened implication then follows from [LTX$^+$22] as observed in Remark 1.3.2. $\qquad\square$

Appendix A. p-adic Abel-Jacobi maps and p-adic heights

We summarize the definitions and results we need from the theory of p-adic heights. For more details or more general setups, see Nekovář's original paper [Nek93] and [DL, Appendix A]; our constructions follow the sign conventions of the latter reference. Nothing in this appendix is new.

We denote by L a finite extension of $\mathbf{Q}_p$, and by Γ a finite-dimensional L-vector space.

A.1. p-adic Abel–Jacobi maps and biextensions.

Let F be a field of characteristic different from p, and let X be a smooth projective scheme over F of pure dimension $m-1 \geq 1$. We denote by $\mathrm{Z}^\bullet(X)_R$ the module of $\bullet$-dimensional algebraic cycles with coefficients in a ring R (omitted

from the notation when $R = \mathbf{Z}$), and by $\mathrm{Ch}^\bullet(X)_R = \mathbf{Z}^\bullet(X)_R/(\text{rational equivalence})$ the Chow R-module. We denote $H^i(F, -) = H^i_{\mathrm{cont}}(G_F, -)$.

A.1.1. *p-adic Abel–Jcobi maps.* Let $0 \le d \le m$ and consider the absolute étale cohomology $H^{2d}(X, L(d))$. By the Hochschild–Serre spectral sequence, it has a filtration $\mathrm{Fil}^\bullet$ with

$$0 \longrightarrow H^1(F, H^{2d-1}(X_{\overline{F}}, L(d))) \longrightarrow H^{2d}(X, L(d))/\mathrm{Fil}^2 \longrightarrow H^0(F, H^{2d}(X_{\overline{F}}, L(d))) \longrightarrow 0.$$

We denote by $\overline{\mathrm{cl}}\colon \mathbf{Z}^d(X)_L \to H^0(G_F, H^{2d}(X_{\overline{F}}, L(d))$ the geometric cycle class map, by $\mathbf{Z}^d(X)^0_L$ its kernel, and we let

$$\widetilde{\mathrm{cl}}\colon \mathbf{Z}_d(X)_L \longrightarrow H^{2d}(X, L(d)))/\mathrm{Fil}^2,$$
$$\mathrm{cl}\colon \mathbf{Z}_d(X)^0_L \longrightarrow H^1(F, H^{2d-1}(X_{\overline{F}}, L(d)))$$

be the absolute cycle class map and the Abel–Jacobi map, respectively. The maps $\overline{\mathrm{cl}}$, $\widetilde{\mathrm{cl}}$ factor through the Chow group $\mathrm{Ch}^d(X)$, and the map cl factors through the image $\mathrm{Ch}^d(X)^0 \subset \mathrm{Ch}^d(X)$ of $\mathbf{Z}^d(X)^0$.

If $M \subset H^{2d-1}(X_{\overline{F}}, L(d))$ is a G_F-stable subspace, we denote by

$$\mathbf{Z}^d_M(X)^0_L, \quad \mathrm{Ch}^d_M(X)^0_L$$

the preimages in $\mathbf{Z}^d(X)^0_L$, $\mathrm{Ch}^d(X)^0_L$ of $H^1(F, M) \subset H^1(F, H^{2d-1}(X_{\overline{F}}, L(d)))$ under the Abel–Jacobi map.

Suppose that F is non-archimedean of residue characteristic ℓ. We will consider subspaces M satisfying the condition:

(1) if $\ell \ne p$: $H^1(F, M) = 0$;

(2) if $\ell = p$: $H^1_{\mathrm{st}}(F, M) = H^1_f(F, M)$.

Remark A.1.1. Since by [NN' 16, Theorem B] the map cl takes values in the subspace

$$H^1_{\mathrm{st}}(F, H^{2d-1-1}(X_{\overline{F}}, L(d))),$$

the conditions above imply that

$$\mathrm{cl}(\mathbf{Z}^d_M(X)^0_L) \subset H^1_f(F, M).$$

If M is pure of weight -1 (as is implied for all $M \subset H^{2d-1}(X_{\overline{F}}, L(d))$ by the weight-monodromy conjecture), then the relevant one among the conditions above is satisfied.
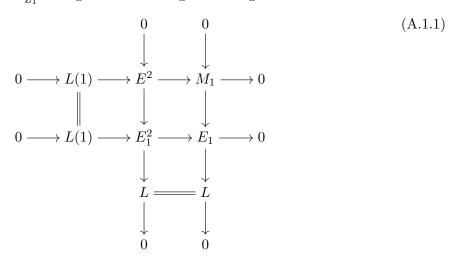
A.1.2. *Biextensions from algebraic cycles.* Let $d_1, d_2 \ge 0$ be integers with $d_1 + d_2 = m$, and let

$$Z_1 \in \mathbf{Z}^{d_1}(X)^0_L, \quad Z_2 \in \mathbf{Z}^{d_2}(X)^0_L$$

be cycles with disjoint supports. Let $M_i := H^{2d_i-1}(X_{\overline{F}}, L(d_i))$. To each Z_i is associated an extension of $L[G_F]$-modules

$$0 \to M_i \to E_i \to L \to 0$$

whose class in $H^1(F, M_i)$ is the p-adic Abel–Jacobi image $\mathrm{cl}(Z_i)$. A further geometric construction yields the *biextension* $E_1^2 = E_{Z_1}^{Z_2}$ fitting in the following exact diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \text{(A.1.1)} \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L(1) & \longrightarrow & E^2 & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L(1) & \longrightarrow & E_1^2 & \longrightarrow & E_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $M_1 = M_2^*(1)$ via Poincaré duality, and $E^2 := E_2^*(1)$. We denote its class by $[E_1^2] \in H^1(F, E^2)$.

A.2. **Height pairings.** We collect some definitions and properties of local and global height pairings.

A.2.1. *Local height pairings of algebraic cycles.* Suppose that F is non-archimedean of residue characteristic ℓ. Let $\lambda \colon F^\times \hat{\otimes} L \to \Gamma$ be an L-linear map.

For $i = 1, 2$ let $M_i \subset H^{2d_i-1}(X_{\overline{F}}, L(d_i))$ be $L[G_F]$-submodules, and denote still by $\langle \, , \, \rangle \colon M_1 \otimes_L M_2 \to L(1)$ the restriction of the Poincaré pairing

$$\langle \, , \, \rangle \colon H^{2d_1-1}(X_{\overline{F}}, L(d_1)) \otimes_L H^{2d_2-1}(X_{\overline{F}}, L(d_2)) \xrightarrow{\cup} H^{2m-2}(X_{\overline{F}}, L(m)) \xrightarrow{\mathrm{Tr}} L(1),$$

where the map Tr is the sum of the trace maps for the connected components of X. Assume that M_1, M_2 satisfy the following conditions:

(1) $\langle \, , \, \rangle \colon M_1 \otimes_L M_2 \to L(1)$ is a perfect pairing;

(2) if $\ell \neq p$, we have $H^0(F, M_i) = 0$ for $i = 1, 2$; this implies condition (1) for M_1, M_2 in § A.1.1, and is implied by the condition that M_i is pure of weight -1;

(3) if $\ell = p$:

  — M_i is crystalline with $\mathrm{D_{crys}}(M_i)^{\varphi=1} = 0$ for $i = 1, 2$; this implies condition (2) for M_1, M_2 in § A.1.1, and is implied by the condition that M_i is crystalline and pure of weight -1;

  — the *Panchishkin condition*: there is a (necessarily unique) extension of crystalline representations

$$0 \to M_i^+ \to M_i \longrightarrow M_i^- \to 0$$

  such that $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{dR}}(M_i^+) = \mathbb{D}_{\mathrm{dR}}(M_i^-)/\mathrm{Fil}^0 \mathbb{D}_{\mathrm{dR}}(M_i^-) = 0$; this implies that the natural map

$$\mathbb{D}_{\mathrm{dR}}(M_i^+) \oplus \mathrm{Fil}^0 \mathbb{D}_{\mathrm{dR}}(M_i) \xrightarrow{\cong} \mathbb{D}_{\mathrm{dR}}(M_i) \tag{A.2.1}$$

  is a splitting of the Hodge filtration on $\mathbb{D}_{\mathrm{dR}}(M_i)$.

Assume that $Z_1 \in \mathrm{Z}^{d_1}_{M_1}(X)^0$, $Z_2 \in \mathrm{Z}^{d_2}_{M_2}(X)^0$. Then the biextension class $[E^{Z_2}_{Z_1}]$ belongs to the preimage $H^1_{M_1\text{-}f}(F, E^2) \subset H^1(F, E^2)$ of $H^1_f(F, M_1)$ under the natural map $H^1_f(F, E^1) \to H^1(F, M_1)$. This group sits in the (pushout) diagram of exact sequences[20]

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(F, L(1)) & \xleftarrow{\;\sigma\;} & H^1_{M_1\text{-}f}(F, E^2) & \longrightarrow & H^1_f(F, M_1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & H^1_f(F, L(1)) & \xleftarrow{\;\sigma_f\;} & H^1_f(F, E^2) & \longrightarrow & H^1_f(F, M_1) & \longrightarrow & 0
\end{array}
\tag{A.2.2}
$$

admitting canonical splittings σ, σ_f. These are obvious if $\ell \neq p$, as then $H^1(F, M_1) = 0$; for $\ell = p$, they are induced by (A.2.1) (see [Nek93, § 4]). Morevoer, the Kummer map identifies $H^1(F, L(1)) \cong F^\times \hat{\otimes} L(1)$.

Definition A.2.1. Let M_1, M_2, Z_1, Z_2 be as above. We define

$$
h_{X,\lambda}(Z_1, Z_2) := \lambda \circ \sigma([E^2_1]) \in \Gamma. \tag{A.2.3}
$$

Remark A.2.2. Since the conditions on the pair (M_1, M_2) are stable under subobejcts and extensions (see [DL, Lemma A.10] for extensions when $\ell = p$), there is a maximal pair satisfying those; in particular we may omit (M_1, M_2) from the notation.

Remark A.2.3. If $\ell = p$, it follows from the previous discussion that $\sigma([E^2_1]) \in \mathscr{O}^\times_F \hat{\otimes} L \subset F^\times \hat{\otimes} L$ if and only if $[E^2_1]$ is crystalline (that is, belongs to $H^1_f(F, E^2)$).

Lemma A.2.4 (Base change)**.** *Consider the setup of Definition A.2.1.*

(1) Let F'/F be a finite extension, and let $\lambda' := \lambda \circ \mathrm{N}_{F'/F}$. Then for any $Z_1 \in \mathrm{Z}^{d_1}_{M_1}(X)^0$, $Z'_2 \in \mathrm{Z}^{d_2}_{M_2}(X_{F'})^0$,

$$
h_{X_F, \lambda}(Z_1, \mathrm{N}_{F'/F}Z'_2) = h_{X_{F'}, \lambda'}(Z_{1,F'}, Z'_2).
$$

(2) Let $u\colon X' \to X$ be a finite étale morphism, and let $Z_1 \in \mathrm{Z}^{d_1}_{M_1}(X)^0$, $Z_2 \in \mathrm{Z}^{d_2}_{M_2}(X)^0$. Denote by Z'_i the pullback of Z_i to X'. Assume $\ell \neq p$. Then

$$
h_{X,\lambda}(Z_1, Z_2) = \frac{1}{\deg u} h_{X', \lambda}(Z'_1, Z'_2).
$$

Proof. Part (1) is [Nek95, (II.1.9.1)]. Part (2) follows from [LL21, Lemma B.3] and [DL, Theorem A.4]. $\qquad\square$

A.2.2. *Global height pairings for Selmer groups.* Let now F be number field and $\lambda\colon \Gamma_{F,L} \to \Gamma$ be an L-linear map.

Let M_1, M_2 be L-vector spaces endowed with continuous G_F-representations that are unramfied at all but finitely many places of F, and de Rham at all the p-adic places. Assume moreover that M_1, M_2 are endowed with a perfect G_F-equivariant pairing $\langle\,,\,\rangle\colon M_1 \otimes_L M_2 \to L(1)$, and that for each i and each finite place w of F, the representation M_i restricted to G_{F_w} satisfies the conditions (2), (3) of § A.2.1.

---

[20]This diagram should also replace an incorrect one in [Dis17, (4.1.4)].

Under these conditions,[21] Nekovář [Nek93] defined a bilinear height pairing on the Bloch–Kato Selmer groups

$$h_{M_1,\lambda}\colon H^1_f(F, M_1) \otimes_L H^1_f(F, M_2) \longrightarrow \Gamma \qquad (\text{A.2.4})$$

as follows. For $i = 1, 2$ pick representatives E_i of the extension classes $[E_i] \in H^1_f(F, M_i)$, and let E^2_1 be a biextension fitting in a diagram (A.1.1) of G_F-representations. For each place w of F, one can then define $h^{E^2_1}_w([E_1], [E_2])$ by the right-hand side of (A.2.3) (where everything is viewed as a representation of G_w); the sum

$$h([E_1], [E_2]) := \sum_w h^{E^2_1}_w([E_1], [E_2])$$

does not depend on the choice of E^2_1.

Lemma A.2.5 (Projection formula). *Let (M_1, M_2) and (M'_1, M'_2) be as above. Let $\phi\colon M'_1 \to M_1$ be a map of G_F-representations, and let $\phi^*(1)\colon M_2 \to M'_2$ be the dual map. Let $[E'_1] \in H^1_f(F, M'_1)$, $[E_2] \in H^1_f(F, M_2)$. Denote by $E'_2 := \phi^*(1)_* E_2$, $E_1 := \phi_* E'_1$ the pushouts. Then*

$$h_{M'_1}([E'_1], [E'_2]) = h_{M_1}([E_1], [E_2]).$$

Proof. Let $E' \in H^1_f(F, E^{2\prime})$ be a biextension (as in (A.1.1)) of E'_1 and $E^{2\prime} := \phi^*(E^*_2(1)) = E'^*_2(1)$. The map $\phi\colon M'^*_2(1) \cong M'_1 \to M_1 \cong M^*_2(1)$ induces by pullback a map $\phi\colon E^{2\prime} \to E^2$. Then a diagram chase shows that $\phi_* E' \in H^1_f(F, E^2)$ is a biextension of E_1 and E^2. $\qquad\square$

A.2.3. *Decomposition in the case of algebraic cycles.* Let X be a proper smooth scheme over F of dimension $m - 1$, and suppose that $M_i \subset H^{2d_i-1}(X_{\overline{F}}, L(d_i))$ are $L[G_F]$-submodules satisfying the above conditions with respect to a pairing $\langle\,,\,\rangle$ that is the restriction of the Poincaré pairing. We then denote $h_{X,\lambda} := h_{M_1,M_2,\lambda}$, for which we have

$$h_{X,\lambda}(\mathrm{cl}(Z_1), \mathrm{cl}(Z_2)) = \sum_{w \nmid \infty} h_{X_w, \lambda_w}(Z_1, Z_2), \qquad (\text{A.2.5})$$

where the sum runs over all the non-archimedean places of F, and $X_w := X_{F_w}$, $\lambda_w := \lambda_{|F^\times_w \hat\otimes L}$.

A.3. **Relation to arithmetic intersection theory.** We collect two results relating local heights away from p, and the crystalline property of biextensions at p, with arithmetic intersections. We start with some preliminaries. For more details on the background, see [LL21, Appendix B] and references therein.

A.3.1. *Extensions of algebraic cycles.* Let $\mathscr{X}$ be a regular scheme; for a closed subset $\mathscr{Y}$ (omitted from the notation if $\mathscr{Y} = \mathscr{X}$) we denote by $K^{\mathscr{Y}}_0(\mathscr{X})$ the K-group of complexes of coherent sheaves on $\mathscr{X}$ with cohomology supported in $\mathscr{Y}$. We denote by $\mathrm{F}^\bullet$ the filtration on $K^{\mathscr{Y}}_0(\mathscr{X})$ by the codimension of support. We have an L-linear map

$$\kappa\colon \mathrm{Z}^d(\mathscr{X})_L \longrightarrow \mathrm{F}^\bullet K_0(\mathscr{X})_L \qquad (\text{A.3.1})$$

such that if $\mathscr{Z} \subset \mathscr{X}$ is an integral subscheme, then $\kappa([\mathscr{Z}]) = [\mathscr{O}_{\mathscr{Z}}]$.

---

[21]These are not the most general possible; for instance, the crystalline condition at p-adic places is not necessary.

Let now F be a nonarchimedean local field and denote by k its residue field. Assume that the regular scheme $\mathscr{X}$ is endowed with a projective and flat map $\pi\colon \mathscr{X} \to \mathscr{O}_F$, and denote by X and $\mathscr{X}_k$ respectively the generic and special fibre of $\mathscr{X}$.

Definition A.3.1. Let $Z \in \mathrm{Z}^d(X)_L$, and denote by $|Z| \subset X$ its support. We say that an element

$$\mathscr{Z} \in \mathrm{F}^d K_0^{\mathscr{X}_k \cup |Z|}(\mathscr{X})_L \subset \mathrm{F}^d K_0(\mathscr{X})_L$$

is an *extension* of Z if $\mathscr{Z}_{|X} \in \mathrm{F}^d K_0(X)_L$ coincides with $\kappa(Z)$.

A.3.2. *Intersection pairing.* Suppose that X has dimension $m - 1 \geq 1$. For a pair of integers $d_1, d_2 \geq 0$ with $d_1 + d_2 = m$, and cycles $\mathscr{Z}_i \in \mathrm{F}^{d_i} K_0(\mathscr{X})$ with $|\mathscr{Z}_1| \cap |\mathscr{Z}_2| \subset |\mathscr{X}_k|$, we define their *intersection* by

$$(\mathscr{Z}_1 \cdot \mathscr{Z}_2) := \chi(\pi_*(\mathscr{Z}_1 \cup \mathscr{Z}_2)),$$

where

$$\cup\colon \mathrm{F}^{d_1} K_0^{|\mathscr{Z}_1|}(\mathscr{X}) \otimes \mathrm{F}^{d_2} K_0^{|\mathscr{Z}_2|}(\mathscr{X}) \longrightarrow \mathrm{F}^m K_0^{\mathscr{X}_k}(\mathscr{X})$$

is the cup product, and $\chi\colon K_0(\mathrm{Spec}\,k) \to \mathbf{Z}$ is the Euler characteristic. The definition is extended linearly to cycles with coefficients in L.

A.3.3. *Arithmetic intersections and the crystalline property at p.* Consider the setup of § A.2.1 with $\ell = p$.

Proposition A.3.2. *Assume that $p > m$ or $m = 2$, and that X admits a proper smooth model $\mathscr{X}/\mathscr{O}_F$. If the supports of the Zariski closures $\mathscr{Z}_1$, $\mathscr{Z}_2$ of Z_1, Z_2 in $\mathscr{X}$ are disjoint, then the biextension $[E_1^2]$ is crystalline.*

Proof. If $p > m$, this is a special case of [DL, Theorem A.6]. If $m = 2$, this is a special case of [Dis17, Proposition 4.3.1]. $\qquad\square$

A.3.4. *Arithmetic intersections and local heights away from p.* We start with a preliminary definition following [LL21, Appendix B].

Definition A.3.3. Let $\mathscr{X}$ be a scheme. A *correspondence* on $\mathscr{X}$ is a diagram of finite morphisms

it is said to be *étale* if both morphisms are étale. Correspondences on $\mathscr{X}$ form a monoidal category under composition. We denote by $\mathrm{Corr}(\mathscr{X})_L$ (respectively $\mathrm{\acute{E}tCorr}(\mathscr{X})_L$) the L-algebra generated by isomorphism classes of correspondences (respectively étale correspondences) on $\mathscr{X}$. It acts (on the right) on cycles and cohomology of $\mathscr{X}$ by pullback and pushforward.

A *commutative L-algebra of étale correspondences* on $\mathscr{X}$ is a commutative L-algebra $\mathbb{T}$ equipped with a homomorphism $\mathbb{T} \to \mathrm{\acute{E}tCorr}(\mathscr{X})_L$.

Consider now the setup of § A.2.1 with $\ell \neq p$.

Proposition A.3.4. *Assume that $m = 2n$ and $d_1 = d_2 = n$. Let $T_1, T_2 \in \text{ÉtCorr}(\mathscr{X})_L$, and assume that $Z_1.T_1$ and $Z_2.T_2$ have disjoint supports. Let $\mathscr{X}$ be a regular flat projective scheme over $\mathscr{O}_F$ with generic fibre X, and let $\mathscr{Z}_i \in \text{F}^{d_i} K_0(\mathscr{X})$ be an extension of Z_i for $i = 1, 2$.*

Suppose that one of the following conditions holds:

(1) $\mathscr{X}$ is smooth over $\mathscr{O}_F$, $\mathscr{Z}_i$ is (the image under $\kappa = $ (A.3.1) of) the Zariski closure of Z_i, and $T_i = \text{id}$;

(2) T_1, T_2 annihilate $H^{2n}(\mathscr{X}, L(n))$.

Then

$$h_\lambda(\mathscr{Z}_1.T_1, \mathscr{Z}_2.T_2) = -((\mathscr{Z}_1.T_1) \cdot (\mathscr{Z}_2.T_2))\, \lambda(\varpi)$$

where $\varpi \in F^\times$ is a uniformizer.

Proof. In case (1), this is a special case of [LL21, Proposition B.10] combined with [DL, Theorem A.4, Remark A.5]. In case (2), this is [LL21, Proposition B.13] combined with [DL, Theorem A.4, Remark A.5].   □

A.3.5. *Correspondences that annihilate the cohomology.* To find a correspondence satisfying condition (2) of the previous proposition, we recall some results from [LL21, LL22]. We assume $\mathscr{X}$ to be a regular scheme, proper and flat over a p-adic integer ring of relative dimension $2n - 1$ (not necessarily strictly semistable), and whose generic fiber X is smooth. Let $\mathbb{T}$ be a commutative L-algebra of étale correspondences on $\mathscr{X}$ with a maximal ideal $\mathfrak{m}$. Let Y denote the *reduced* special fiber of $\mathscr{X}$. Assume that there is a stratification $Y = Y^{[m]} \supset \cdots \supset Y^{[0]}$ by closed subschemes and, for each $0 \le i \le d$, a refinement of $Y^{(i)} := Y^{[i]} \setminus Y^{[i-1]}$ as a disjoint union of open and closed subschemes of $Y^{(i)}$ of pure dimension d_i:

$$Y^{(i)} = \coprod_{M \in \mathfrak{S}^i} Y^{(M)},$$

over a finite set of indices $\mathfrak{S}^i$, such that

(1) For every i and $M \in \mathfrak{S}^i$, denoting by $Y^{[M]}$ the Zariski closure of $Y^{(M)}$, then $Y^{[M]}$ is *smooth* and is a disjoint union $\coprod_{M' \in \mathfrak{S}_M} Y^{(M')}$ where $\mathfrak{S}_M$ is a subset of $\mathfrak{S} := \coprod \mathfrak{S}^{(i)}$;

(2) For every i and $M \in \mathfrak{S}^i$, the scheme $Y^{(M)}$ is stable under the action of $\mathbb{T}$.

Theorem A.3.5 (Li–Liu)**.** *Under the above assumptions, if we further assume either of the following two holds*

(1) $H^j(Y^{[M]} \otimes_k \overline{k}, L)_\mathfrak{m} = 0$ whenever $j \ne \dim Y^{[M]}$ for every $M \in \mathfrak{S}$.

(2) $H^{2n}(X, L(n))_\mathfrak{m} = 0$ and $H^j(Y^{(i)} \otimes_k \overline{k}, L)_\mathfrak{m} = 0$ whenever $j \le \dim Y^{(i)} - \text{codim}_{\mathscr{X}} Y^{(i)}$ for every i.

then $H^{2n}(\mathscr{X}, L(n))_\mathfrak{m} = 0$ holds.

Proof. Case (1): The vanishing assumption $H^j(Y^{[M]} \otimes_k \overline{k}, L)_\mathfrak{m} = 0$ is the assertion of [LL22, Prop. 4.25]. The proof of [LL22, Theorem 4.21] applies verbatim to show that the assumptions imply the desired vanishing $H^{2n}(\mathscr{X}, L(n))_\mathfrak{m} = 0$.

Case (2): This is [LL21, Corollary B.15]. We sketch their proof.

By the assumption $H^{2n}(X, L(n))_{\mathfrak{m}} = 0$ and the exact sequence

$$H^{2n}_Y(\mathscr{X}) \longrightarrow H^{2n}(\mathscr{X}) \longrightarrow H^{2n}(X)$$

it suffices to show $H^{2n}_Y(\mathscr{X})_{\mathfrak{m}} = 0$. This follows from an induction using

− the exact sequences

$$H^{2n}_{Y_{j+1}}(\mathscr{X}) \longrightarrow H^{2n}_{Y_j}(\mathscr{X}) \longrightarrow H^{2n}_{Y_j^\circ}(\mathscr{X} \setminus Y_{j+1}),$$

− the absolute purity theorem of Gabber $H^{2n}_{Y_j^\circ}(\mathscr{X} \setminus Y_{j+1}) \simeq H^{2n-2n_j}(Y_j^\circ)$ for the regular local immersion $Y_j^\circ \hookrightarrow \mathscr{X} \setminus Y_{j+1}$ of codimension n_j,

− the Hochschild–Serre spectral sequence $H^r(k, H^s(Y_j^\circ \otimes_k \overline{k})(n)) \Longrightarrow H^{r+s}(Y_j^\circ)(n)$. In particular, it suffices to replace (3) by a weaker assumption $H^{2n-2n_j}(Y_j^\circ \otimes_k \overline{k}, L)_{\mathfrak{m}} = H^{2n-2n_j-1}(Y_j^\circ \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0$ (namely $H^{d_{Y_j}-c_{Y_j}-i}(Y_j^\circ \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0$ for $i = 0, 1$ where d_{Y_j} and c_{Y_j} denote respectively the dimension of Y_j and the codimension of Y_j in $\mathscr{X}$.

<div align="right">□</div>

References

[AG08] Avraham Aizenbud and Dmitry Gourevitch, *Schwartz functions on Nash manifolds*, Int. Math. Res. Not. IMRN **5** (2008), Art. ID rnm 155, 37, DOI 10.1093/imrn/rnm155. MR2418286 ↑18

[AGRS10] Avraham Aizenbud, Dmitry Gourevitch, Stephen Rallis, and Gérard Schiffmann, *Multiplicity one theorems*, Ann. of Math. (2) **172** (2010), no. 2, 1407–1434, DOI 10.4007/annals.2010.172.1413. MR2680495 ↑19

[Ana] U.K. Anandavardhanan, *Distinction and Base Change*, arXiv:2102.10602. ↑19

[BZ76] I. N. Bernstein and . V. Zelevinskii, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70 (Russian). MR0425030 ↑52

[BP21] Raphaël Beuzart-Plessis, *Comparison of local relative characters and the Ichino-Ikeda conjecture for unitary groups*, J. Inst. Math. Jussieu **20** (2021), no. 6, 1803–1854, DOI 10.1017/S1474748019000707. MR4332778 ↑14, 23, 30, 32, 34

[BP16] ———, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes unitaires*, Mém. Soc. Math. Fr. (N.S.) **149** (2016), vii+191, DOI 10.24033/msmf.457 (French, with English and French summaries). MR3676153 ↑10, 19

[BP20] ———, *A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the Archimedean case*, Astérisque **418** (2020), ix + 305, DOI 10.24033/ast (English, with English and French summaries). MR4146145 ↑10, 19

[BP21a] ———, *Plancherel formula for $\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(E)$ and applications to the Ichino-Ikeda and formal degree conjectures for unitary groups*, Invent. Math. **225** (2021), no. 1, 159–297, DOI 10.1007/s00222-021-01032-6. MR4270666 ↑14, 18, 33, 34

[BP21b] ———, *A new proof of the Jacquet-Rallis fundamental lemma*, Duke Math. J. **170** (2021), no. 12, 2805–2814, DOI 10.1215/00127094-2020-0090. MR4305382 ↑14, 34

[BPLZZ21] Raphaël Beuzart-Plessis, Yifeng Liu, Wei Zhang, and Xinwen Zhu, *Isolation of cuspidal spectrum, with application to the Gan-Gross-Prasad conjecture*, Ann. of Math. (2) **194** (2021), no. 2, 519–584, DOI 10.4007/annals.2021.194.2.5. MR4298750 ↑3, 12, 21, 24, 33, 34, 37, 41, 42, 43, 44

[BPCZ22] Raphaël Beuzart-Plessis, Pierre-Henri Chaudouard, and Michal Zydor, *The global Gan-Gross-Prasad conjecture for unitary groups: the endoscopic case*, Publ. Math. Inst. Hautes Études Sci. **135** (2022), 183–336, DOI 10.1007/s10240-021-00129-1. MR4426741 ↑3, 24

[BW80] Armand Borel and Nolan R. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Annals of Mathematics Studies, No. 94, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980. MR554917 ↑92

[BG14] Kevin Buzzard and Toby Gee, *The conjectural connections between automorphic representations and Galois representations*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 135–187, DOI 10.1017/CBO9781107446335.006. MR3444225 ↑

[Car12] Ana Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413, DOI 10.1215/00127094-1723706. MR2972460 ↑6, 20, 93

[Car14] _____, *Monodromy and local-global compatibility for $l = p$*, Algebra Number Theory **8** (2014), no. 7, 1597–1646, DOI 10.2140/ant.2014.8.1597. MR3272276 ↑54, 93

[Cas80] W. Casselman, *The unramified principal series of $\mathfrak{p}$-adic groups. I. The spherical function*, Compositio Math. **40** (1980), no. 3, 387–406. MR571057 ↑52

[Cas89] _____, *Introduction to the Schwartz space of $\Gamma\backslash G$*, Canad. J. Math. **41** (1989), no. 2, 285–320, DOI 10.4153/CJM-1989-015-6. MR1001613 ↑18

[CZ21] Pierre-Henri Chaudouard and Michal Zydor, *Le transfert singulier pour la formule des traces de Jacquet-Rallis*, Compos. Math. **157** (2021), no. 2, 303–434, DOI 10.1112/S0010437X20007599 (French, with English and French summaries). MR4234897 ↑30

[CH13] Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations, II*, Camb. J. Math. **1** (2013), no. 1, 53–73, DOI 10.4310/CJM.2013.v1.n1.a2. MR3272052 ↑41

[Clo90] Laurent Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159 (French). MR1044819 ↑20, 43

[Coa91] John Coates, *Motivic p-adic L-functions*, L-functions and arithmetic (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, pp. 141–172. MR1110392 ↑6

[Dan] Gefei Dang, *Newforms and relative local characters*, MIT Thesis, in preparation. ↑8, 105

[Dis17] Daniel Disegni, *The p-adic Gross-Zagier formula on Shimura curves*, Compos. Math. **153** (2017), no. 10, 1987–2074, DOI 10.1112/S0010437X17007308. MR3692745 ↑3, 7, 11, 111, 113

[Dis20] _____, *Local Langlands correspondence, local factors, and zeta integrals in analytic families*, J. Lond. Math. Soc. (2) **101** (2020), no. 2, 735–764, DOI 10.1112/jlms.12285. MR4093973 ↑

[Dis] _____, *The p-adic Gross–Zagier formula on Shimura curves, II: nonsplit primes*, J. Inst. Math. Jussieu, to appear, DOI 10.1017/S1474748021000608. ↑7

[Dis22] _____, *The universal p-adic Gross–Zagier formula*, Invent. math **230** (2022), 509–649. ↑7

[DL] Daniel Disegni and Yifeng Liu, *A p-adic arithmetic inner product formula*, preprint. ↑7, 12, 14, 64, 92, 93, 101, 108, 111, 113, 114

[DZ] Daniel Disegni and Wei Zhang, *On the regular non-semisimple orbital integrals of Jacquet–Rallis*, in preparation. ↑39

[Eme06] Matthew Emerton, *Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 5, 775–839, DOI 10.1016/j.ansens.2006.08.001 (English, with English and French summaries). MR2292633 ↑51

[Fli88] Yuval Z. Flicker, *Twisted tensors and Euler products*, Bull. Soc. Math. France **116** (1988), no. 3, 295–313 (English, with French summary). MR984899 ↑

[GGP12] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups*, Astérisque **346**

(2012), 1–109 (English, with English and French summaries). Sur les conjectures de Gross et Prasad. I. MR3202556 ↑3, 4, 8, 10, 22

[GRS11] David Ginzburg, Stephen Rallis, and David Soudry, *The descent map from automorphic representations of* GL(n) *to classical groups*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. MR2848523 ↑21

[Gol14] Wushi Goldring, *Galois representations associated to holomorphic limits of discrete series*, Compos. Math. **150** (2014), no. 2, 191–228, DOI 10.1112/S0010437X13007355. With an appendix by Sug-Woo Shin. MR3177267 ↑22

[GL21] Harald Grobner and Jie Lin, *Special values of L-functions and the refined Gan-Gross-Prasad conjecture*, Amer. J. Math. **143** (2021), no. 3, 859–937, DOI 10.1353/ajm.2021.0022. MR4270260 ↑5

[Gro97] Benedict H. Gross, *On the motive of a reductive group*, Invent. Math. **130** (1997), no. 2, 287–313, DOI 10.1007/s002220050186. MR1474159 ↑16, 17

[Gro04] _____, *Heegner points and representation theory*, Heegner points and Rankin L-series, Math. Sci. Res. Inst. Publ., vol. 49, Cambridge Univ. Press, Cambridge, 2004, pp. 37–65, DOI 10.1017/CBO9780511756375.005. MR2083210 (2006h:11068) ↑3

[Gro] _____, *Incoherent definite spaces and Shimura varieties*, Relative Trace Formulas, Simons Symposia, Springer, Cham, 2021. ↑78

[GS95] B. H. Gross and C. Schoen, *The modified diagonal cycle on the triple product of a pointed curve*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 3, 649–679 (English, with English and French summaries). MR1340948 ↑87

[GZ86] Benedict H. Gross and Don B. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), no. 2, 225–320, DOI 10.1007/BF01388809. MR833192 (87j:11057) ↑3

[Har14] R. Neal Harris, *The refined Gross-Prasad conjecture for unitary groups*, Int. Math. Res. Not. IMRN **2** (2014), 303–389, DOI 10.1093/imrn/rns219. MR3159075 ↑3

[Har01] Urs T. Hartl, *Semi-stability and base change*, Arch. Math. (Basel) **77** (2001), no. 3, 215–221, DOI 10.1007/PL00000484. MR1865862 ↑85, 87

[Hid98] Haruzo Hida, *Automorphic induction and Leopoldt type conjectures for* GL(n), Asian J. Math. **2** (1998), no. 4, 667–710, DOI 10.4310/AJM.1998.v2.n4.a5. Mikio Sato: a great Japanese mathematician of the twentieth century. MR1734126 ↑52, 55

[II10] Atsushi Ichino and Tamutsu Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425, DOI 10.1007/s00039-009-0040-4. MR2585578 ↑3

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464, DOI 10.2307/2374264. MR701565 ↑39, 40

[JS81] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*, Amer. J. Math. **103** (1981), no. 3, 499–558, DOI 10.2307/2374103. MR618323 ↑

[JR11] Hervé Jacquet and Stephen Rallis, *On the Gross-Prasad conjecture for unitary groups*, On certain L-functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 205–264. MR2767518 (2012d:22026) ↑4

[Jan16] Fabian Januszewski, *p-adic L-functions for Rankin-Selberg convolutions over number fields*, Ann. Math. Qué. **40** (2016), no. 2, 453–489, DOI 10.1007/s40316-016-0061-y (English, with English and French summaries). MR3529190 ↑6

[Jan] _____, *Non-abelian p-adic Rankin-Selberg L-functions and non-vanishing of central L-values*, preprint available at arXiv:1708.02616. ↑51

[KMSW] Tasho Kaletha, Alberto Minguez, Sug Woo Shin Shin, and Paul-James White, *Endoscopic classification of representations: Inner forms of unitary groups*, preprint. ↑8, 18, 20, 21, 22, 34

[KSZ] Mark Kisin, Sug-Woo Shin, and Yihang Zhu, *The stable trace formula for Shimura varieties of abelian type*, arXiv:2110.05381. ↑8

[Kob13]   Shinichi Kobayashi, *The p-adic Gross-Zagier formula for elliptic curves at supersingular primes*, Invent. Math. **191** (2013), no. 3, 527–629, DOI 10.1007/s00222-012-0400-9. MR3020170 ↑7

[Kol88]   V. A. Kolyvagin, *Finiteness of $E(\mathbf{Q})$ and $SH(E,\mathbf{Q})$ for a subclass of Weil curves*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 3, 522–540, 670–671 (Russian); English transl., Math. USSR-Izv. **32** (1989), no. 3, 523–541. MR954295 (89m:11056) ↑3, 7

[Jan]   Fabian Januszewski, *Non-abelian p-adic Rankin–Selberg L-functions and non-vanishing of central L-values*, preprint. ↑49, 50

[LL21]   Chao Li and Yifeng Liu, *Chow groups and L-derivatives of automorphic motives for unitary groups*, Ann. of Math. (2) **194** (2021), no. 3, 817–901, DOI 10.4007/annals.2021.194.3.6. MR4334978 ↑14, 89, 90, 98, 111, 112, 113, 114

[LL22]   ———, *Chow groups and L-derivatives of automorphic motives for unitary groups, II*, Forum Math. Pi **10** (2022), Paper No. e5, 71, DOI 10.1017/fmp.2022.2. MR4390300 ↑88, 89, 114

[LLS]   Jian-Shu Li, Dongwen Liu, and Binyong Sun, *Ordinary p-adic period integrals*, in preparation. ↑5

[LM]   Qirui Li and Andreas Mihatsch, *On the Linear AFL: The Non-Basic Case*, arXiv:2208.10144v1. ↑85

[LTX+22] Yifeng Liu, Yichao Tian, Liang Xiao, Wei Zhang, and Xinwen Zhu, *On the Beilinson-Bloch-Kato conjecture for Rankin-Selberg motives*, Invent. Math. **228** (2022), no. 1, 107–375, DOI 10.1007/s00222-021-01088-4. MR4392458 ↑3, 7, 10, 19, 21, 22, 86, 92, 108

[Loe21]   David Loeffler, *Spherical varieties and norm relations in Iwasawa theory*, J. Théor. Nombres Bordeaux **33** (2021), no. 3, 1021–1043, DOI 10.1007/s10884-020-09844-5 (English, with English and French summaries). MR4402388 ↑50

[Mok15]   Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248, DOI 10.1090/memo/1108. MR3338302 ↑8, 18, 20, 21, 34, 92

[MZ]   Andreas Mihatsch and Wei Zhang, *On the Arithmetic Fundamental Lemma conjecture over a general p-adic field*, JEMS, to appear, arXiv:2104.02779. ↑15, 101, 104

[MS82]   James S. Milne and Kuang-yen Shih, *Conjugates of Shimura varieties*, Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin-New York, 1982, pp. 280–356. MR654325 ↑78

[Nek93]   Jan Nekovář, *On p-adic height pairings*, Séminaire de Théorie des Nombres, Paris, 1990–91, Progr. Math., vol. 108, Birkhäuser Boston, Boston, MA, 1993, pp. 127–202, DOI 10.1007/s10107-005-0696-y. MR1263527 ↑11, 94, 108, 111, 112

[Nek95]   ———, *On the p-adic height of Heegner cycles*, Math. Ann. **302** (1995), no. 4, 609–686, DOI 10.1007/BF01444511. MR1343644 ↑7, 100, 111

[Nek00]   ———, *p-adic Abel-Jacobi maps and p-adic heights*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), CRM Proc. Lecture Notes, vol. 24, Amer. Math. Soc., Providence, RI, 2000, pp. 367–379, DOI 10.1090/crmp/024/18. MR1738867 ↑

[NN' 16] Jan Nekovář and Wiesława Nizioł, *Syntomic cohomology and p-adic regulators for varieties over p-adic fields*, Algebra Number Theory **10** (2016), no. 8, 1695–1790, DOI 10.2140/ant.2016.10.1695. With appendices by Laurent Berger and Frédéric Déglise. MR3556797 ↑109

[NT]   James Newton and Jack Thorne, *Adjoint Selmer groups of automorphic Galois representations of unitary type*, J. Eur. Math. Soc., to appear. ↑8

[Oes84]   Joseph Oesterlé, *Nombres de Tamagawa et groupes unipotents en caractéristique p*, Invent. Math. **78** (1984), no. 1, 13–88, DOI 10.1007/BF01388714 (French). MR762353 ↑16

[PR87]   Bernadette Perrin-Riou, *Points de Heegner et dérivées de fonctions L p-adiques*, Invent. Math. **89** (1987), no. 3, 455–510, DOI 10.1007/BF01388982 (French). MR903381 (89d:11034) ↑3, 7

[Rag16]   A. Raghuram, *Critical values of Rankin-Selberg L-functions for $GL_n \times GL_{n-1}$ and the symmetric cube L-functions for GL_2*, Forum Math. **28** (2016), no. 3, 457–489, DOI 10.1515/forum-2014-0043. MR3510825 ↑5

[RS08]   A. Raghuram and Freydoon Shahidi, *On certain period relations for cusp forms on* GL_n, Int. Math. Res. Not. IMRN, posted on 2008, Art. ID rnn 077, 23, DOI 10.1093/imrn/rnn077. MR2439563 ↑25

[RS]   Steve Rallis and Gerard Schiffmann, *Multiplicity one Conjectures*, arXiv:0705.2168. ↑66

[Ram]   Dinakar Ramakrishnan, *A theorem on* $GL(n)$ *à la Tchebotarev*, preprint. ↑20, 44

[RSZ18]   M. Rapoport, B. Smithling, and W. Zhang, *Regular formal moduli spaces and arithmetic transfer conjectures*, Math. Ann. **370** (2018), no. 3-4, 1079–1175, DOI 10.1007/s00208-017-1526-2. MR3770164 ↑104

[RSZ20]   _____, *Arithmetic diagonal cycles on unitary Shimura varieties*, Compos. Math. **156** (2020), no. 9, 1745–1824, DOI 10.1112/s0010437x20007289. MR4167594 ↑14, 76, 79, 83, 84, 85, 99, 102

[RSZ21]   _____, *On Shimura varieties for unitary groups*, Pure Appl. Math. Q. **17** (2021), no. 2, 773–837, DOI 10.4310/PAMQ.2021.v17.n2.a8. MR4257601 ↑14, 76, 77, 79, 80, 81, 82, 83

[ST]   Jack Sempliner and Richard Taylor, *On the formalism of Shimura varieties*, preprint. ↑78

[Shi78]   Goro Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45** (1978), no. 3, 637–679. MR507462 ↑5

[Shn16]   Ariel Shnidman, *p-adic heights of generalized Heegner cycles*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 3, 1117–1174 (English, with English and French summaries). MR3494168 ↑7

[TY07]   Richard Taylor and Teruyoshi Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493. MR2276777 ↑85, 93

[Tho]   Jack Thorne, *On the vanishing of adjoint Bloch–Kato Selmer groups of irreducible automorphic Galois representations*, preprint. ↑8

[Urb11]   Eric Urban, *Eigenvarieties for reductive groups*, Ann. of Math. (2) **174** (2011), no. 3, 1685–1784, DOI 10.4007/annals.2011.174.3.7. MR2846490 ↑13

[Yun11]   Zhiwei Yun, *The fundamental lemma of Jacquet and Rallis*, Duke Math. J. **156** (2011), no. 2, 167–227, DOI 10.1215/00127094-2010-210. With an appendix by Julia Gordon. MR2769216 ↑14, 34

[Wal85a] J.-L. Waldspurger, *Quelques propriétés arithmétiques de certaines formes automorphes sur* GL(2), Compositio Math. **54** (1985), no. 2, 121–171 (French). MR783510 ↑

[Wal85b] _____, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), no. 2, 173–242 (French). MR783511 (87g:11061b) ↑3

[Wei40]   André Weil, *Une lettre et un extrait de lettre à Simone Weil [1940a]*, Oeuvres Scientifiques. Collected papers. Vol. I (1926–1951), Springer-Verlag, New York-Heidelberg, 1979, pp. 244-255. ↑3

[Xue19]   Hang Xue, *Arithmetic theta lifts and the arithmetic Gan-Gross-Prasad conjecture for unitary groups*, Duke Math. J. **168** (2019), no. 1, 127–185, DOI 10.1215/00127094-2018-0039. MR3909895 ↑3, 11

[YZZ12]   Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang, *The Gross-Zagier Formula on Shimura Curves*, Annals of Mathematics Studies, vol. 184, Princeton University Press, Princeton, NJ, 2012. ↑3

[YZZ]   _____, *Triple product L-series and Gross-Kudla-Schoen cycles*, preprint. ↑3

[Zha12]   Wei Zhang, *On arithmetic fundamental lemmas*, Invent. Math. **188** (2012), no. 1, 197–252, DOI 10.1007/s00222-011-0348-1. MR2897697 ↑4, 12, 14, 15, 33

[Zha14a] _____, *Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups*, Ann. of Math. (2) **180** (2014), no. 3, 971–1049, DOI 10.4007/annals.2014.180.3.4. MR3245011 ↑24, 27, 35

[Zha14b] _____, *Automorphic period and the central value of Rankin-Selberg L-function*, J. Amer. Math. Soc. **27** (2014), no. 2, 541–612, DOI 10.1090/S0894-0347-2014-00784-0. MR3164988 ↑17, 23, 25, 26, 27, 28, 29, 31, 32, 33, 34

[Zha21]   _____, *Weil representation and arithmetic fundamental lemma*, Ann. of Math. (2) **193** (2021), no. 3, 863–978, DOI 10.4007/annals.2021.193.3.5. MR4250392 ↑15, 104

[Zha]   _____, *High dimensional Gross–Zagier formula*, Lecture notes in IHES summer school, 2022. ↑3

[ZZh]   Zhiyu Zhang, *Maximal parahoric arithmetic transfers, resolutions and modularity*, preprint. ↑15, 86, 101, 104, 105

Department of Mathematics, Ben-Gurion University of the Negev, Be'er Sheva 84105, Israel
E-mail address: `disegni@bgu.ac.il`

Massachusetts Institute of Technology, Department of Mathematics, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
E-mail address: `weizhang@mit.edu`