GAN–GROSS–PRASAD CYCLES AND DERIVATIVES OF p-ADIC L-FUNCTIONS

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ABSTRACT. We study the p-adic analogue of the arithmetic Gan–Gross–Prasad (GGP) conjectures for unitary groups. Let Π be a hermitian cuspidal automorphic representation of $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$ over a CM field, which is algebraic of minimal regular weight at infinity. We first show the rationality of twists of the ratio of L-values of Π appearing in the GGP conjectures. Then, when Π is p-ordinary at a prime p, we construct a cyclotomic p-adic L-function $\mathscr{L}_p(\mathrm{M}_\Pi)$ interpolating those twists. Finally, under some local assumptions, we prove a precise formula relating the first derivative of $\mathscr{L}_p(\mathrm{M}_\Pi)$ to the p-adic heights of Selmer classes arising from arithmetic diagonal cycles on unitary Shimura varieties. We deduce applications to the p-adic Beilinson–Bloch–Kato conjecture for the motive attached to Π. All proofs are based on some relative-trace formulas in p-adic coefficients.

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1. Introduction

The pioneering formulas of Gross–Zagier and Perrin-Riou, [GZ86, PR87], revealed a remarkable relation between Heegner points and derivatives of complex and p-adic L-functions. They had immediate applications to the (classical and p-adic) Birch and Swinnerton-Dyer conjectures, soon strengthened by the Selmer-group bounds proved by Kolyvagin [Kol88].

A "furtive caress"[1] between those formulas and one by Waldspurger on central L-values, [Wal85b], did not escape Gross; and in [Gro04], he blessed it into a representation-theoretic marriage, which would blossom in [YZZ12] (and later p-adically in [Dis17]).

The seeds for a new generation were sown in a paper by Gan, Gross, and Prasad [GGP12]. Their influential work conjectured a pair of non-vanishing criteria in the context of embeddings of unitary groups: one for automorphic periods, in terms of Rankin–Selberg L-values (generalizing [Wal85b]); and one for algebraic cycles in Shimura varieties, in terms of (complex) L-derivatives (the *arithmetic* GGP conjecture, generalizing [GZ86]).

The conjecture on automorphic periods was refined to an exact formula by Ichino–Ikeda and N. Harris [II10, Har14], and recently proved in this form in [BPLZZ21, BPCZ22]. On the other hand, despite considerable progress (see [Zha] for a review), the arithmetic GGP conjecture remains open outside of cases where it can be reduced to Heegner points [YZZ12, Xue19].[2]

The purpose of this work is to formulate and, under some local assumptions, prove a p-adic variant of the arithmetic GGP conjecture. The result in fact takes the form of a precise formula, in the spirit of [PR87, Dis17, II10, Har14]. It has immediate applications to the p-adic Beilinson–Bloch–Kato conjecture for the relevant motives, which can be further strengthened by the Selmer bounds recently established in [LTX+22, LaSk].

[1] Words borrowed from [Wei40].

[2] The analogous conjecture for orthogonal groups is also known for 1-cycles in threefolds [YZZ].

(Indeed, one advantage of working in p-adic rather than archimedean coefficients is that we obtain a nonvanishing criterion in Selmer groups, rather than Chow groups: while the p-adic Abel-Jacobi map from the latter to the former should be injective, this is not known beyond cycles of codimension one.)

In the rest of this introduction, we state our main results, discuss their history and context, and give some ideas on the proofs.

In § 1.1, we describe our p-adic L-function (Theorem B), preceded by a rationality result for twisted Rankin–Selberg L-values (Theorem A) that should be of independent interest.

In § 1.2 we state our applications to the p-adic Beilinson–Bloch–Kato conjecture (Theorem C; the order of presentation is dictated by ease of exposition rather than logic). In § 1.3 we define the Gan–Gross–Prasad cycles and state our formula for their p-adic heights (Theorem D).

In § 1.4, we give a sketch of our methods: inspired by the strategy proposed by Jacquet–Rallis for the Ichino–Ikeda conjecture [JR11], and by one of us [Zha12] for the arithmetic GGP conjecture (in archimedean coefficients), we construct a p-adic relative-trace formula from which we extract the p-adic L-function; then, we compare it to another relative-trace formula encoding the p-adic heights of GGP cycles.

1.1. **The p-adic L-function.** Let F_0 be a number field, and denote by $\mathbf{A}$ the adèles of F_0, by $D_{F_0} = \prod_{v\nmid\infty} D_{F_{0,v}}$ the discriminant of F_0 (here $D_{F_{0,v}}$ is the norm of the different ideal of $F_{0,v}$). Let F be a quadratic extension of F_0, let $\mathrm{c} \in \mathrm{Gal}(F/F_0)$ be the nontrivial element, and let $\eta\colon F_0^\times\backslash\mathbf{A}^\times \to \{\pm 1\}$ be the associated quadratic character. Define a reductive group over F_0 by

$$\mathrm{G}' := (\mathrm{Res}_{F/F_0}\mathrm{GL}_n \times \mathrm{Res}_{F/F_0}\mathrm{GL}_{n+1})/(\mathrm{GL}_1\times\mathrm{GL}_1),$$

where $\mathrm{GL}_1\times\mathrm{GL}_1$ is the split center of G'. Let $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ be an (irreducible) automorphic representation of $\mathrm{G}'(\mathbf{A})$. Define[3] a Rankin–Selberg and an Asai L-function[4] for Π and a character χ of $F_0^\times\backslash\mathbf{A}^\times$ by

$$L(s,\Pi\otimes\chi) := L(s,\Pi_n\times(\Pi_{n+1}\otimes\chi\circ\mathrm{Nm}_{F/F_0}))$$
$$L(s,\Pi,\mathrm{As}^\star) := L(s,\Pi_n,\mathrm{As}^{(-1)^n})L(s,\Pi_{n+1},\mathrm{As}^{(-1)^{n+1}}).$$

We say that a cuspidal automorphic representation Π_ν is *hermitian* if $\Pi\circ\mathrm{c}\cong\Pi^\vee$ and $L(s,\Pi_\nu,\mathrm{As}^{(-1)^\nu})$ is regular at $s=1$. We say that $\Pi=\Pi_n\boxtimes\Pi_{n+1}$ is hermitian if Π_n, Π_{n+1} are. For such a representation Π, we define

$$\mathscr{L}(s,\Pi_v,\chi_v) := D_{F_{0,v}}^{n+1}\prod_{i=1}^{n+1} L(i,\eta_v^i)\cdot\frac{L(s,\Pi_v\otimes\chi_v)}{L(1,\Pi_v,\mathrm{As}^\star)}, \tag{1.1.1}$$

and

$$\mathscr{L}(s,\Pi,\chi) := \prod_{v\nmid\infty}\mathscr{L}(s,\Pi_v,\chi_v).$$

Here, the abelian factor may be interpreted in terms of L-values of motives of unitary groups (§ 2.2.1).

[3]Throughout the introduction (but *differently* from the rest of the paper) L-functions do not include archimedean factors.

[4]See [GGP12, § 7] for the definition of $L(s,\Pi_n,\mathrm{As}^\pm)$.

1.1.1. *Rationality of $\mathscr{L}$.* Assume from now on that F_0 is totally real and F is CM. Let $\arg(z) \coloneqq z/|z|$ (a character of $\mathbf{C}^\times$), let $\Pi^\circ_{\nu,\mathbf{R}}$ be the representation of $\mathrm{GL}_\nu(\mathbf{C})/\mathrm{GL}_1(\mathbf{R})$ induced by the character $\arg^{\nu-1}\otimes\arg^{\nu-3}\otimes\ldots\otimes\arg^{1-\nu}$ of the torus $(\mathbf{C}^\times)^\nu$, and define the representation

$$\Pi^\circ_\infty = \bigotimes_{v|\infty}\Pi^\circ_{\mathbf{R}} \coloneqq \bigotimes_{v|\infty}\Pi^\circ_{n,\mathbf{R}}\otimes\Pi^\circ_{n+1,\mathbf{R}}$$

of $\mathrm{G}'(F_{0,\infty})$. Let us also denote by $\mathbf{1}_\infty$ the trivial representation of $\mathrm{G}'(F_{0,\infty})$ over $\mathbf{Q}$.

Let $\Pi = \Pi^\infty\otimes\mathbf{1}_\infty$ be a representation of $\mathrm{G}'(\mathbf{A})$ on a characteristic-zero field L (admitting embeddings into $\mathbf{C}$). We say that Π is a *trivial-weight (algebraic) cuspidal automorphic representation* if for every $\iota\colon L\hookrightarrow\mathbf{C}$, the representation $\Pi^\iota\coloneqq\iota\Pi^\infty\otimes\Pi^\circ_\infty$ is an (irreducible) cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$. (It is known that every cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over $\mathbf{C}$ such that $\Pi_\infty\cong\Pi^\circ_\infty$ arises in this manner for some number field L.) We say that Π is hermitian if Π^ι is for some (equivalently, every) ι.

We first prove the following strong rationality property for the values of $\mathscr{L}$. For an ideal $m\subset\mathscr{O}_{F_0}$, let $Y(m)_{/\mathbf{Q}}$ be the finite étale scheme of characters of $F_0^\times\backslash\mathbf{A}^\times/F_{0,\infty}^\times(\widehat{\mathscr{O}}^\times_{F_0}\cap 1+m\widehat{\mathscr{O}}_{F_0})$. Let $Y\coloneqq\varinjlim_m Y(m)$, the ind-finite scheme over $\mathbf{Q}$ of locally constant characters of $F_0^\times\backslash\mathbf{A}^\times/F_{0,\infty}^\times$.

Theorem A. *Let Π be a trivial-weight hermitian cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ defined over a characteristic-zero field L. Then there is an element*

$$\mathscr{L}(\mathrm{M}_\Pi,\cdot)\in\mathscr{O}(Y_L). \tag{1.1.2}$$

such that

$$\mathscr{L}(\mathrm{M}_\Pi,\chi)=\frac{\mathscr{L}(1/2,\Pi^\iota,\chi)}{\varepsilon(\frac12,\chi^2)^{\binom{n+1}{2}}}$$

for all $\chi\in Y_L(\mathbf{C})$ with underlying embedding $\iota\colon L\hookrightarrow\mathbf{C}$.

For the notation 'M_Π', see Remark 1.2.2.

Remark 1.1.1. For $n=1$, Theorem A is a variant of a classical result of Shimura [Shi78]. A conditional proof of the rationality of $\mathscr{L}(1/2,\Pi,\mathbf{1})$ for a more general class of Π was recently obtained by Grobner and Lin [GL21, Theorem C]. (In fact, their rationality result is also a consequence of the Ichino–Ikeda conjecture, but the method of [GL21] is different.) See also [Rag16] for a related result, and [GHL] for relations to Deligne's conjecture.

1.1.2. *p-adic interpolation.* Fix from now on a rational prime p. For $v|p$ a place of F_0, let $N_v^\circ\subset G'_v\coloneqq\mathrm{G}'(F_{0,v})$ be the subgroup of integral unipotent upper-triangular matrices, and let $T_v^+\subset G'_v$ be the monoid of diagonal matrices such that $tN_v^\circ t^{-1}\subset N_v^\circ$. Let Π be a trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p$. We say that Π is v-*ordinary* if $\Pi_v^{N_v^\circ}$ contains a nonzero vector (necessarily unique up to scalar multiple) on which all the operators $U_{t,v}\coloneqq\sum_{x\in N_v^\circ/tN_v^\circ t^{-1}}[xt]$, for $t\in T_v^+$, act by units in $\mathscr{O}_L$. We say that Π is *ordinary* if it is v-ordinary for all $v|p$.

For any number field E, denote $\Gamma_E\coloneqq E^\times\backslash\mathbf{A}_E^{\infty\times}/\prod_{w\nmid p}\mathscr{O}^\times_{E,w}$, and let

$$\mathscr{Y}\coloneqq\operatorname{Spec}\mathbf{Z}_p[\![\Gamma_{F_0}]\!]\otimes_{\mathbf{Z}_p}\mathbf{Q}_p.$$

We have a natural map $Y(p^\infty)\coloneqq\varinjlim_r Y(p^r)\hookrightarrow\mathscr{Y}$.

If L'/L is a field extension and S/L is an (ind-) scheme, we denote $S_{L'} := S \times_{\operatorname{Spec} L} \operatorname{Spec} L'$.

Theorem B. *Let Π be an ordinary, hermitian, trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p$. Assume that for each place $v|p$ of F_0 that does not split in F, the representation Π_v is unramified.*

There exists a unique function

$$\mathscr{L}_p(\mathrm{M}_\Pi) \in \mathscr{O}(\mathscr{Y}_L)$$

whose restriction to $Y(p^\infty)_L$ satisfies

$$\mathscr{L}_p(\mathrm{M}_\Pi)(\chi) = e_p(\mathrm{M}_{\Pi\otimes\chi})\, \mathscr{L}(\mathrm{M}_\Pi, \chi) \tag{1.1.3}$$

where $\mathscr{L}(\mathrm{M}_\Pi)$ is as in (1.1.2)*, and $e_p(\mathrm{M}_{\Pi\otimes\chi}) = \prod_{v|p} e(\Pi_v, \chi_v)$ is the product of the explicit local terms* (5.3.5)*.*

Remark 1.1.2. We conjecture that the theorem remains true without the non-ramification condition at nonsplit p-adic places.

Remark 1.1.3. We say that Π is *non-exceptional* if $e_p(\mathrm{M}_\Pi) \neq 0$. By a recent result of Liu and Sun (Proposition 5.2.6), the factor $e_p(\mathrm{M}_{\Pi\otimes\chi})$ is as conjectured by Coates and Perrin-Riou [Coa91]; this implies that if Π_v is an irreducible principal series for all $v|p$, then Π is non-exceptional (see Remark 5.3.3).

Remark 1.1.4. Januszewski [Jan16] has proven a variant of Theorem B in a more general context, by the method of modular symbols (see also the substantial improvements in [LiSu]). Our method is similar locally at p but very different globally (and at archimedean places), see § 1.4.2 below.

Remark 1.1.5. Other authors have studied the variation of the above L-values (and in fact their 'square roots') in anticyclotomic or more general self-dual p-adic families, see [HY, Liu, Dim]. It is of course expected that these values can be interpolated into a function over the entire ordinary deformation space, that specializes to the functions of these works in self-dual subspaces and to our $\mathscr{L}_p(\mathrm{M}_\Pi)$ in the cyclotomic direction. (The case of 'two' abelian variables explicitly conjectured in [Liu, Hypothesis 7.12] could be achieved by the method of this paper, but we chose not to address it in order to bound the technical aspects.)

1.2. **On the p-adic Beilinson–Bloch–Kato conjectures for Rankin–Selberg motives.** Before moving to discuss our main result, we give its main arithmetic application, which can be stated without much further background.

Let

$$\Pi = \Pi_n \boxtimes \Pi_{n+1}$$

be a hermitian trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p$. Denote by G_F the absolute Galois group of F, by $\overline{\mathbf{Q}}_p$ an algebraic closure of L and let $\rho_{\Pi_\nu,\overline{\mathbf{Q}}_p} \colon G_F \to \mathrm{GL}_\nu(\overline{\mathbf{Q}}_p)$ be the semisimple representation attached to Π_ν by the global Langlands correspondence (as described in [Car12, Theorem 1.1]). Assuming that $\varepsilon(\Pi) := \varepsilon(1/2, \Pi_n^\iota \times \Pi_{n+1}^\iota) = -1$ for any (equivalently, all) $\iota\colon L \hookrightarrow \mathbf{C}$, we construct a continuous representation

$$\rho_\Pi \colon G_F \longrightarrow \mathrm{GL}_{n(n+1)}(L) \tag{1.2.1}$$

whose base-change $\rho_\Pi \otimes_L \overline{\mathbf{Q}}_p$ is isomorphic, up to semisimplication, to $\rho_{\Pi_n,\overline{\mathbf{Q}}_p} \otimes \rho_{\Pi_{n+1},\overline{\mathbf{Q}}_p}(n)$ (Remark 11.1.3). It satisfies $\rho_\Pi^{\mathrm{c}} \cong \rho_\Pi^*(1)$, where $\rho^{\mathrm{c}}(g) \coloneqq \rho(c^{-1}gc)$ for any lift $c \in G_F$ of c.

The Beilinson–Bloch–Kato (BBK) conjecture relates the dimension of the Bloch–Kato Selmer group

$$H^1_f(F, \rho_\Pi)$$

to the order of vanishing of $\mathscr{L}(s, \Pi^\iota)$ at $s = 1/2$, for any $\iota\colon L \hookrightarrow \mathbf{C}$. Assuming that Π is ordinary, we can consider a variant in terms of

$$\mathrm{ord}_{\chi=\mathbf{1}}\mathscr{L}_p(\mathrm{M}_\Pi) \coloneqq \sup\{r \mid \mathscr{L}_p(\mathrm{M}_\Pi) \in \mathfrak{m}_{\mathbf{1}}^r \subset \mathscr{O}(\mathscr{Y}_L)\},$$

where $\mathfrak{m}_{\mathbf{1}}$ is the ideal of functions vanishing at $\chi = \mathbf{1}$. We prove the following.

Theorem C. *Let Π be an ordinary, hermitian, trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a finite extension L of $\mathbf{Q}_p$. Assume that $\varepsilon(\Pi) = -1$, and that the following further conditions are satisfied:*

- *F/F_0 is unramified; in particular, $F_0 \neq \mathbf{Q}$;*
- *all places $v|2$ are split in F/F_0;*
- *$p > 2n$ if $n > 1$;*
- *for every place $v|p$ of F_0, we have that v splits in F and Π_v is unramified;*
- *for every place v of F_0 that splits in F, at least one of $\Pi_{n,v}$ and $\Pi_{n+1,v}$ is unramified;*
- *for every place v of F_0 that is inert in F, each of $\Pi_{n,v}$ and $\Pi_{n+1,v}$ is either unramified or almost unramified (namely, the base change of an almost unramified representation of the unitary group), and if $\Pi_{n,v}$ is almost unramified then $\Pi_{n+1,v}$ is also almost unramified;*
- *Hypothesis 12.2.1 on the nonvanishing of certain local spherical characters holds true.*

Then

$$\mathrm{ord}_{\chi=\mathbf{1}}\mathscr{L}_p(\mathrm{M}_\Pi) = 1 \implies \dim_L H^1_f(F, \rho_\Pi) \geq 1. \tag{1.2.2}$$

If moreover p is an admissible prime for Π in the sense of [LTX+22, Definition 8.1.1], *then*

$$\mathrm{ord}_{\chi=\mathbf{1}}\mathscr{L}_p(\mathrm{M}_\Pi) = 1 \implies \dim_L H^1_f(F, \rho_\Pi) = 1. \tag{1.2.3}$$

Here a representation of a unitary group $U(\nu)$ over a non-archimedean local field is called *almost unramified* if it has a non-zero vector fixed by the stabilizer of a vertex lattice of type 1 or $\nu - 1$; see [Liu22] for the case when ν is even. For a comment on the reason for this condition, see Remark 1.4.2 below.

This result is a consequence of a non-vanishing criterion for certain explicit elements of $H^1_f(F, \rho_\Pi)$ arising as classes of algebraic cycles, which we describe in the rest of this section. The stronger (1.2.3) follows from combining that criterion with the Selmer bounds of [LTX+22] (whose admissibility condition is expected to be mild, see *ibid.* Remark 1.1.5) or [LaSk] (under different and often milder conditions on p). In particular, in this case we have that $H^1_f(F, \rho_\Pi)$ is generated by the class of an algebraic cycle – a result analogous to the finiteness of the p^∞-torsion of the Tate–Shafarevich group of an elliptic curve.

Remark 1.2.1. The history of theorems of type (1.2.2) consists of several works for similar 2-dimensional Galois representations over CM fields (starting with [PR87] and continuing with [Nek95, Kob13, Shn16, Dis17, Dis23, Dis22]), together with a very recent result by Y. Liu ad one of us for a family of higher-dimensional representations [DL24, Theorem 1.7]. The result (1.2.3) appears to be the first one for higher-dimensional representations, *ex aequo* with the main result of [Dis] building on [DL24]; previously, only 2-dimensional cases were known, based on generalizations of [Kol88].

Remark 1.2.2. Our notation (and the definitions going back to (1.1.1)) suggest that one may think of $\mathscr{L}(\mathrm{M}_\Pi)$, $\mathscr{L}_p(\mathrm{M}_\Pi)$ as attached to the virtual motive M_Π over F_0 whose p-adic realization is (up to abelian factors)

$$\mathrm{M}_{\Pi,p} \coloneqq (\mathrm{Ind}_{G_F}^{G_{F_0}} \rho_\Pi) \ominus \mathrm{As}^\star(\rho_\Pi).$$

Here, $\mathrm{As}^\star(\rho_\Pi) = \mathrm{As}^\star(\rho_{\Pi_n}) \oplus \mathrm{As}^\star(\rho_{\Pi_{n+1}})$ with the factors defined by

$$\begin{aligned} \mathrm{As}^\pm(\rho_{\Pi_\nu})\colon G_{F_0} &\longrightarrow \mathrm{GL}(L^\nu \otimes_L L^\nu) \\ G_F \ni g &\longmapsto \rho_{\Pi_\nu}(g) \otimes \rho_{\Pi_\nu}^{\mathrm{c}}(g), \\ c &\longmapsto (x \otimes y \longmapsto \pm y \otimes x) \end{aligned}$$

and the sign $\star = (-1)^\nu$ on the ν-factor.

Then the p-adic BBK conjecture would rather relate $\mathrm{ord}_{\chi=\mathbf{1}}\mathscr{L}_p(\mathrm{M}_\Pi)$ with

$$\dim_L H^1_f(F_0, \mathrm{Ind}_{G_F}^{G_{F_0}} \rho_\Pi) - \dim_L H^1_f(F_0, \mathrm{As}^\star(\Pi)).$$

The first term equals $\dim_L H^1_f(F, \rho_\Pi)$. Under our assumption that Π is hermitian, $\mathrm{As}^\star(\Pi_\nu)$ coincides with the adjoint representation defined in the opening paragraphs of [NT] (cf. [GGP12, Proposition 7.4]). By the results obtained there and in [Tho], under some irreducibility assumptions on ρ_{Π_ν}, we have $H^1_f(F_0, \mathrm{As}^\star(\rho_\Pi)) = 0$.

Remark 1.2.3. Theorem C and Theorem D below rely on a decomposition of the tempered part of the cohomology of unitary Shimura varieties (Hypothesis 11.1.2), which is expected to be proven in a sequel to [KSZ]. (At a more basic level, we also freely use the results of [Mok15, KMSW] on automorphic representations of unitary groups.)

Remark 1.2.4. Part of Hypothesis 12.2.1 has been proven in [Dan].

In the next subsection we describe, after some preliminaries, the construction of the Selmer classes of interest and our formula relating those to the derivative of $\mathscr{L}_p(\mathrm{M}_\Pi)$ (Theorem D).

1.3. **The p-adic arithmetic Gan–Gross–Prasad conjecture.** The cycles of interest arise from Shimura varieties attached to certain unitary group. We start by describing the representation-theoretic background.

1.3.1. *Incoherent unitary groups and their representations.* For a place v of F_0, denote by $\mathscr{V}_v$ the set of isomorphism classes of pairs $V_v = (V_{n,v}, V_{n+1,v})$ of (non-degenerate) $F_v/F_{0,v}$-hermitian spaces over F_v, where $V_{n,v}$ has rank n and $V_{n+1,v} = V_{n,v} \oplus F_v e$ with e a vector of norm 1. Let

$\mathscr{V}^\circ$ be the set of collections $V = (V_v)_v$ with $V_v \in \mathscr{V}_v$ such that $V_{n,v}$ is positive-definite for all archimedean places, and for all but finitely many places v, the Hasse–Witt invariant

$$\epsilon(V_v) \coloneqq \eta_v((-1)^{\binom{n}{2}} \det V_{n,v}) \tag{1.3.1}$$

equals $+1$.

We say that $V \in \mathscr{V}^\circ$ is *coherent* if there exists a (unique up to isomorphism) pair of F/F_0-hermitian spaces, still denoted $V = (V_n, V_{n+1})$, whose v-localization is V_v. This holds if and only if $\epsilon(V) \coloneqq \prod_v \epsilon(V_v)$ equals $+1$. When $\epsilon(V)$ equals -1, we refer to V as an *incoherent* pair of F/F_0-hermitian spaces. For $V \in \mathscr{V}^\circ$, we denote by

$$\mathrm{H}_v^{V_v} \coloneqq \mathrm{U}(V_{n,v}) \subset \mathrm{G}_v^{V_v} \coloneqq \mathrm{U}(V_{n,v}) \times \mathrm{U}(V_{n+1,v}), \tag{1.3.2}$$

(where the embedding is diagonal), by $H_v^{V_v} \subset G_v^{V_v}$ their $F_{0,v}$-points. When V is coherent, these are localizations of unitary groups $\mathrm{H}^V \coloneqq \mathrm{U}(V_n) \hookrightarrow \mathrm{G}^V \coloneqq \mathrm{U}(V_n) \times \mathrm{U}(V_{n+1})$ over F_0. When V is incoherent, we still use the notation

$$\mathrm{H}^V \subset \mathrm{G}^V$$

for the collections (1.3.2), which we refer to as *incoherent* unitary groups over F_0, and we denote $\mathrm{G}^V(\mathbf{A}^S) = \prod'_{v\notin S} G_v^{V_v}$.

In § 2.2, for each $V_v \in \mathscr{V}_v$, we fix measures dh_v on $H_v = H_v^{V_v}$ such that (i) if v is finite, dh_v is $\mathbf{Q}$-valued; (ii) if v is archimedean and V_v is positive definite, $\mathrm{vol}(H_v, dh_v) \in \mathbf{Q}^\times$; (iii) if $V \in \mathscr{V}^\circ$ is coherent, $\prod_v dh_v$ is the Tamagawa measure on $\mathrm{H}^V(\mathbf{A})$. We also have measures dg_v on $G_v = G_v^{V_v}$ with the analogous properties.

Suppose that $V \in \mathscr{V}^\circ$ is incoherent. If v is a place of F_0 non-split in F, we let $V(v) \in \mathscr{V}$ be the coherent collection with $V(v)_w = V_w$ if $w \neq v$, and $V(v)_v \in \mathscr{V}_v$ is the unique element different from V_v if v is non-archimedean, and the element such that $V(v)_{n,v}$ has signature $(n-1,1)$ if v is archimedean. We let $\mathrm{G}^{(v)} = \mathrm{G}^{V(v)}$.

Let $\underline{\pi}^\circ_{\mathbf{R}}$ be the set of (isomorphism classes of) tempered representations of the real group $U(n-1,1) \times U(n,1)$ whose base-change to $\mathrm{GL}_n(\mathbf{C})/\mathbf{R}^\times \times \mathrm{GL}_{n+1}(\mathbf{C})/\mathbf{R}^\times$ is Π°_∞. For a characteristic-zero field L and an incoherent $\mathrm{G} = \mathrm{G}^V$, a *cuspidal automorphic representation of* $\mathrm{G}(\mathbf{A})$ *over* L *trivial at infinity* is a representation $\pi = \pi^\infty \otimes \mathbf{1}_\infty$ of $\mathrm{G}(\mathbf{A})$ over L, such that for every $\iota\colon L \hookrightarrow \mathbf{C}$, every $v|\infty$, and some (equivalently, every) $\pi_v^\circ \in \underline{\pi}^\circ_{\mathbf{R}}$, the complex representation of $\mathrm{G}^{(v)}(\mathbf{A})$

$$\pi^{\iota,v} \coloneqq \iota\pi^v \otimes \pi_v^\circ$$

is irreducible, cuspidal and automorphic. If each π^ι is tempered and admits a *cuspidal* automorphic base-change to $\mathrm{G}'(\mathbf{A})$, we say that π is *stable*; the base-change of π^ι is necessarily of the form Π^ι for a trivial-weight representation Π over L that we call the base-change of π and denote $\mathrm{BC}(\pi)$.

1.3.2. *Arithmetic diagonal cycles.* When V is incoherent, we may attach to $\mathrm{G} = \mathrm{G}^V$ a tower of Shimura varieties $(X_K)_{K\subset \mathrm{G}(\mathbf{A}^\infty)}$ over F of dimension $2n-1$, and to $\mathrm{H} = \mathrm{H}^V$ a tower of Shimura varieties $(Y_{K'})_{K'\subset \mathrm{H}(\mathbf{A}^\infty)}$ over F of dimension $n-1$. They are proper provided that $F_0 \neq \mathbf{Q}$, a condition that we henceforth assume.

The embedding $\jmath\colon \mathrm{H}(\mathbf{A}) \to \mathrm{G}(\mathbf{A})$ induces a morphism of Shimura varieties still denoted by $\jmath$. Consider the (well-defined) normalized fundamental class $[Y]^\circ := \lim_{K'} \mathrm{vol}(K')[Y_{K'}] \in \varprojlim_{K'} \mathrm{Ch}^0(Y_{K'})_{\mathbf{Q}}$ and the *arithmetic diagonal cycle* $\jmath_*([Y]^\circ) \in \varprojlim_K \mathrm{Ch}^n(X_K)_{\mathbf{Q}}$ (where $\mathrm{Ch}^i(Z)_{\mathbf{Q}}$ denotes the Chow group of codimension-i cycles on Z with rational coefficients). The p-adic absolute cycle class of $\jmath_*([Y]^\circ)$ can be projected to an element

$$Z \in H^1_f(F, M^{\mathrm{temp}})$$

where $M^{\mathrm{temp}} = \varprojlim_K H^{2n-1}_{\text{ét}}(X_{K,\overline{F}}, \mathbf{Q}_p(n))^{\mathrm{temp}}$, and the superscript 'temp' refers to the tempered part of cohomology (see § 11.1.3).

1.3.3. *Gan–Gross–Prasad cycles.* Let π be a stable,[5] cuspidal automorphic representation of $\mathrm{G}(\mathbf{A})$ trivial at infinity, over some finite extension L of $\mathbf{Q}_p$; let $\Pi = \mathrm{BC}(\pi)$. According to Hypothesis 11.1.2, there is an injective map

$$\pi \longrightarrow \mathrm{Hom}_{\mathbf{Q}_p[G_F]}(M^{\mathrm{temp}}, \rho_\Pi),$$

well-defined uniquely up to scalar multiples. We identify π with the image of this map, and define the *Gan–Gross–Prasad functional*

$$\begin{aligned} Z_\pi \colon \pi &\longrightarrow H^1_f(F, \rho_\Pi) \\ \phi &\longmapsto Z_\pi(\phi) := \phi_* Z. \end{aligned} \tag{1.3.3}$$

We call elements in its image *Gan–Gross–Prasad cycles.*

1.3.4. *The p-adic arithmetic Gan–Gross–Prasad conjecture.* By construction, we have

$$Z_\pi \in \mathrm{Hom}_{\mathrm{H}^V(\mathbf{A})}(\pi, L) \otimes_L H^1_f(F, \rho_\Pi).$$

The space $\mathrm{Hom}_{\mathrm{H}^V(\mathbf{A})}(\pi, L)$ is known to be of dimension 0 or 1; in the latter case, π is said to be *distinguished*. By the local Gan–Gross–Prasad conjecture proved in [BP16, BP20], for a given representation Π over L as in Theorem A, there exists a unique (up to isomorphism) pair (V, π) where $V \in \mathscr{V}^\circ$ and π is a representation of $\mathrm{G}^V(\mathbf{A})$ as above that is distinguished. Moreover, π can be defined over L, and V is incoherent if and only if $\varepsilon(\Pi) = -1$ (see § 2.5.4).

The following is a p-adic analogue of the arithmetic Gan–Gross–Prasad conjecture [GGP12, Conjecture 27.1] for unitary groups.

Conjecture 1.3.1. *Let Π be a representation as in Theorem B. Assume that $\varepsilon(\Pi) = -1$ and that Π is not exceptional. The following conditions are equivalent:*

(1) $\mathrm{ord}_{\chi=\mathbf{1}} \mathscr{L}_p(\mathrm{M}_\Pi) = 1$*;*

(2) for the unique distinguished π with $\mathrm{BC}(\pi) = \Pi$, we have

$$Z_\pi \neq 0.$$

Remark 1.3.2. According to the p-adic BBK conjecture, both conditions are also equivalent to

(3) $\dim_L H^1_f(F, \rho_\Pi) = 1$.

[5]If π is only assumed to be tempered but not stable, we can still define Z_π with values in the Selmer group of a certain Galois representation (see § 11.2.3).

The implication *(2)* $\Longrightarrow$ *(3)* is [LTX+22, Theorem 1.1.9] or [LaSk, Theorem 1.4] (each under suitable conditions on p; see remark (1) following [LaSk, Theorem 1.4] for a comparison of the two sets of conditions).

As a refinement of Conjecture 1.3.1, we prove (under some conditions) a formula that 'measures' the product $Z_\pi \otimes Z_{\pi^\vee}$ in terms of the derivative of $\mathscr{L}_p(\mathrm{M}_\Pi)$; in order to state it, we need to define some pairings.

1.3.5. *Dualities.* Continue with the setup of § 1.3.3. Fix a non-degenerate pairing

$$\langle\, ,\, \rangle_\Pi\colon \rho_\Pi \otimes_L \rho_{\Pi^\vee} \longrightarrow L(1),$$

and for a compact open subgroup $K \subset \mathrm{G}(\mathbf{A}^\infty)$, let $\langle\, ,\, \rangle_K\colon M_K^{\mathrm{temp}} \otimes M_K^{\mathrm{temp}} \to L(1)$ be the pairing induced by Poincaré duality. Then we (well-)define a pairing

$$(\, ,\,)_\pi\colon \pi \otimes \pi^\vee \longrightarrow L \tag{1.3.4}$$

by $(\phi, \phi')_\pi \coloneqq \mathrm{vol}(K)^{-1}\, \phi \circ u_K(\phi'^*(1))$ for any $K \subset \mathrm{G}(\mathbf{A}^\infty)$ fixing ϕ, ϕ'. Here, $\phi'^*(1)\colon \rho^*_{\Pi^\vee}(1) \to M_K^{\mathrm{temp},*}(1)$ is the transpose, the volume uses the measure $\prod_v dg_v$, and $u_K\colon M_K^{\mathrm{temp},*}(1) \to M_K^{\mathrm{temp}}$ is the isomorphism induced by $\langle\, ,\, \rangle_K$.

1.3.6. *Invariant functionals.* If π is distinguished, there is a canonical generator

$$\alpha \in \mathrm{Hom}_{\mathrm{H}^V(\mathbf{A})}(\pi, L) \otimes_L \mathrm{Hom}_{\mathrm{H}^V(\mathbf{A})}(\pi^\vee, L)$$

defined as follows. Pick a factorization $(\, ,\,)_\pi = \prod_v (\, ,\,)_{\pi_v}$, where each factor is a pairing on $\pi_v \otimes \pi_v^\vee$. Then α is defined on factorizable elements $\phi = \otimes_{v\nmid\infty}\phi_v$, $\phi' = \otimes_{v\nmid\infty}\phi'_v$ by the product of absolutely convergent integrals

$$\iota\alpha(\phi, \phi') \coloneqq \mathrm{vol}(H^V_\infty, dh_\infty) \cdot \prod_{v\nmid\infty} \mathscr{L}(1/2, \iota\Pi_v)^{-1} \int_{H_v} \iota(\pi(h)\phi, \phi')_\pi \, dh_v, \tag{1.3.5}$$

where $\iota\colon L \hookrightarrow \mathbf{C}$ is any embedding, $\mathrm{vol}(H^V_\infty, dh_\infty) = \prod_{v\mid\infty} \mathrm{vol}(H_v^{V_v}, dh_v) \in \mathbf{Q}^\times$, and almost all factors are equal to 1.

1.3.7. *p-adic heights and main result.* Assume that Π is ordinary. Then ρ_Π is Panchishkin-ordinary in the sense of [Nek93] (recalled in § 10.2.1). By Nekovář's theory (see [Nek93] or § 10), the pairing $\langle\, ,\, \rangle_\Pi$ and the natural projection $\lambda\colon \Gamma_F \to \Gamma_{F_0}$ induce a height pairing

$$h_\pi\colon H^1_f(F, \rho_\Pi) \otimes_L H^1_f(F, \rho_{\Pi^\vee}) \longrightarrow \Gamma_{F_0} \hat{\otimes} L.$$

For $\mathscr{L} \in \mathscr{O}(\mathscr{Y})_L$, set

$$\partial\mathscr{L} \coloneqq [\mathscr{L} - \mathscr{L}(\mathbb{1})] \in T^*_{\mathbb{1}}\mathscr{Y}_L = \mathfrak{m}_{\mathbb{1}}/\mathfrak{m}_{\mathbb{1}}^2 \otimes_{\mathbf{Q}_p} L = \Gamma_{F_0} \hat{\otimes} L.$$

The following is a p-adic analogue of the refined arithmetic Gan–Gross–Prasad conjecture (cf. [Xue19, Conjecture 5.1]), in the spirit of the Ichino–Ikeda refinement of the usual Gan–Gross–Prasad conjecture. The case $n = 1$ is essentially equivalent to the p-adic Gross–Zagier formula as in [Dis17].

Conjecture 1.3.3. *Let $V \in \mathscr{V}^\circ$ be an incoherent pair, and let π be a distinguished, stable, ordinary, cuspidal automorphic representation of $\mathrm{G}^V(\mathbf{A})$, trivial at infinity, over a finite extension L of $\mathbf{Q}_p$. Let $\Pi \coloneqq \mathrm{BC}(\pi)$ and assume that it is ordinary and non-exceptional. Then for all $\phi \in \pi$, $\phi' \in \pi^\vee$, we have*

$$h_\pi(Z_\pi(\phi), Z_{\pi^\vee}(\phi')) = e_p(\mathrm{M}_\Pi)^{-1} \cdot \frac{1}{4}\partial\mathscr{L}_p(\mathrm{M}_\Pi) \cdot \alpha(\phi, \phi')$$

in $\Gamma_{F_0}\hat{\otimes}L$.

Remark 1.3.4. This conjecture implies the direction (1) $\Longrightarrow$ (2) in Conjecture 1.3.1; the converse implication is reduced to the conjectural non-degeneracy of h_π.

We have the following theorem, confirming the above refined conjecture in certain cases.

Theorem D. *Conjecture 1.3.3 holds if we further assume that:*

- *F/F_0 is unramified; in particular, $F_0 \neq \mathbf{Q}$;*
- *all places $v|2$ are split in F/F_0;*
- *$p > 2n$ if $n > 1$;*
- *for every place $v|p$ of F_0, we have that v splits in F and π_v is unramified;*
- *for every finite place v of F_0 that splits in F/F_0, at least one of $\pi_{n,v}$ and $\pi_{n+1,v}$ is unramified;*
- *for every finite place v of F_0 that is inert in F/F_0, $\pi_{n,v}$ and $\pi_{n+1,v}$ are either unramified or almost unramified, and if $\pi_{n,v}$ is almost unramified then $\pi_{n+1,v}$ is also almost unramified;*
- *Hypothesis 12.2.1 holds true.*

Remark 1.3.5. Besides the p-adic Gross–Zagier results mentioned in Remark 1.2.1, the only other p-adic height formula in the literature is the recent [DL24, Theorem 1.8]. While our setup and global approach to the proof are different, a theorem on p-local heights in [DL24] is essential for us.

1.4. **p-adic relative trace formulas and the proofs.** Our approach to Theorem D is based on the comparison of a pair of relative-trace formulas with p-adic coefficients, analogously to the approach proposed by one of us [Zha12] over archimedean coefficients. In fact, Theorem A and Theorem B are also proved by constructing rational and p-adic relative-trace formulas. We give a brief overview; unexplained terminology will be defined in the main body of the paper.

1.4.1. *Rationality.* Let us first explain the proof of Theorem A. For each $\chi \in Y(\mathbf{C})$, we have a Jacquet–Rallis relative-trace distribution

$$I(-,\chi)\colon \mathscr{H}(\mathrm{G}'(\mathbf{A}), \mathbf{C}) \longrightarrow \mathbf{C}$$

on the Hecke algebra for G'. For a 'regular' $f' \in \mathscr{H}(\mathrm{G}(\mathbf{A}), \mathbf{C})$, it admits a spectral and a geometric expansion

$$\sum_\Pi \frac{1}{4}\mathscr{L}(1/2, \Pi, \chi) \prod_v I_{\Pi_v}(f'_v, \chi_v) = I(f', \chi) = \sum_{\gamma \in \mathrm{B}'(F_0)} I_\gamma(f', \chi), \tag{1.4.1}$$

where: Π ranges over isomorphism classes of cuspidal representations of $\mathrm{G}'(\mathbf{A})$; the I_{Π_v} are local spherical characters; the variety $\mathrm{B}'_{/F_0} = \mathrm{H}'_1\backslash \mathrm{G}'/\mathrm{H}'_2$ for certain reductive subgroups $\mathrm{H}'_1, \mathrm{H}'_2 \subset \mathrm{G}'$; and the I_γ are products of local orbital integrals.

The only possible sources of irrationality in the right-hand side of (1.4.1) are essentially the archimedean orbital integrals. However, there is a particularly well-behaved class of $f'_\infty \in \mathscr{H}(\mathrm{G}'(F_{0,\infty}))$ (and corresponding $f' \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$), the so-called (rational) Gaussians, whose orbital integrals are controlled. Building on [BPLZZ21], we are able to show that for Π as in Theorem A, there exist L-rational Gaussians f' annihilating every automorphic representation of $\mathrm{G}'(\mathbf{A})$ but Π. Moreover, we need to show that one can pick f' to be 'regular' (that is, supported on suitably regular elements for the group action of $\mathrm{H}'_1 \times \mathrm{H}'_2$): this could be quickly done by invoking the results of [Zha14a, Appendix A], but we do it in a more explicit way as described in § 1.4.6. Then the rationality of $\mathscr{L}(1/2, \Pi, \chi)$ can be deduced from (1.4.1).

1.4.2. *p-adic analytic distribution.* We have a p-adic variant of $I(-,\chi)$, that we describe at first in a slightly idealized form. For any 'convenient' subgroup $K'_p \subset \mathrm{G}'(F_{0,p})$, we construct a distribution

$$\mathscr{I} = \mathscr{I}_{K'_p} \colon \mathscr{H}(\mathrm{G}'(\mathbf{A}^p))^\circ_{K'_p,\mathrm{rs},\mathrm{qc}} \longrightarrow \mathscr{O}(\mathscr{Y})$$

on a certain space of regularly supported, $\mathbf{Q}_p$-rational Gaussian elements of the Hecke algebra away from p. It admits a spectral and a geometric expansion

$$\sum_{\Pi} \frac{1}{4}\mathscr{L}_p(\mathrm{M}_\Pi, \chi) \prod_{v\nmid p} \mathscr{I}_{\Pi_v}(f'_v, \chi_v) = \mathscr{I}(f'^p, \chi) = \int_{\mathrm{B}'_{\mathrm{rs}}(F_0)} \mathscr{I}_\gamma(f'^p, \chi)\, dI^{\mathrm{ord}}_{\gamma, K'_p, p}, \tag{1.4.2}$$

where Π ranges over representations as in Theorem B with nontrivial K'_p-invariants; the $\mathscr{I}_{\Pi_v}$, $\mathscr{I}_\gamma$ are $\mathscr{O}(\mathscr{Y})$-valued spherical characters and orbital integrals, respectively; and finally, $dI^{\mathrm{ord}}_{-,K'_p,p}$ is a certain generalized Radon measure on the rational points of $\mathrm{B}'_{\mathrm{rs}} \subset \mathrm{B}'$, the open subvariety of regular semisimple orbits. In fact, we construct $\mathscr{I}$ from its geometric expansion, and prove Theorem B by extracting $\mathscr{L}_p(\mathrm{M}_\Pi)$ from $\mathscr{I}$.

Remark 1.4.1. This appears to be a new method for constructing p-adic L-functions. Let us linger on the archimedean input: while previous works relied on the nonvanishing of zeta integrals for explicit cohomological test vectors (as proved by Sun in [Sun17]), we use instead the 'spectral matching' property (proved by Beuzart-Plessis [BP21a]), which relates the value of I_{Π_v} on Gaussians with spherical characters of constant Hecke measures on a definite unitary group, whose computation is trivial.

Under some conditions on K'_p, we can relax the conditions of regularity on f'^p by using the recent work of Lu [Lu]; then the orbital integrals corresponding to non-semisimple orbits need an interesting regularization featuring Deligne–Ribet p-adic L-functions.

We note that Urban [Urb11, § 6] has constructed a p-adic Arthur–Selberg trace formula; it would be interesting to compare or combine our two approaches.

1.4.3. *The derivative.* For suitable f'^p, we then have a similar expansion for the derivative of $\mathscr{I}$. We will be especially interested in those f'^p that 'purely match' an $f^p \in \mathscr{H}(\mathrm{G}^V(\mathbf{A}^p))$ for some incoherent V, in the following sense. We have a 'matching of orbits' map for all places v

$$\underline{\delta}\colon \mathrm{B}_{\mathrm{rs}}(F_{0,v}) \longrightarrow \bigsqcup_{V'_v \in \mathscr{V}_v} \mathrm{H}^{V'_v}(F_{0,v}) \backslash \mathrm{G}^{V'_v}(F_{0,v}) / \mathrm{H}^{V'_v}(F_{0,v})$$

with image the set of regular semisimple orbits on the right hand side. The matching condition on f^p, f'^p is that, defining unitary-group orbital integrals by

$$J_\delta(f_v) = \int_{\mathrm{H}^{V_v}(F_{0,v})^2} f_v(h^{-1}\gamma h')\, dh dh',$$

we should have $I_\gamma(f'_v, \mathbf{1}) = J_{\underline{\delta}(\gamma)}(f_v)$ if $\underline{\delta}(\gamma)$ belongs to $\mathrm{H}^{V_v}(F_{0,v})\backslash \mathrm{G}^{V_v}(F_{0,v})/\mathrm{H}^{V_v}(F_{0,v})$, and $I_\gamma(f'_v, \mathbf{1}) = 0$ otherwise.

For such f'^p, we have $\mathscr{I}(f'^p, \mathbf{1}) = 0$ and the $\Gamma_{F_0} \hat{\otimes} \mathbf{Q}_p$-valued expansions

$$\sum_{\Pi} \frac{1}{4} \partial \mathscr{L}_p(\mathrm{M}_\Pi) \prod_{v \nmid p} \mathscr{I}_{\Pi_v}(f'_v, \chi_v) = \partial \mathscr{I}(f'^p) = \int_{\mathrm{B}'_{\mathrm{rs}}(F_0)} \partial \mathscr{I}_\gamma(f')\, dI^{\mathrm{ord}}_{\gamma, K'_p, p} \tag{1.4.3}$$

for the derivative. Moreover

$$\partial \mathscr{I}_\gamma(f'^p) = \sum_{v \nmid p\infty \text{ nonsplit in } F} I_\gamma(f'^{vp})\, \partial \mathscr{I}_\gamma(f'_v)$$

with $I_\gamma(f'^{vp}) = \mathscr{I}_\gamma(f'^{vp}, \mathbf{1})$. The v-component of the sum can be nonzero only if γ matches an orbit δ of $\mathrm{H}^{V(v)}(\mathbf{A}^p)\backslash \mathrm{G}^{V(v)}(\mathbf{A}^p)/\mathrm{H}^{V(v)}(\mathbf{A}^p)$ for the coherent pair $V(v) \in \mathscr{V}^\circ$ that is locally isomorphic to V at all places except v.

In practice, unless K'_p is suitably symmetric, we are only able to prove the geometric expansion in (1.4.2) after specialization at a $\chi \in Y(p^\infty)$, and with a generalized Radon measure $I^{\mathrm{ord}}_{-,K'_p,p}(\chi_p)$ depending on χ_p; nevertheless we can show that (1.4.3) still holds with $I^{\mathrm{ord}}_{-,K'_p,p} := I^{\mathrm{ord}}_{-,K'_p,p}(\mathbf{1})$.

1.4.4. *Arithmetic distribution.* Let $V \in \mathscr{V}^\circ$ be incoherent, $\mathrm{G} = \mathrm{G}^V$. For a convenient subgroup $K_p \subset \mathrm{G}(F_{0,p})$, we define another $\Gamma_{F_0} \hat{\otimes} \mathbf{Q}_p$-valued distribution on a suitable subset of $\mathscr{H}(\mathrm{G}(\mathbf{A}^p))$ by

$$\mathscr{J}_{K_p}(f^p) = h(Z^{\mathrm{ord}}_{K_p} T(f^p), Z^{\mathrm{ord}}_{K_p}),$$

where $Z^{\mathrm{ord}}_{K_p}$ is an ordinary modification of the arithmetic diagonal cycle in level K_p, and h is a limit of height pairings on the Selmer group of the tempered, ordinary part of $H^{2n-1}(X_{K^pK_p, \overline{F}_0}, \mathbf{Q}_p(n))$.

When the cycles have disjoint support on the generic fiber, the p-adic height pairing admits an expansion $h = \sum_{v \nmid \infty} h_v$ into local height pairings. The disjointness is guaranteed if f has regular support at some place v_0.

By results in [DL24, LL21], the local height pairing at a place v away from p is related to the arithmetic intersection pairing on a regular v-integral model, at least after applying suitable Hecke correspondences to the cycles, and under some vanishing condition for the absolute cohomology of the model (upon localizing at a non-Eisenstein ideal). After a base change, for suitable levels we may use the models constructed in the previous work of Rapoport, Smithling and the second author [RSZ20, RSZ21]; here, a technical difficulty is to verify the vanishing of cohomology in the case of non-trivial level structure, as required both in order to treat the ramification of Π in Theorem D, and for the place v_0 of regular support. Once this is settled:

– for *split* places away from p, we can show that the local arithmetic intersection numbers vanish, by refining an argument of [Zha12, RSZ20];

– for *inert* places v (thus away from p), by results in [Zha12] and [RSZ20], the local arithmetic intersection numbers admit geometric expansions over the orbits δ for $V(v)$, whose terms are products of local orbital integrals $J_\delta(f_{v'})$ ($v' \neq v$) and arithmetic intersection numbers $\mathscr{J}_\delta(f_v)$ in a certain v-adic Rapoport–Zink space.

On the other hand, the contribution of *p-adic* places vanishes: this is proved by a variant of an argument of Perrin-Riou, which in our higher-dimensional case relies on a recent foundational result of Y. Liu and the first author in [DL24].

We then obtain a spectral and a geometric expansion

$$\sum_\pi \mathscr{J}_\pi(f^p) = \mathscr{J}_{K_p}(f^p) = \int_{\mathrm{B}'_{\mathrm{rs}}(F_0)} \sum_{v \nmid p\infty \ \text{nonsplit}} \mathbf{1}_{V(v)}(\gamma) J^{vp}_{\underline{\delta}(\gamma)}(f^{vp}) \mathscr{J}_{\underline{\delta}(\gamma),v}(f_v)\, dI^{\mathrm{ord}}_{\gamma,p,K'_p},$$

where: π ranges over equivalence classes of automorphic representations as in Theorem D; the geometric expansion is pulled back to $\mathrm{B}'_{\mathrm{rs}}$ via the 'matching of orbits' map $\underline{\delta}$, and $\mathbf{1}_{V(v)}$ is the indicator function of those orbits matching one on $\mathrm{G}^{V(v)}$; and finally, $dI^{\mathrm{ord}}_{\gamma,p,K'_p}$ is as in (1.4.3).

1.4.5. *Comparison.* Theorem D is eventually deduced from the spectral sides of an equality

$$\mathscr{J}_{K_p}(f^p) = \partial \mathscr{I}_{K'_p}(f'^p) \tag{1.4.4}$$

for suitable matching f^p, f'^p.

We prove (1.4.4) by comparing the geometric expansions. By the definitions of local matching of Hecke elements (which can be globally assembled thanks to the Fundamental Lemma [Yun11, BP21b]), orbital integrals on either side are the same, thus we are reduced to identities

$$\mathscr{J}_{\underline{\delta}(\gamma),v}(f_v) = \partial \mathscr{I}_\gamma(f'_v) \tag{1.4.5}$$

for inert places v. For the spherical f'_v, f_v, the identity (1.4.5) is the Arithmetic Fundamental Lemma proposed by one of us [Zha12] and then proved in [Zha21, MZ]; for certain f'_v, f_v of maximal parahoric level, (1.4.5) is the arithmetic transfer conjecture recently proved by Z. Zhang [ZZh].

Remark 1.4.2. We point out the main obstacle to removing the condition of our representations being almost unramified at inert places from our main theorems. The condition comes from working with Shimura varieties at "almost self-dual" levels (namely, for vertex-parahoric subgroups of type 1 or $\nu - 1$). Although Z. Zhang's result on the arithmetic transfer conjecture [ZZh] holds in greater generality (at maximal parahoric levels), we can only show the vanishing result for absolute cohomology of integral models alluded to in § 1.4.4 in almost self-dual levels, see Proposition 9.4.2. The proof of that proposition relies on a refined understanding of the cohomology of the irreducible components of the special fiber, which currently seems only available at almost self-dual levels. The generalization to other maximal parahoric levels seems a very interesting yet challenging question.

1.4.6. *Construction of test Gaussians.* In order to deduce Theorem D from the comparison, we need to pick suitable matching f^p, f'^p that annihilate all terms in the spectral expansions but those corresponding to π, Π; then we may use the comparison in [BP21a, BP21] of the functionals I_{Π_v} with corresponding ones, J_{π_v}, that are related to the local components of $\alpha = (1.3.5)$.

The most challenging requirement for the Gaussian f'^p is that the spherical character $\otimes_{v\nmid p\infty} I_{\Pi_v}(f'^p)$ should not vanish, its non-regular-semisimple orbital integrals should vanish, while at the same time its level should be controlled in order to allow working with nice integral models on the arithmetic side. This turns out to be a rather hard semi-local problem, which is solved by an explicit construction of a pair of elements $f'_{v,\pm}$ of Iwahori level (to be used at a pair of split non-p-adic places), and two explicit local computations: one on the spectral side, which is Proposition 5.2.6 (a result of Liu–Sun); and one on the geometric side, which is part of Proposition 6.1.2, whose proof occupies the entire § 6. It is curious to note that $f'_{v,+}$ also occurs in the construction of the p-adic relative-trace formula (and in fact, this is how we discovered it).

1.4.7. *Organization of the paper.* After some preliminaries in § 2, this paper is divided into two parts and en epilogue. In Part 1, we construct the analytic distribution $\mathscr{I}$ and prove the associated RTF, as well as Theorems A and B. In Part 2, we construct the distribution $\mathscr{J}$ and prove the associated RTF. In the epilogue, we compare the two RTFs to prove Theorems D and C. More details on the contents of the two parts are provided at the beginning of each.

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2. Notation and preliminaries

2.1. **Basic notation.** We set up some notation to be used throughout the paper unless otherwise noted.

2.1.1. *Fields.* We denote by $F \supset F_0$ a quadratic extension of number fields, as in the introduction, and by $\mathrm{c} \in \mathrm{Gal}(F/F_0)$ the conjugation. We denote by $\mathbf{A}$ the adèles of F_0. From § 4 on, we will assume that F_0 is totally real and F is CM.

We denote by c the nontrivial automorphism of F/F_0, and by

$$\eta\colon F_0^\times \backslash \mathbf{A}^\times \longrightarrow \{\pm 1\}$$

the quadratic character associated with F/F_0. We fix an auxiliary element $\tau \in F$ such that $\tau^{\mathrm{c}} = -\tau$, and an extension $\eta'\colon F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ of η.

If F' is a number field and S is a finite set of places of F', we denote by $F'_S = \prod_{v\in S} F'_v$, and by $\mathbf{A}_{F'}^S = \prod'_{v\notin S} F'_v$. If $F'' \subset F'$ is a subfield and ℓ is a place of of F'', for notational purposes we identify ℓ with the set of places of F' above ℓ.

2.1.2. *L-functions.* In the rest of the paper (*unlike* in the introduction), all global ζ- and L-functions valued in the complex numbers are complete including the archimedean factors (this also includes the ratio of L-functions $\mathscr{L}(1/2,\Pi,\chi)$). If $L^S(s)$ is a global L-function, we denote by

$$L^{S,*}(s_0)$$

its leading term at $s=s_0$.

2.1.3. *Groups.* We now recall the groups under consideration in this paper, then discuss local and global base-change from unitary groups to general linear groups. We denote by $\mathbf{G}_m = \operatorname{Spec}\mathbf{Q}[T^{\pm 1}]$ the multiplicative group over $\mathbf{Q}$. If G is a (usually, group-) scheme over a global field F_0 and v is a place of F_0, we denote $G_v := \mathrm{G}(F_{0,v})$ with its v-adic topology. We also denote

$$[G] = \mathrm{G}(F_0)\backslash \mathrm{G}(\mathbf{A}).$$

For $*=\emptyset, 0$ (where in this type of context, '$\emptyset$' will always mean 'no subscript') and $\nu\in\mathbf{N}$, let $\mathrm{G}'_{\nu,*} := \operatorname{Res}_{F_*/F_0}\mathrm{GL}_\nu$. We consider

$$\mathrm{G}' := \mathrm{G}'_n/\mathrm{G}'_{1,0}\times \mathrm{G}'_{n+1}/\mathrm{G}'_{1,0}, \tag{2.1.1}$$

where $G'_{1,0}$ is the F_0-split center of G'_ν, and its subgroups

$$j_1\colon \mathrm{H}'_1 := \mathrm{G}'_n \hookrightarrow \mathrm{G}',$$

where $j_1(h) := [(\mathrm{diag}(h,1),h)]$, and

$$j_2\colon \mathrm{H}'_2 := \mathrm{G}'_{n,0}/\mathrm{G}'_{1,0}\times \mathrm{G}'_{n+1,0}/\mathrm{G}'_{1,0} \hookrightarrow \mathrm{G}',$$

where j_2 is induced by $F_0\hookrightarrow F$.

For unitary groups, we use the notation H^V, G^V introduced in § 1.3.1. We denote by $\mathscr{V}$ the set of isomorphism classes of pairs $V=(V_n, V_{n+1}=V_n\oplus Fe)$ of F/F_0-hermitian spaces with $(e,e)=1$. When F_0 is totally real and F is CM, we denote by $V_\infty^\circ = (V_v^\circ)_{v|\infty}$ the pair such that $V_{n,v}$ is positive-definite, and by $\mathscr{V}^\circ\subset\mathscr{V}$ the set of (coherent or incoherent) pairs (V_v) such that $V_v = V_v^\circ$ for all $v|\infty$. We partition

$$\mathscr{V}^\circ = \mathscr{V}^{\circ,+}\sqcup \mathscr{V}^{\circ,-},$$

where $V\in\mathscr{V}^{\circ,\epsilon}$ if and only if $\epsilon(V)=\epsilon$.

2.2. **Measures.** Let F_0 be a number field, and let $D=|D_{F_0}|$ be the absolute value of its discriminant. Fixing a nontrivial character $\psi\colon F_0\backslash\mathbf{A}\to\mathbf{C}^\times$, we denote by $dx=\prod_v dx_v$ the self-dual measure on $\mathbf{A}$ with respect to ψ; it satisfies $\mathrm{vol}(F_0\backslash\mathbf{A}, dx)=1$. For a finite place v, let d_v be a generator of the different ideal of $F_{0,v}$ and let $D_v := |d_v|^{-1}$. Assume for definiteness that $\mathrm{Ker}(\psi_v) = d_v^{-1}\mathscr{O}_{F_{0,v}}$ for all finite places v; then we have $\mathrm{vol}(\mathscr{O}_{F_{0,v}}, dx_v) = D_v^{-1/2}$. We have $D=\prod_{v\nmid\infty} D_v$, and for a finite set of places S of F_0 we define $D^S := \prod_{v\nmid S\infty} D_v$.

2.2.1. *Tamagawa measures.* If G is a reductive group over a local or global field E, we denote by M_{G} the Artin–Tate motive attached to (the quasi-split inner form of) G by Gross [Gro97]. If E

is a local field, let

$$\Delta_{\mathrm{G}} \coloneqq D_v^{\dim \mathrm{G}/2} L(M_{\mathrm{G}}^\vee(1)).$$

Then the abelian term in (1.1.1) (including the factor $D_{F_{0,v}}^{n+1}$) equals $\Delta_{G^{V_v}}/\Delta_{H^{V_v}}$ where H^{V_v}, G^{V_v} are is in (1.3.2) (for any $V_v \in \mathscr{V}_v$.)

Assume from now on that E is the global field F_0. For a finite set S of places of F_0, let

$$\Delta_{\mathrm{G}}^S \coloneqq (D^S)^{\dim \mathrm{G}/2}\, L^{S,*}(M_{\mathrm{G}}^\vee(1), 0).$$

Let ω be any non-zero top-degree invariant differential form on G. We denote by

$$d_\omega g_v \coloneqq |\omega|_v$$

its modulus with respect to dx_v ([Oes84, §4]), a Haar measure on $\mathrm{G}(F_{0,v})$. We define

$$d^\natural g_v \coloneqq \Delta_{\mathrm{G},v}\, d_\omega g_v$$

Then for all finite places v and any open compact subgroup $K_v \subset G_v$, we have $\mathrm{vol}(K_v, d^\natural g_v) \in \mathbf{Q}^\times$. Moreover if G_v is unramified and K_v is hyperspecial, we have $\mathrm{vol}(K_v, d^\natural g_v) = 1$. The Tamagawa measure on G is

$$dg \coloneqq \Delta_{\mathrm{G}}^{-1} \prod_v d^\natural g_v. \tag{2.2.1}$$

2.2.2. *Variants.* We define a variant

$$dg_v = \begin{cases} d^\natural g_v & \text{if } v \nmid \infty \\ \Delta_{\mathrm{G}}^{-1} d^\natural g_v = \Delta_{\mathrm{G}}^{\infty,-1} d_\omega g_v & \text{if } v = \infty \end{cases} \tag{2.2.2}$$

so that $dg = \prod_v dg_v$. The 'rationale' for this choice is the following.

Lemma 2.2.1. *Suppose G_∞ is compact. Then* $\mathrm{vol}(G_\infty, dg_\infty)$ *is rational.*

Proof. We say that two measures μ, μ' are commensurable if $\mu = c\mu'$ for some $c \in \mathbf{Q}^\times$. Let $\mu \coloneqq \prod_v \mu_v$ be the measure on $\mathrm{G}(\mathbf{A})$ considered in § 9 of [Gro97], to which all citations in this proof will refer. The measure μ is nonzero by Propositions 9.4, 9.5. For almost all finite v, $\mu_v = dg_v$; for all finite v, μ_v gives rational volume to compact open subgroups (equation (5.2)), hence it is commensurable with dg_v; and μ is commensurable with dg (Theorem 9.9). It follows that dg_∞ is commensurable with μ_∞, which (again by equation (5.2)), gives rational volume to G_∞. □

We also consider a different measure, for comparison with some of the literature (notably [Zha14a, § 2]). Let Z be the center of G, let $\mathrm{G}^{\mathrm{ad}} \coloneqq \mathrm{G}/\mathrm{Z}$; put

$$\zeta_{\mathrm{G},v}(1) \coloneqq D_v^{-\dim \mathrm{Z}/2} \Delta_{\mathrm{Z},v}, \qquad \zeta_{\mathrm{G}}^{S,*}(1) \coloneqq D^{-\dim \mathrm{Z}/2} \Delta_{\mathrm{Z}}^S,$$

sot that $D_v^{\dim \mathrm{Z}/2} \zeta_{\mathrm{G},v}(1) \Delta_{\mathrm{G}^{\mathrm{ad}},v} = \Delta_{\mathrm{G},v}$. Then we set

$$d^* g_v \coloneqq \zeta_{\mathrm{G},v}(1)^1\, d^\tau g_v, \qquad d^* g = \prod_v d^* g_v$$

so that

$$dg = \zeta_{\mathrm{G},v}^*(1)^{-1} \prod_v d^* g_v$$

and for finite v, $dg_v = D_v^{\dim \mathrm{Z}/2}\Delta_{\mathrm{G}^{\mathrm{ad}},v} d^*g_v$.

2.2.3. *Local and incoherent measures.* The global measures do not depend on ω, but the local ones do. We fix the following explicit choices:

- if $\mathrm{G} = \mathrm{GL}_\nu$, we take
$$\omega := \det(g)^{-\nu} \wedge_{i,j} dg_{ij}.$$
- if G is a (product of) unitary groups over a local or a global field, we fix ω as in [Zha14b, § 2].

If G is a (product of) incoherent unitary groups, we then get a measure on $\mathrm{G}(\mathbf{A})$ by (2.2.1), with a factorization $dg = \prod_v dg$ as in (2.2.2).

2.3. **Hecke algebras.** Let G be a reductive group over a number field F_0, let v be a place of F_0, and let L be a characteristic-zero field, with $L = \mathbf{C}$ if v is archimedean or G_v is not compact. We denote by $\mathcal{S}(G_v, L)$ the space of Schwartz functions on G_v valued in L: when $F_{0,v}$ is non-archimedean, this is the same as the smooth compactly supported L-valued functions, whereas when $F_{0,v}$ is archimedean this is defined in [Cas89, AG08]. We denote by $\mathscr{H}(G_v, L)$ the space of *Schwartz measures* on G_v: those are measures of the form $\dot{f}\, dg$ where $\dot{f} \in \mathcal{S}(G_v, L)$ and dg is the Haar measure fixed above. The field L will be omitted when it is unimportant or understood from context. For an open compact $K_v \subset G_v$, we denote
$$e_{K_v} := \frac{1}{\mathrm{vol}(K_v, dg_v)} \mathbf{1}_{K_v}\, dg_v$$
for any Haar measure dg_v; it is an idempotent in $\mathscr{H}(G_v)$.

When G'_v is the group (2.1.1), we define the *standard hyperspecial subgroup* $K_v^\circ \subset G'_v$ to be the image of $\mathrm{GL}_n(\mathscr{O}_{F_v}) \times \mathrm{GL}_{n+1}(\mathscr{O}_{F,v})$; when $G_v^{V_v}$ is the product of unramified unitary groups from (1.3.2), a *relative hyperspecial subgroup* $K_v^\circ \subset G_v^{V_v}$ is one of the form $\mathrm{U}(\Lambda_v) \times \mathrm{U}(\Lambda_v \oplus \mathscr{O}_{F,v} e)$ for some self-dual lattice $\Lambda_v \subset V_{n,v}$. For S a finite set of places of F_0, and G denoting either G' or G^V for some $V \in \mathscr{V}^\circ \cup_{\mathscr{V}^\circ,+} \mathscr{V}$, we consider the Hecke algebra
$$\mathscr{H}(\mathrm{G}(\mathbf{A}^S)) := {\bigotimes_{v\notin S}}' \mathscr{H}(G_v),$$
where the restricted tensor product is with respect to
$$f_v^\circ := e_{K_v^\circ} \tag{2.3.1}$$
for some relative hyperspecial $K_v^\circ \subset G_v$. If $K = \prod_v K_v \subset \mathrm{G}(\mathbf{A}^S)$ is an open compact subgroup, we denote $e_K := \prod_v e_{K_v}$.We say that an element $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}^\infty))$ is *supported in the set S* if we can write $f = f_S \otimes \prod_{v \notin S\infty} f_v^\circ$ for some $f_S \in \mathscr{H}(G_S)$.

For $f \in \mathscr{H}(G_v)$, we denote $f^\vee(x) := f(x^{-1})$. We denote by $\star$ the convolution operation $f_1 \star f_2(x) := \int_{G_v} f_1(xg) f_2(g^{-1})$, and sometimes omit this symbol is omitted.

2.3.1. *Convention.* We stipulate that groups and Hecke algebras act on locally symmetric spaces, Shimura varieties, and their homology and algebraic cycles on the right; on automorphic forms on the left.

2.4. **Local base-change and distinction.** Let v be a place of F_0. If v is nonarchimedean, $\mathrm{G}^?$ is a reductive group over $F_{0,v}$, and L is a field admitting embeddings into $\mathbf{C}$, we say that an absolutely irreducible (that is, $\pi \otimes_L \overline{L}$ is irreducible) smooth admissible representation π of $G_v^?$ over L is *tempered* if $\pi \otimes_{L,\iota} \mathbf{C}$ is tempered for every $\iota\colon L \hookrightarrow \mathbf{C}$.

Let $\mathscr{V}_v$ be the set of isomorphism classes of pairs $V_v = (V_{n,v}, V_{n+1,v} = V_{n,v} \oplus F_{0,v}e)$ of hermitian spaces over $F_v/F_{0,v}$. Let $\mathrm{Temp}(G_v^?)(L)$ be the set of $\mathrm{Gal}(\overline{L}/L)$-orbits of isomorphism classes of irreducible tempered representations of $G_v^?$ over L.

2.4.1. *Local base-change.* Let v be a place of F_0. Thanks to [Mok15, KMSW], we have a local base-change map BC from complex irreducible admissible representations of $G_v^{V_v}$ to complex irreducible admissible representations of G'_v, whose definition is recalled in [BP21a, § 2.10]. It has the following properties:

(1) it restricts to a map
$$\mathrm{BC}\colon \mathrm{Temp}(G_v^{V_v})(\mathbf{C}) \longrightarrow \mathrm{Temp}(G'_v)(\mathbf{C}); \tag{2.4.1}$$

(2) being defined by a map of L-groups, it is rational in the sense that it yields a map
$$\mathrm{BC}\colon \mathrm{Temp}(G_v^{V_v})(L) \longrightarrow \mathrm{Temp}(G'_v)(L)$$
for any characteristic-zero field L;

(3) when v splits in F, we simply have $\mathrm{BC}(\pi) \coloneqq \pi \boxtimes \pi^\vee$ if we identify $G_v^{V_v} \cong G'_{n,0,v} \times G'_{n+1,0,v}$ for the unique $V_v \in \mathscr{V}_v$;

(4) when $G_v^{V_v} = U(n) \times U(n+1)$ over $\mathbf{R}$, the preimage of $\Pi^\circ_{\mathbf{R}}$ under (2.4.1) consists of the trivial representation only;

(5) when $G_v^{V_v} = U(n-1,1) \times U(n,1)$ over $\mathbf{R}$, the preimage
$$\underline{\pi}^\circ_{\mathbf{R}} \coloneqq \mathrm{BC}^{-1}(\Pi^\circ_{\mathbf{R}}) \tag{2.4.2}$$
consists of the $n(n+1)$ discrete series representations having the Harish-Chandra parameter $\{\frac{1-\nu}{2}, \frac{3-\nu}{2}, \ldots, \frac{\nu-1}{2}\}$ on the $U(\nu-1,1)$-component. (See [LTX⁺22, Proposition C.3.1].)

If v is non-archimedean and π_v, respectively Π_v, is a representation of $G_v^{V_v}$, respectively G'_v, over a field L admitting embeddings into $\mathbf{C}$, we will write $\mathrm{BC}(\pi_v) = \Pi_v$ if $\mathrm{BC}(\iota\pi_v) = \iota\Pi_v$ for every embedding $\iota\colon L \hookrightarrow \mathbf{C}$.

2.4.2. *Hermitian representations.* We will say that a tempered representation Π_v of G'_v is *hermitian* if the space $\mathrm{Hom}_{H'_{2,v}}(\Pi_v, \eta_v^n \boxtimes \eta_v^{n-1})$ is nonzero. By the local Flicker–Rallis conjecture proved by Matringe, Mok, and others (see [Ana, §3.1] and references therein), a representation Π_v over $\mathbf{C}$ is hermitian if and only if it is in the image of base-change for some $V_v \in \mathscr{V}_v$.

2.4.3. *Distinction and the local Gan–Gross–Prasad conjecture.* Let v be a place of F_0 and let L be a field of characteristic zero; we restrict to $L = \mathbf{C}$ if v is archimedean and V_v is not definite. We say that a tempered representation π of $G_v^{V_v}$ over L is *distinguished* if the space $\mathrm{Hom}_{H_v^{V_v}}(\pi_v, L)$ is nonzero, and by the *multiplicity-one* result of [AGRS10], this space is one-dimensional if nonzero.

It is clear that distinction is a $\mathrm{Gal}(\overline{L}/L)$-invariant property. We denote by

$$\mathrm{Temp}(H_v^{V_v}\backslash G_v^V)(L) \subset \mathrm{Temp}(G_v^V)(L)$$

the subset of orbits of distinguished tempered representations.

The following fundamental result is the local Gan–Gross–Prasad conjecture for unitary groups.

Proposition 2.4.1. *Let Π_v be a hermitian tempered representation of G'_v over a characteristic zero field L; we restrict to $L = \mathbf{C}$ if v is archimedean. There exists a unique pair (V_v, π_v) with $V_v \in \mathscr{V}_v$ and $\pi_v \in \mathrm{Temp}(H_v^{V_v}\backslash G_v^{V_v})(L)$ such that $\Pi_v = \mathrm{BC}(\pi_v)$.*

Proof. If $L = \mathbf{C}$, this is proved in [BP16, BP20]. In general, we may assume that there is an embedding $\iota\colon L \hookrightarrow \mathbf{C}$ and apply the result to $\iota\Pi_v$ to obtain a pair $(V_v, \pi_v^{\mathbf{C}})$. By uniqueness, $\pi_v^{\mathbf{C}}$ is isomorphic to its $\mathrm{Aut}(\mathbf{C}/\iota L)$-conjugates. □

2.5. **Automorphic base-change.**

2.5.1. *Rational spaces of automorphic representations.* The following discussion is based on [Clo90, Théorème 3.1.3]. Let L be a field admitting embeddings into $\mathbf{C}$, and let $\Pi = \Pi^\infty \otimes \mathbf{1}_\infty$ be an absolutely irreducible representation of $\mathrm{G}'(\mathbf{A})$ over an L-vector space. We say that Π is cuspidal automorphic of trivial weight if for every (equivalently, some) embedding $\iota\colon L \hookrightarrow \mathbf{C}$, the representation $\Pi^\iota := \iota\Pi^\infty \otimes \Pi^\circ_\infty$ is cuspidal and automorphic. Every cuspidal automorphic representation $\Pi_{\mathbf{C}}$ of $\mathrm{G}'(\mathbf{A})$ such that $\Pi_{\mathbf{C},\infty} \cong \Pi^\circ_\infty$ arises as Π^ι for some Π defined over a number field; the smallest such number field $\mathbf{Q}(\Pi^\iota) =: \iota\mathbf{Q}(\Pi^\infty)$ depends only on $\Pi_{\mathbf{C}}$, and Π is unique up to $\mathbf{Q}(\Pi)$-isomorphism.

Denote by $\widetilde{\mathscr{C}}(\mathrm{G}')(L)$ the set of isomorphism classes of trivial-weight cuspidal automorphic representations defined over L, and by $\mathscr{C}(\mathrm{G}')(L) := \widetilde{\mathscr{C}}(\mathrm{G}')(\overline{L})/G_L$, where we recall that $G_L := \mathrm{Gal}(\overline{L}/L)$. By [Car12], for every $\Pi \in \widetilde{\mathscr{C}}(\mathrm{G}')$ and every finite place v, the representation Π_v of G'_v is tempered.

Lemma 2.5.1. *The natural map*

$$\widetilde{\mathscr{C}}(\mathrm{G}') \longrightarrow \mathscr{C}(\mathrm{G}')$$

is an isomorphism.

Proof. This follows from the above discussion and [Clo90, Proposition 3.1, Théorème 3.1.3]. □

2.5.2. *Ramakrishnan's automorphic Tchebotarev theorem.* We will use the following special case of [Ram, Theorem A].

Proposition 2.5.2. *Let Π, Π' be two cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$. Assume that $\Pi_w \cong \Pi'_w$ for all but finitely many primes w of F split over F_0. Then $\Pi \cong \Pi'$.*

2.5.3. *Base change.* Let $V \in \mathscr{V}$ or, if F_0 is totally real and F is CM, let $V \in \mathscr{V} \cup_{\mathscr{V}^\circ,+} \mathscr{V}^\circ$. Let $\mathrm{G} = \mathrm{G}^V$, $\mathrm{H} = \mathrm{H}^V$. For a field L admitting embeddings into $\mathbf{C}$, denote by

$$\widetilde{\mathscr{C}}(\mathrm{G})(L) \quad \supset \quad \widetilde{\mathscr{C}}(\mathrm{H}\backslash\mathrm{G})(L)$$

the set of isomorphism classes of tempered[6] cuspidal automorphic representations of $\mathrm{G}(\mathbf{A})$ that are trivial at infinity, and its subset of representations that are $\mathrm{H}(\mathbf{A})$-distinguished. We also put

$$\mathscr{C}(\mathrm{G})(L) \coloneqq \widetilde{\mathscr{C}}(\mathrm{G})(\overline{L})/G_L \quad \supset \quad \mathscr{C}(\mathrm{H}\backslash\mathrm{G})(L) \coloneqq \quad \widetilde{\mathscr{C}}(\mathrm{H}\backslash\mathrm{G})(\overline{L})/G_L.$$

We will view $\mathscr{C}(\mathrm{G}')$, $\mathscr{C}(\mathrm{G})$ and $\mathscr{C}(\mathrm{H}\backslash\mathrm{G})$ as ind-finite schemes over $\mathbf{Q}$.

Definition 2.5.3. Let $V \in \mathscr{V}$, and let $\mathrm{G} = \mathrm{G}^V$. Let π be a (complex) automorphic representation of $\mathrm{G}(\mathbf{A})$ which is tempered everywhere, and let Π be an automoprhic representation of $\mathrm{G}'(\mathbf{A})$. We say that Π is a *weak automorphic base-change of* π, and write $\Pi \cong \mathrm{BC}(\pi)$, if for all but finitely many places v of F_0 split in F, we have $\Pi_v \cong \mathrm{BC}(\pi_v)$ for the local base-change of (2.4.1). We say that Π is a *strong automorphic base-change* of π if $\Pi_v \cong \mathrm{BC}(\pi_v)$ for all places v.

Remark 2.5.4. By Proposition 2.5.2, a weak automorphic base-change of π is unique up to isomorphism if it exists, which justifies the notation. Moreover by [Mok15, KMSW], if Π is a weak automorphic base-change of π, then Π is a strong base-change of π. From now we will simply write *the (automorphic) base-change* without adjectives.

Suppose now that F_0 is totally real and F is CM. Let $V \in \mathscr{V}^\circ$, let $\mathrm{G} = \mathrm{G}^V$, and let L be a characteristic-zero field. Let π be a cuspidal automorphic representation of $\mathrm{G}(\mathbf{A})$ over L which is trivial at infinity, and let Π be a trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over L. We say that Π is the cuspidal *automorphic base-change* of π, and write

$$\Pi \cong \mathrm{BC}(\pi),$$

if for every $\iota\colon L \hookrightarrow \mathbf{C}$ and every finite place v, we have $\iota\Pi_v \cong \mathrm{BC}(\iota\pi_v)$. We say that π is *stable* if it admits a cuspidal automorphic base-change over L; we denote by

$$\widetilde{\mathscr{C}}(\mathrm{G})(L)^{\mathrm{st}} \subset \widetilde{\mathscr{C}}(\mathrm{G})(L), \qquad \mathscr{C}(\mathrm{G})(L)^{\mathrm{st}} \subset \mathscr{C}(\mathrm{G})(L)$$

the subsets consisting of (orbits of) isomorphism classes of representations that are stable.

Note that by the definitions and the rationality of local base-change maps observed in § 2.4.1, the stability condition is Galois-invariant, so that the above definition makes sense.

2.5.4. *Hermitian automorphic representations as the image of base-change.*

Proposition 2.5.5. *Let* Π *be a cuspidal automorphic representation of* $\mathrm{G}'(\mathbf{A})$ *with* $\Pi_\infty \cong \Pi_\infty^\circ$. *The following are equivalent:*

(1) Π *is hermitian;*

(2) for every $V \in \mathscr{V}$ *such that* G^V *is quasi-split at all places, there exists a cuspidal automorphic representation* π' *of* $\mathrm{G}^V(\mathbf{A})$ *such that* $\Pi \cong \mathrm{BC}(\pi')$*;*

(3) for some $V \in \mathscr{V}^\circ$, *there exists a cuspidal automorphic representation* π *of* $\mathrm{G}^V(\mathbf{A})$ *over* $\mathbf{C}$, *trivial at infinity and tempered everywhere, such that* $\Pi^\infty \otimes \mathbf{1}_\infty \cong \mathrm{BC}(\pi)$.

(4) there exists a unique pair (V, π) *with* $V \in \mathscr{V}^\circ$ *and* π *an* $\mathrm{H}^V(\mathbf{A})$*-distinguished cuspidal automorphic representation* π *of* $\mathrm{G}^V(\mathbf{A})$ *over* $\mathbf{C}$, *trivial at infinity and tempered everywhere, such that* $\Pi^\infty \otimes \mathbf{1}_\infty \cong \mathrm{BC}^\circ(\pi)$.

[6] That is, tempered at all finite places.

Proof. That (1) implies (2) is the automorphic descent of [GRS11]. Assume (2) holds for the representation π' of $\mathrm{G}^{V'}(\mathbf{A})$, and let $V \in \mathscr{V}^\circ$ agree with V' at all finite places. If $V \in \mathscr{V}^{\circ,+}$, let $V'' = V$ and let $\pi'' = \pi'$; if $V \in \mathscr{V}^{\circ,-}$, let v be an archimedean place of F_0, let $V'' = V(v)$, and let $\pi'' = \pi'^v \otimes \pi_v^\circ$ for any $\pi_v^\circ \in \underline{\pi}_{\mathbf{R}}^\circ = (2.4.2)$, a representation of $\mathrm{G}^{V''}(\mathbf{A})$. Then $\Pi_v = \mathrm{BC}(\pi_v'')$ for all v, so that by [LTX$^+$22, Proposition C.3.1.1 (1)], π'' is automorphic with base-change Π. Let $\pi = \pi''^\infty \otimes \mathbf{1}_\infty$, which is a representation of $\mathrm{G}^V(\mathbf{A})$ over $\mathbf{C}$ trivial at infinity. Then by definition, $\mathrm{BC}(\pi) = \Pi^\infty \otimes \mathbf{1}_\infty$, so that (3) holds. The implication (3) $\Rightarrow$ (1) follows from [Mok15, KMSW] together with the special cases of base-change for real groups stated in § 2.4.1 (4)-(5); a simpler alternative proof, when Π is supercuspidal at some split place, is given in [BPLZZ21, Theorem 4.12 (2)].

Suppose now that (1) and (3) hold. By Proposition 2.4.1 and [LTX$^+$22, Proposition C.3.1.1 (1)], we can modify the pair (V, π) of part (3) locally at finitely many places so that the resulting representation satisfies the properties of (4). $\square$

Corollary 2.5.6. *There is a sub-ind-scheme*

$$\mathscr{C}(\mathrm{G}')^{\mathrm{her}} \quad \subset \quad \mathscr{C}(\mathrm{G}')$$

parametrising those trivial-weight cuspidal automorphic representation of Π *of* $\mathrm{G}'(\mathbf{A})$ *that are hermitian. Moreover, the base-change map gives an isomorphism of* $\mathbf{Q}$*-ind-schemes*

$$\mathrm{BC}\colon \coprod_{V \in \mathscr{V}^\circ} \mathscr{C}(\mathrm{H}^V \backslash \mathrm{G}^V)^{\mathrm{st}} \longrightarrow \mathscr{C}(\mathrm{G}')^{\mathrm{her}}. \tag{2.5.1}$$

Proof. This follows from the equivalence (1) $\Leftrightarrow$ (4) in Proposition 2.5.5. $\square$

Remark 2.5.7. For $\epsilon \in \{\pm\}$, let $\mathscr{C}(\mathrm{G}')^{\mathrm{her},\epsilon} \subset \mathscr{C}(\mathrm{G}')^{\mathrm{her}}$ be the subset of those representations with $\epsilon(\Pi) \coloneqq \varepsilon(\Pi_n \times \Pi_{n+1}, 1/2) = \epsilon$. By [GGP12, § 26, discussion of Question (1)], we have

$$\mathscr{C}(\mathrm{G}')^{\mathrm{her},\epsilon} = \mathrm{BC}\left(\coprod_{V \in \mathscr{V}^{\circ,\epsilon}} \mathscr{C}(\mathrm{H}^V \backslash \mathrm{G}^V)^{\mathrm{st}}\right).$$

Remark 2.5.8. Similarly to the above, for a characteristic-zero field L we may define the notions of *discrete* (rather than tempered cuspidal), trivial-at-infinity automorphic representation of $\mathrm{G}^V(\mathbf{A})$ over L, and of *isobaric* (rather than cuspidal) trivial-weight automoprhic representation of $\mathrm{G}'(\mathbf{A})$ over L; Proposition 2.5.2 remains true with 'cuspidal' replaced by 'isobaric'. Denote the corresponding sets of isomorphism classes by $\mathscr{C}^\sharp(\mathrm{G}^V)(L)$, $\mathscr{C}^\sharp(\mathrm{G}')(L)$. By the variant of Shin's result in [Gol14, Theorem A.1] stated in [LTX$^+$22, Proposition 3.2.8], we have a base-change map $\mathrm{BC}\colon \mathscr{C}^\sharp(\mathrm{G}^V)(L) \to \mathscr{C}^\sharp(\mathrm{G}')(L)$.

2.6. **Relative traces.**

Definition 2.6.1. Let L be a normed field. Suppose given data $D = (\Pi_1, \Pi_2; \vartheta, \beta, T)$ consisting of:

- L-vector spaces Π_1, Π_2;
- a bilinear form $\vartheta\colon \Pi_1 \otimes \Pi_2 \to L$;
- a bilinear form $\beta\colon \Pi_1 \otimes \Pi_2 \to \Gamma$, where Γ is a finite-dimensional L-vector space;

– a map $T\colon \Pi_1 \to \Pi_1$,

satisfying:

– for $i = 1, 2$ we can write $\Pi_i = \varinjlim_{\lambda\in\Lambda} \Pi_{i,\lambda}$ as a filtered direct limit of finite-dimensional L-vector spaces and injective maps, in such a way that:
– for every $\lambda \in \Lambda$, $\vartheta_{|\Pi_{1,\lambda}\otimes\Pi_{2,\lambda}}$ is a perfect pairing.

Let us say that a basis $\{\phi\}$ of Π_1 is *admissible* if there is a presentation $\Pi_1 = \varinjlim_{\lambda\in\Lambda} \Pi_{1,\lambda}$ with the above properties, such that $\{\phi\} \cap \Pi_{1,\lambda}$ is a basis of $\Pi_{1,\lambda}$ for all $\lambda \in \Lambda$; if this is the case we denote by $\{\phi^\vee\}$ the basis of Π_2 whose restriction to $\Pi_{2,\lambda}$ is the ϑ-dual basis of $\{\phi\} \cap \Pi_{1,\lambda}$.

We define the *trace of T relative to β, ϑ* to be

$$\mathrm{Tr}^\beta_\vartheta(T) := \sum_\phi \beta(T\phi, \phi^\vee), \tag{2.6.1}$$

provided the sum is absolutely convergent and is independent of the choice of an admissible basis $\{\phi\}$ of Π_1.

Remark 2.6.2. If $\Gamma = L$ and $\beta = \vartheta$, we recover the usual notion of trace. In the examples of interest to us:

– when L is not $\mathbf{C}$, the sum (2.6.1) will have only finitely many nonzero terms;
– we will have $\beta = h \circ (P_1 \boxtimes P_2)$ for some linear functionals $P_i\colon \Pi_i \to S_i$ valued in an L-vector space S_i, and some bilinear form $h\colon S_1 \otimes S_2 \to \Gamma$. (In fact, in the first part of the paper we will only consider $S_1 = S_2 = \Gamma = L$, and h equal to the multiplication map.)

2.6.1. *Relations between different relative traces.* We give a preliminary definition. In the situation of Definition 2.6.1, let $\alpha_2 \in \mathrm{End}_L(\Pi_2)$. Let $\mu\colon \Lambda \to \Lambda$ be a strictly increasing function with cofinal image such that $\alpha_2(\Pi_{2,\lambda}) \subset \Pi_{2,\mu(\lambda)}$. We define the *$\vartheta$-transpose* of α_2 to be the unique $\alpha_2^\vartheta \in \mathrm{End}_L(\Pi_1)$ whose restriction to $\Pi_{1,\mu(\lambda)}$ is the transpose of $\alpha_{2|\Pi_{2,\lambda}}$ for the restriction of ϑ.

Lemma 2.6.3. *Let $D = (\Pi_1, \Pi_2; \vartheta, \beta, T)$ and $D' = (\Pi_1', \Pi_2'; \vartheta', \beta', T')$ be data as in Definition 2.6.1. In each of the following, suppose that all the data in D, D' are equal except for the indicated differences.*

(1) Suppose that $\beta' = \beta \circ (1 \boxtimes \alpha_2)$ for some $\alpha_2 \in \mathrm{End}_L(\Pi_2)$. Then

$$\mathrm{Tr}^\beta_\vartheta(T) = \mathrm{Tr}^{\beta'}_{\vartheta'}(T'\alpha_2^\vartheta),$$

where $\alpha_2^\vartheta \in \mathrm{End}_L(\Pi_1)$ is the ϑ-transpose of α_2.

(2) Suppose that $\vartheta' = \vartheta \circ (\alpha_1 \boxtimes \mathrm{id})$ and $T' = T\alpha_1$ for some L-isomorphism $\alpha_1 : \Pi_1' \to \Pi_1$. Then

$$\mathrm{Tr}^\beta_\vartheta(T) = \mathrm{Tr}^{\beta'}_{\vartheta'}(T').$$

(3) Suppose that $\Pi_i' \subset \Pi_i$ are all direct summands, that $\vartheta' := \vartheta_{|\Pi_1'\otimes\Pi_2'}$ is a perfect pairing (in the sense that it satisfies the condition of Definition 2.6.1), and that $T(\Pi_1) \subset \Pi_1'$. If $\beta' = \beta_{|\Pi_1'\otimes\Pi_2'}$ and $T' = T_{|\Pi_1'}$, then

$$\mathrm{Tr}^\beta_\vartheta(T) = \mathrm{Tr}^{\beta'}_{\vartheta'}(T').$$

The proof is elementary linear algebra.

Part 1. p-adic L-functions and the analytic relative-trace formula

We study Rankin–Selberg L-functions and the related Jacquet–Rallis relative-trace formulas in a sequence of contexts. In § 3, we review the theory in complex coefficients. In § 4, we construct a Jacquet–Rallis RTF in rational coefficients and at the same time prove Theorem A on the rationality of twisted Rankin–Selberg L-values. The construction relies in particular on the existence of suitable Gaussians, obtained from a refinement of the results of [BPLZZ21]. In § 7, we construct an RTF in p-adic coefficients and at the same time prove Theorem B on the existence of p-adic L-functions. The construction relies on some local theory and in particular on a suitable family of explicit test Hecke measures at p-adic places: the theory is developed on the spectral side in § 5 (whose centerpiece is an explicit calculation of Liu–Sun) and on the geometric side in § 6 (whose centerpiece is a new explicit calculation of orbital integrals for the aforementioned test Hecke measures).

3. Jacquet–Rallis relative-trace formulas

We consider the traces of Hecke operators relative to two period functionals and the Petersson inner product on automorphic forms for G', and compare (the resulting local terms) with a parallel relative-trace distribution for G. The substance of this section is not new, rather it recalls some related work done by previous authors, particularly [JR11, Zha14b, BPLZZ21]. We omit detailed discussions of convergence issues, for which we refer to [Zha14b] or [BP21, Appendix A].

3.1. **Period functionals and the distribution.** Let $\mathscr{A}(\mathrm{G}')$ be the space of automorphic forms on $\mathrm{G}'(\mathbf{A})$, and let $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')$ be its cuspidal subspace. We endow $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')$ with the bilinear Petersson product

$$\vartheta(\phi_1,\phi_2) \coloneqq \int_{[\mathrm{G}']} \phi_1(g)\phi_2(g)\,dg.$$

3.1.1. *Period functionals.* We define two functionals on $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A}))$.

For $\chi \in Y(\mathbf{C})$, the (χ-twisted) *Rankin–Selberg period* is the functional

$$P_{1,\chi}(\phi) \coloneqq \int_{[\mathrm{H}'_1]} \phi(h_1)\chi(h_1)\,dh_1.$$

where $\chi(h_1) \coloneqq \chi(N_{F/F_0}\det h_1)$,

The *Flicker–Rallis period* is the functional

$$P_2(\phi) \coloneqq \int_{[\mathrm{H}'_2]} \phi(h_2)\eta(h_2)\,dh_2,$$

where $\eta(h_2) \coloneqq \eta(\det(h_n)^{n+1}\det(h_{n+1})^n)$ if $h_2 = ([h_n],[h_{n+1}])$.

3.1.2. *Relative-trace distribution.* We say that $f' \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$ is *quasicuspidal* if $R(f')$ sends $\mathscr{A}(\mathrm{G}')$ to $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')$ (cf. [BPLZZ21, Definition 3.2]), and we denote by $\mathscr{H}(\mathrm{G}'(\mathbf{A}))_{\mathrm{qc}}$ the space of quasicuspidal Hecke measures.

Definition 3.1.1. We define a relative-trace distribution on $\mathscr{H}(\mathrm{G}(\mathbf{A}))_{\mathrm{qc}} \times Y(\mathbf{C})$ by

$$I(f', \chi) \coloneqq C \cdot \mathrm{Tr}_{\vartheta}^{P_{1,\chi} \otimes P_2}(R(f')),$$

where the constant

$$C \coloneqq \frac{\Delta_{\mathrm{G}}}{\Delta_{\mathrm{H}}^2} \frac{\Delta_{\mathrm{H}_1'} \Delta_{\mathrm{H}_2'}}{\Delta_{\mathrm{G}'}} \tag{3.1.1}$$

is motivated by rationality considerations.

We note that the above definition does fit within the setup of Definition 2.6.1: we may write

$$\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}') = \varinjlim_{(K,\mathfrak{a})} \mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')^{K,\mathfrak{a}=0}$$

as K varies among compact open subgroups of $\mathrm{G}'(\mathbf{A}^\infty)$ and $\mathfrak{a}$ among finite-codimension ideals in the center of the universal enveloping algebra of $\mathrm{Lie}\, G'_\infty$. The relative trace is well-defined by (the proof of) [Zha14a, Theorem 2.3]. (See also [BPCZ22, Proposition 2.8.4.1] for a more general result in a framework similar to ours.)

In the next two subsections we discuss the two expansions of I: a spectral expansion, in terms of automorphic representations, and a geometric expansion, in terms of orbits (double-cosets).

3.2. **Spectral expansion.** Let Π be a cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$, which by multiplicity one we may and do identify with a subspace of $\mathscr{A}_{\mathrm{cusp}}(\mathrm{G}')$. We define a distribution on $\mathscr{H}(\mathrm{G}'(\mathbf{A}))$ by

$$I_\Pi(f', \chi) \coloneqq C \cdot \mathrm{Tr}_{\vartheta_\Pi}^{P_{1,\Pi,\chi} \otimes P_{2,\Pi^\vee}}(\Pi(f')),$$

where we use subscripts to indicate the restriction of period functionals and Petersson product to Π, $\Pi^\vee$, $\Pi \otimes \Pi^\vee$.

We define some local periods, in order to factorize I_Π.

3.2.1. *Whittaker models and rational structures.* Let $\psi\colon F_0\backslash \mathbf{A} \to \mathbf{C}^\times$ be a nontrivial character, and let

$$\psi_F \coloneqq \psi(\frac{1}{2}\mathrm{Tr}_{F/F_0}(\cdot))\colon F\backslash \mathbf{A}_F \longrightarrow \mathbf{C}^\times.$$

We inflate ψ_F to a character of $\mathrm{N}_n(\mathbf{A}_{F_0})$ by $\psi_{F,n}(u) = \psi_F(\sum_{i=1}^{n-1} u_{i,i+1})$. Let Π_ν be an automorphic representation of $\mathrm{G}_\nu(\mathbf{A})$. Its ψ-Whittaker model $\mathscr{W}_\psi(\Pi_\nu)$ is the image of the map

$$\begin{aligned} \mathscr{W}: \Pi &\longrightarrow C^\infty(\mathrm{N}_\nu(\mathbf{A})\backslash \mathrm{GL}_\nu(\mathbf{A}), \psi_{F,\nu}) \\ \phi &\longmapsto W_\phi(g) \coloneqq \int_{\mathrm{N}_\nu(\mathbf{A})} \phi(ug)\overline{\psi}_{F,\nu}(u)\, du. \end{aligned} \tag{3.2.1}$$

The ψ-Whittaker model of $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ is $\mathscr{W}_\psi(\Pi) = \mathscr{W}_{\overline{\psi}}(\Pi_n) \boxtimes \mathscr{W}_\psi(\Pi_{n+1})$; it has a $\mathrm{G}'(\mathbf{A})$-factorization $\mathscr{W}_\psi(\Pi) = \bigotimes_v \mathscr{W}_{\psi_v}(\Pi_v)$.

We now consider rational structures, along the lines of [RS08, § 3.2]. Let v be a finite place of F_0 with underlying rational prime ℓ, and suppose that Π_v is a smooth irreducible admissible representation of G'_v over a subfield $L \subset \mathbf{C}$. For $\sigma \in \mathrm{Aut}(\mathbf{C}/L)$, let $a_\sigma \in \mathbf{Z}_\ell^\times$ be its image under the composition

$$\mathrm{Aut}(\mathbf{C}/L) \longrightarrow \mathrm{Gal}(L(\mu_{\ell^\infty})/L) \longrightarrow \mathbf{Z}_\ell^\times$$

of the restriction and the cyclotomic character. Let $t_{\sigma,\nu} := \mathrm{diag}(a_\sigma^{\nu-1},\ldots,1)$ and let $t_\sigma := (t_{\sigma,n}, t_{\sigma,n+1}) \in G'_v$. Then we may define an action of $\mathrm{Aut}(\mathbf{C}/L)$ on $\mathscr{W}_{\psi_v}(\Pi_v \otimes_L \mathbf{C})$ by

$$W^\sigma(g) := \sigma(W(t_\sigma^{-1} g)); \tag{3.2.2}$$

we will denote by $\mathscr{W}_{\psi_v}(\Pi_v)$ the space of $\mathrm{Aut}(\mathbf{C}/L)$-invariants; it is an $L[G'_v]$-module satisfying $\mathscr{W}_{\psi_v}(\Pi_v)\otimes_L \mathbf{C} \cong \mathscr{W}_{\psi_v}(\Pi_v\otimes_L \mathbf{C})$ (see [RS08, Lemma 3.2]).

3.2.2. *Factorizations of the periods and Petersson product.* For the following factorization results, see [Zha14b, § 3] and references therein. Let $\epsilon'_\nu(\tau) := \mathrm{diag}(\tau^{\nu+\epsilon-1}, \tau^{\nu+\epsilon-2}, \ldots, \tau^{\epsilon-1}) \in \mathrm{GL}_\nu(F_v)$, where $\epsilon \in \{0,1\}$ has the same parity as ν.

For $W = W_n \otimes W_{n+1} \in \mathscr{W}_{\psi,v}(\Pi_v)$, define[7]

$$\begin{aligned}
P_{1,\Pi_v,\chi_v}(W) &:= \frac{\varepsilon(\frac12,\chi_v^2,\psi_v)^{\binom{n+1}{2}}}{L(1/2,\Pi_v\otimes\chi_v)} \int_{\mathrm{N}_n(F_v)\backslash \mathrm{GL}_n(F_v)} W(j_1(h_1))\chi_v(h_1)\, d^\natural h_1,\\
P_{2,\Pi_v}(W) &:= \frac{\varepsilon(\frac12,\eta_v,\psi_v)^{\binom{n+1}{2}}}{L(1,\Pi_v,\mathrm{As}^{-\star})}\, P^\sharp_{2,\Pi_{n,v}}(W_n) P^\sharp_{2,\Pi_{n+1,v}}(W_{n+1}),\\
P^\sharp_{2,\Pi_{\nu,v}}(W_\nu) &:= \int_{\mathrm{N}_{\nu-1}(F_{0,v})\backslash \mathrm{GL}_{\nu-1}(F_{0,v})} W_\nu\left(\begin{pmatrix} \epsilon'_{\nu-1}(\tau) h_{2,\nu-1} & \\ & 1\end{pmatrix}\right) \eta_v(\det h_{2,\nu-1})^{\nu-1}\, d^\natural h_{2,\nu-1}.
\end{aligned} \tag{3.2.3}$$

where $L(1,\Pi_v,\mathrm{As}^{-\star}) = \prod_{\nu=n}^{n+1} L(1,\Pi_{\nu,v},\mathrm{As}^{(-1)^{\nu-1}})$.

For $W\in \mathscr{W}_{\psi_v}(\Pi_v)$, $W^\vee \in \mathscr{W}_{\overline{\psi}_v}(\Pi_v^\vee)$, define

$$\vartheta_{\Pi_v}(W,W^\vee) := L(1,\Pi_v\times\Pi_v^\vee)^{-1} \prod_{\nu=n}^{n+1} \int_{\mathrm{N}_{\nu-1}(F)\backslash \mathrm{GL}_{\nu-1}(F)} W_\nu\left(\begin{pmatrix} g_{\nu-1} & \\ & 1\end{pmatrix}\right) W_\nu^\vee\left(\begin{pmatrix} g_{\nu-1} & \\ & 1\end{pmatrix}\right) d^\natural g_{\nu-1}.$$

Remark 3.2.1. With our normalizations, when all the data are unramified and $W(1) = W^\vee(1) = 1$, we have

$$P_{1,\Pi_v,\chi_v}(W) = P_{2,\Pi_v}(W) = \vartheta_{\Pi_v}(W,W^\vee) = 1.$$

Moreover, the three functionals are rational in the following sense. If Π_v is defined over a subfield $L\subset \mathbf{C}$, by (3.2.2) and a change of variable we see that for every $\sigma\in\mathrm{Aut}(\mathbf{C}/L)$ we have

$$P_{1,\Pi_v,\chi_v^\sigma}(W^\sigma) = \sigma P_{1,\Pi_v,\chi_v}(W), \qquad P_{2,\Pi_v}(W^\sigma) = \sigma P_{2,\Pi_v}(W), \qquad \vartheta_{\Pi_v}(W^\sigma, W^{\vee,\sigma}) = \sigma\vartheta_{\Pi_v}(W,W^\vee).$$

We may now state the factorization (see [Zha14b, §3] and § 3.2.4 below): for any $\phi\in\Pi$ with factorizable ψ-Whittaker function $W = \otimes_v W_v \in \mathscr{W}_\psi(\Pi)$, and $\phi^\vee\in\Pi^\vee$ with factorizable

[7]Our definition of $P_{2,v}$ differs form the one of [Zha14b] by the factor $\varepsilon(\frac12,\eta,\psi)^{\binom{n+1}{2}}$, cf. § 3.5.1 below.

$\overline{\psi}$-Whittaker function $W^\vee = \otimes_v W_v^\vee \in \mathscr{W}_{\overline{\psi}}(\Pi^\vee)$, we have

$$\begin{aligned} P_{1,\Pi,\chi}(\phi) &= \frac{L(1/2, \Pi\otimes\chi)}{\Delta_{\mathrm{H}'_1}\cdot \varepsilon(\frac12, \chi^2, \psi)^{\binom{n+1}{2}}} \prod_v P_{1,\Pi_v,\chi_v}(W_v) \\ P_{2,\Pi}(\phi^\vee) &= \frac{n(n+1)L^*(1,\Pi, \mathrm{As}^{-\star})}{\Delta_{\mathrm{H}'_2}} \prod_v P_{2,\Pi_v}(W_v^\vee) \\ \vartheta_\Pi(\phi,\phi^\vee) &= \frac{4n(n+1)L^*(1,\Pi\times\Pi^\vee)}{\Delta_{\mathrm{G}'}} \prod_v \vartheta_{\Pi_v}(W_v, W_v^\vee), \end{aligned} \tag{3.2.4}$$

where 4 is the Tamagawa number of G'. In the factorization of P_2, we have used that $\varepsilon(\frac12,\eta,\psi) = 1$.

3.2.3. *Local spherical character.* We define a character

$$I_{\Pi_v}(f'_v,\chi_v) = I_{\Pi_v}(f'_v,\chi_v,\psi_v) = \mathrm{Tr}^{P_{1,\Pi_v,\chi_v}\otimes P_{2,\Pi_v^\vee}}_{\vartheta_{\Pi_v}}(R(f'_v))$$

on $\mathscr{H}(\mathrm{G}'_v)$. This is the same as in [Zha14b], except for Petersson inner product on G rather than on G^{ad}, and our normalization of measures (so we have the same factors $\Delta_{\mathrm{H}'^{\mathrm{ad}}_{i,v}}$).

3.2.4. *Comparison with the normalization of* [Zha14b]. Let

$$\widetilde{\mathrm{G}}' \coloneqq \mathrm{G}'_n\times \mathrm{G}'_{n+1},$$

and let us identify representations of G'_v with representations of $\widetilde{G}'_v$ whose central character is trivial on $(F_{0,v}^\times)^2$. In [Zha14b], one considers a distribution $\widetilde{I}_{\Pi_v}$ on $\mathcal{S}(\widetilde{G}'_v)$ (denoted there by $I^\natural_{\Pi_v}$), and a global distribution $\widetilde{I}_\Pi$ on $\mathcal{S}(\widetilde{\mathrm{G}}'(\mathbf{A}))$ (denoted there by I_Π).[8] If $f=\otimes_v f_v\in\mathscr{H}(G'_v)$ and $\dot f = \otimes_v \dot f_v\in\mathcal{S}(\widetilde{G}'_v)$ are related by (3.3.14), then

$$\begin{aligned} I_{\Pi_v}(f_v,\chi_v) &= \frac{\Delta_{\mathrm{H}_1'^{\mathrm{ad}},v}\Delta_{\mathrm{H}_2'^{\mathrm{ad}},v}}{\Delta_{\mathrm{G}^{\mathrm{ad}},v}}\cdot\left(\varepsilon(\tfrac12,\eta_v,\psi_v)\varepsilon(\tfrac12,\chi_v^2,\psi_v)\right)^{\binom{n+1}{2}}\cdot \widetilde{I}_{\Pi_v}(\dot f_v,\chi_v), \\ I_\Pi(f,\chi) &= \frac{1}{\mathrm{vol}([\mathrm{Z}_{\mathrm{G}'}], d^*z)}\frac{\zeta^*_{\mathrm{G}'}(1)}{\zeta^*_{\mathrm{H}'_1}(1)\zeta^*_{\mathrm{H}'_2}(1)}\,\widetilde{I}_\Pi(\dot f,\chi) = \frac14\frac{1}{\zeta^*_{\mathrm{H}'_1}(1)\zeta^*_{\mathrm{H}'_2}(1)}\,\widetilde{I}_\Pi(\dot f,\chi), \end{aligned} \tag{3.2.5}$$

where the factor $\mathrm{vol}([\mathrm{Z}_{\mathrm{G}'}], d^*z) = 4L(1,\eta)^2 = 4\zeta^*_{\mathrm{G}'}(1)$ accounts for the fact that the Petersson product in [Zha14b] is defined via integration on $[\mathrm{G}'^{\mathrm{ad}}] = [\widetilde{\mathrm{G}}'^{\mathrm{ad}}]$ and not $[\mathrm{G}']$.

3.2.5. *Factorization of the spherical character.* Define

$$\mathscr{L}(1/2,\Pi,\chi)\coloneqq \frac{\Delta_{\mathrm{G}}}{\Delta_{\mathrm{H}}}\frac{L(1/2,\Pi\otimes\chi)}{L(1,\Pi,\mathrm{As}^\star)},$$

which agrees with the definition made in the introduction as noted in § 2.2.1. We use the analogous notation relative to the constituents Π_v, χ_v for v a finite place of F_0 or $v=\infty$.

[8]Strictly speaking only $\chi_v = \mathbf{1}_v$ is considered in [Zha14b], but the definition remains valid in our more general case too. When this is again the case in the rest of the paper, we will simply cite [Zha14b] without repeating this remark.

Proposition 3.2.2. *For all $f' = \otimes_v f'_v \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$, there is a factorization*

$$I_\Pi(f',\chi) = \frac{1}{4}\frac{\mathscr{L}(1/2,\Pi,\chi)}{\Delta_{\mathrm{H}}\cdot\varepsilon(\frac{1}{2},\chi^2)^{\binom{n+1}{2}}}\prod_v I_{\Pi,v}(f'_v,\chi_v)$$

Proof. Using (3.2.5), the factorization in [Zha14b, Proposition 3.6] is equivalent to

$$I_\Pi(f',\chi) = C\frac{1}{4}\frac{\Delta_{\mathrm{G}'}}{\Delta_{\mathrm{H}'_1}\Delta_{\mathrm{H}'_2}}\cdot\varepsilon(\tfrac{1}{2},\chi^2)^{-\binom{n+1}{2}}\cdot\frac{L(1/2,\Pi\otimes\chi)}{L(1,\Pi,\mathrm{As}^\star)}\cdot\prod_v I_{\Pi_v}(f'_v,\chi_v).$$

By the definition of C in (3.1.1) and of $\mathscr{L}$, this is equivalent to the asserted formula. (Equivalently, the factorization follows from (3.2.4).) $\square$

We will state the spectral expansion $I = \sum_\Pi I_\Pi$ as part of Proposition 3.3.6 below.

3.3. **Geometric expansion.** The distribution I also admits an expansion as a sum of orbital integrals, which we review.

3.3.1. *Orbit varieties.* Let

$$\mathrm{B}' \coloneqq \mathrm{H}'_1\backslash\mathrm{G}'/\mathrm{H}'_2$$

be the categorical quotient, which is an affine variety over F_0, cf. [Zha14a]. Let

$$\mathrm{S} \coloneqq \{\gamma\in\mathrm{G}'_{n+1} \ \mid \gamma\overline{\gamma} = 1_n\}.$$

The maps

$$g=(g_n,g_{n+1})\longmapsto g_\star \coloneqq g_n^{-1}g_{n+1}, \qquad g_{n+1}\longmapsto g_{n+1}g_{n+1}^{\mathrm{c},-1} \tag{3.3.1}$$

induce maps and isomorphisms

$$s\colon \widetilde{\mathrm{G}}'\longrightarrow \mathrm{G}'_{n+1}/\mathrm{G}'_{n+1,0}\cong\mathrm{S}, \qquad \mathrm{B}'\cong\mathrm{G}'_{n,0}\backslash\mathrm{G}'_{n+1}/\mathrm{G}'_{n+1,0}\cong\mathrm{G}'_{n,0}\backslash\mathrm{S}.$$

The second map in (3.3.1) also yields a bijection on F'-points $\mathrm{G}'_{n+1}(F')/\mathrm{G}'_{n+1,0}(F')\to\mathrm{S}(F')$ for any field $F'\supset F_0$.

Representing a point of B' by a matrix in S given in $(n,1)\times(n,1)$ block decomposition, the invariant map

$$\begin{aligned}\mathrm{inv}\colon \mathrm{B}' &\longrightarrow \mathrm{Res}_{F/F_0}\mathbb{A}^{2n+1}\\ \begin{pmatrix} A & b\\ c & d\end{pmatrix} &\longmapsto ((\mathrm{Tr}(\wedge^i A))_{i=1}^n, (cA^{j-1}b)_{j=1}^n, d)\end{aligned} \tag{3.3.2}$$

gives an embedding into affine space.

3.3.2. *Regular, plus-regular, and semisimple orbits.* We define three (quasi-)invariant functions on S (hence on $\widetilde{\mathrm{G}}'$) by

$$\begin{aligned} D^+(s) &\coloneqq \det(e_{n+1}^{\mathrm{t}}, e_{n+1}^{\mathrm{t}}s,\ldots,e_{n+1}^{\mathrm{t}}s^n)\\ D^-(s) &\coloneqq \det(e_{n+1}, se_{n+1},\ldots,s^ne_{n+1})\\ D(s) &\coloneqq \det((e_{n+1}^{\mathrm{t}}s^{i+j}e_{n+1})_{0\le i,j\le n}) = D^+D^-(s),\end{aligned} \tag{3.3.3}$$

where $e_{n+1} = (0, \dots, 0, 1)^{\mathrm{t}} \in F^{n+1}$. We denote by $\widetilde{\mathrm{G}}'_{\mathrm{rs}} \subset \widetilde{\mathrm{G}}'_{\mathrm{reg}^\pm} \subset \widetilde{\mathrm{G}}'$ the open subschemes defined, respectively, by $D \neq 0$ and $D^\pm \neq 0$, and by

$$\mathrm{G}'_{\mathrm{rs}} \subset \mathrm{G}'_{\mathrm{reg}^\pm} \subset \mathrm{G}'$$

the respective images in G'; thus $\mathrm{G}'_{\mathrm{rs}} = \mathrm{G}'_{\mathrm{reg}^+} \cap \mathrm{G}'_{\mathrm{reg}^-}$. The function D descends to B' and we denote by $\mathrm{B}'_{\mathrm{rs}}$ its non-vanishing locus, whose preimage in G' is $\mathrm{G}'_{\mathrm{rs}}$.

Remark 3.3.1. The involution $g^\diamond := g^{\mathrm{c},-1,\mathrm{t}}$ on $\widetilde{\mathrm{G}}'$ satisfies $D^\pm(s(g^\diamond)) = D^\mp(s(g))$, and it descends to G'; in particular, it swaps $\mathrm{G}'_{\mathrm{reg}^+}$ and $\mathrm{G}'_{\mathrm{reg}^-}$.

Let $F' \supset F_0$ be a field. An $\mathrm{H}'_1(F') \times \mathrm{H}'_2(F')$-orbit in $\mathrm{G}'(F')$ is said to be *regular* if its stabilizer is trivial; *semisimple* if the orbit is Zariski-closed.

The regular semisimple orbits in $\mathrm{G}'(F')$ are in bijection with $\mathrm{B}'_{\mathrm{rs}}(F')$, and the preimage in $\mathrm{G}'(F')$ of any $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F')$ consists of a single orbit. The preimage of a general $\gamma \in \mathrm{B}'(F')$ contains finitely many regular orbits (but possibly infinitely many orbits), of which exactly one belongs to $\mathrm{G}'_{\mathrm{reg}^+}(F')$ and exactly one belongs to $\mathrm{G}'_{\mathrm{reg}^-}(F')$ (these two coincide precisely when $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F')$). We will call the elements in $\mathrm{G}'_{\mathrm{reg}^+}$ (respectively $\mathrm{G}'_{\mathrm{reg}^-}$) *plus-regular* (respectively *minus-regular*). We refer to [Lu, § 2.4] for more details.

3.3.3. *Local orbital integrals.* Let v a place of F_0, and let $\gamma' \in G'_{\mathrm{reg}^\pm, v}$. Then for all $\chi_v \colon F_{0,v}^\times \to \mathbf{C}^\times$, we define the orbital integral

$$I^\natural_{\gamma'}(f'_v, \chi_v) := \int_{H'_{1,v}} \int_{H'_{2,v}} f'_v(h_{1,v}^{-1}\gamma' h_{2,v})\chi_v(h_{1,v})\eta(h_{2,v})\, \frac{d^\natural h_{1,v} d^\natural h_{2,v}}{d^\natural g_v}, \tag{3.3.4}$$

where we recall that $f'_v/d^\natural g_v$ is a function. If $\gamma' \in G'_{\mathrm{rs},v}$ or f'_v is supported in the regular locus of G'_v, the integral is absolutely convergent. In general, Lu proved that the integral is convergent when χ is the product of a unitary character and $|\cdot|^s$ for $\pm\mathrm{Re}(s) < -1$ [Lu, Lemma 5.14], and gave the following regularization.

Proposition 3.3.2 ([Lu, Prop. 5.12])**.** *Let $\mathscr{R}^\pm_{v,0}$ be the set of functions, on the space of smooth characters of $F_{0,v}^\times$, of the form*

$$\chi_v \longmapsto \prod_{j=1}^m L(\pm 1 \mp j, (\chi_v\eta_v)^{\mp j} \circ N_{F'_0/F_0}), \tag{3.3.5}$$

for varying integers $m \geq 1$ and finite field extensions F'_0/F_0; and let $\mathscr{R}^\pm_v$ be the set of finite products of functions in $\mathscr{R}^\pm_{v,0}$.

Let $\gamma' \in G'_{\mathrm{reg}^\pm, v}$. Then one can define an element

$$L_{\gamma'} \in \mathscr{R}^\pm_v,$$

such that the following hold.

(1) $L_{\gamma'}$ only depends on the $H_{1,v} \times H'_{2,v}$-orbit of γ', and it equals 1 if and only if $\gamma' \in G'_{\mathrm{rs},v}$,

(2) For unramified data (the precise meaning is given in [Lu, Prop. 5.12 (4)]*), we have*

$$I^\natural_{\gamma'}(f'_v, \chi_v) = L_{\gamma'}(\chi_v). \tag{3.3.6}$$

(3) Define a normalized orbital integral by

$$I_{\gamma'}(f'_v, \chi_v) := \frac{I^{\sharp}_{\gamma'}(f'_v, \chi_v)}{L_{\gamma'}(\chi_v)}. \tag{3.3.7}$$

Then for every fixed character χ_v° of $F_{0,v}^\times$:

(a) if v is archimedean, the function $s \mapsto I_{\gamma'}(f'_v, \chi_v^\circ|\cdot|^s)$ extends to an entire function on $\mathbf{C}$;

(b) if v is non-archimedean, the function

$$\chi \longmapsto I_{\gamma'}(f'_v, \chi_v^\circ \chi)$$

on the space of unramified characters of $F_{0,v}^\times$ is a polynomial in $\chi(\varpi_v)$, whose coefficients are rational over the field of rationality of f' and χ_0.

Remark 3.3.3. The work of Lu, [Lu], treats all regular orbits, and all the results of the present paper involving plus-regular orbital integrals could in principle be extended to general regular orbits as well. Nevertheless, for our purposes in the general construction of the p-adic relative-trace formula, we will only need to consider plus-regular orbital integrals. For this reason, we will restrict our attention to plus-regular orbits, which introduces some simplifications. (In fact, for the applications in the proofs of our main theorems, we will not need to consider any regularized divergent orbital integrals; however we consider the more general p-adic relative-trace formula to be of independent interest.)

Let v be a place of F_0, and let $\gamma' \in G'_{\mathrm{reg}^+,v}$. For $h_1 \in H'_{1,v}$, $h'_2 \in H'_{2,v}$, we have $I_{h_1\gamma' h_2}(-, \chi_v) = \chi_v(h_1)\eta_v(h_2) I_{\gamma'}(-, \chi_v)$. We then add a renormalization factor to the orbital integral so that, when $\chi_v = \mathbf{1}$, it only depends on the orbit of γ'. Let $\eta' \colon F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ be a character such that $\eta'_{|\mathbf{A}^\times} = \eta$. With the notation $\gamma'_\star$ and $s = s(\gamma')$ as in (3.3.1), we define a multiple of the invariant $D_v^+(s)$ by

$$\kappa_v(\gamma') := \eta' \left(\det(\gamma'_\star)^\epsilon \det s^{-(n+\epsilon)/2} \det(e^t_{n+1}, e^t_{n+1}s, \ldots, e^t_{n+1}s^n)\right) \tag{3.3.8}$$

where $\epsilon := 0$ if n is even, $\epsilon := 1$ if n is odd. This equals the transfer factor denoted by Ω_v in [Zha14b, (4.12)-(4.13)] (cf. § 3.5.3 below), and it satisfies

$$\kappa_v(h_1\gamma' h_2, \chi_v) = \eta_v(h_2)\kappa_v(\gamma', \chi_v),$$

and

$$\prod_v \kappa_v(\gamma') = 1 \tag{3.3.9}$$

for all $\gamma' \in \mathrm{G}'_{\mathrm{reg}^+}(F_0)$. We also record the following rationality property.

Lemma 3.3.4. *Let $\gamma' \in G'_{v,\mathrm{reg}^+}$. Then $\kappa_v(\gamma')$ is a square root of $\eta_v(-1)^{-\binom{n+1}{2}}$.*

Proof. With notation as above (but dropping the apex from $\gamma'_\star$ for lightness), write $n = 2m + \epsilon$ and let

$$a := \det(e^t_{n+1}, \ldots, e^t_{n+1}s^n) \det s^{-(m+\epsilon)} \det(\gamma_\star)^\epsilon = \det(e^t_{n+1}s^{-m-\epsilon}, \ldots, e^t_{n+1}s^m) \det \gamma_\star^\epsilon,$$

which satisfies $\kappa_v(\gamma', \mathbf{1}) = \eta'_v(a)$. Using $s^{\mathrm{c}} = s^{-1}$ and $\gamma_\star^{\mathrm{c}} = s^{-1}\gamma_\star$, we find

$$a^{\mathrm{c}} = (-1)^{\binom{n+1}{2}} \det(e^t_{n+1}s^{-m-\epsilon}, \ldots, e^t_{n+1}s^m) \det \gamma_\star = (-1)^{\binom{n+1}{2}} a$$

where $(-1)^{\binom{n+1}{2}}$ is the sign of the longest permutation on $n+1$ elements. The assertion of the lemma follows. $\square$

We now define, for $\gamma \in B'_v$,

$$\begin{aligned} L_\gamma(\chi_v) &\coloneqq L_{\gamma'}(\chi_v) \\ I^\sharp_\gamma(f'_v, \chi_v) &\coloneqq \kappa_v(\gamma')\, I^\sharp_{\gamma'}(f'_v, \chi_v) \\ I_\gamma(f'_v, \chi_v) &\coloneqq \kappa_v(\gamma')\, I_{\gamma'}(f'_v, \chi_v). \end{aligned} \tag{3.3.10}$$

for any γ' in the unique plus-regular orbit above γ. When $\chi_v = \mathbf{1}$, it is straightforward to check the right hand side is independent of the choice of γ'; in general, our notation is somewhat abusive, but the ambiguity can be cancelled out in the global context as discussed next.

3.3.4. *Global orbital integrals.* Let $\mathscr{R}_0^+$ be the set of functions on Hecke characters of F_0 of the form

$$\chi \longmapsto \prod_{j=1}^{m} L(1-j, (\chi\eta)^{-j} \circ N_{F'_0/F_0}), \tag{3.3.11}$$

for varying integers $m \geq 1$ and finite field extensions F'_0/F_0, and let $\mathscr{R}^+$ be the set of finite product of functions in $\mathscr{R}_0^+$.

For any $\gamma \in \mathrm{B}'(F_0)$, by [Lu, §6] we can define an element $L_\gamma = \prod_v L_{\gamma_v}$ in $\mathscr{R}$. For any $f' \coloneqq \otimes_v f'_v \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$, and any character $\chi \in Y(\mathbf{C})$, we put

$$I_\gamma(f', \chi) \coloneqq C \frac{\Delta_{\mathrm{G}'}}{\Delta_{\mathrm{H}_1}\Delta_{\mathrm{H}'_2}}\, L_\gamma(\chi) \prod_v I_\gamma(f'_v, \chi_v) = \frac{\Delta_{\mathrm{G}}}{\Delta_{\mathrm{H}}^2} L_\gamma(\chi) \prod_v I_\gamma(f'_v, \chi_v), \tag{3.3.12}$$

where all but finitely many factors equal 1 (the finite set of exceptions depends on γ); we take the orbital integrals in the product to be defined as in (3.3.10) by means of a common *rational* plus-regular lift $\gamma' \in \mathrm{G}'(F_0)$ of γ, which ensures that the product is well-defined. When $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)$, it is clear that we have

$$I_\gamma(f', \chi) \coloneqq C \cdot \int_{\mathrm{H}'_1(\mathbf{A})} \int_{\mathrm{H}'_2(\mathbf{A})} f'(h_1^{-1}\gamma' h_2)\chi(h_1)\eta(h_2)\, \frac{dh_1 dh_2}{dg}.$$

In fact, this last formula makes sense for any locally constant function $\chi\colon F_0^\times \backslash \mathbf{A}^\times \to \mathbf{C}$.

3.3.5. *Comparison with the normalization of* [Zha14b]. In [Zha14b, §4], one considers the distribution on Hecke functions (and not measures) on $\widetilde{G}'_{\mathrm{rs},v}$, defined by

$$\widetilde{I}_{\gamma'}(\dot{f}'_v, \chi_v) \coloneqq \int_{H_{1,v}} \int_{H_{2,v}} \dot{f}'_v(h_{1,v}^{-1}\gamma' h_{2,v})\chi(h_{1,v})\eta(h_{2,v})\, \frac{d^*h_{1,v} d^*h_{2,v}}{d^*g}$$

(and denoted there by $O(\gamma, f')$), and the global analogue

$$\widetilde{I}_{\gamma'}(\dot{f}', \chi) \coloneqq \prod_v \widetilde{I}_{\gamma'}(\dot{f}'_v, \chi_v).$$

Let $\mathrm{p}\colon \widetilde{G}'_v \to G'_v$ be the projection, and let

$$\begin{aligned}\mathrm{p}_*\colon \mathcal{S}(\widetilde{G}'_v) &\longrightarrow \mathcal{S}(G'_v)\\ \dot{f}' &\longmapsto \left(g=[\widetilde{g}]\longmapsto \int_{F_{0,v}^{\times,2}} \dot{f}'(z\widetilde{g})\, d^*z\right).\end{aligned} \tag{3.3.13}$$

Suppose that $f'=\otimes_v f'_v \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$ and $\dot{f}'=\otimes_v \dot{f}'_v \in \mathcal{S}(\widetilde{\mathrm{G}}'(\mathbf{A}))$ are related by

$$f'_v = \mathrm{p}_*(\dot{f}'_v)\, d^*g_v. \tag{3.3.14}$$

Then

$$\begin{aligned}I_\gamma(f'_v,\chi_v) &= \kappa_v(\gamma')\,\frac{\Delta_{\mathrm{H}_1'^{\mathrm{ad}},v}\Delta_{\mathrm{H}_2'^{\mathrm{ad}},v}}{\Delta_{\mathrm{G}'^{\mathrm{ad}},v}}\,\widetilde{I}_{\gamma'}(\dot{f}'_v,\chi_v),\\ I_\gamma(f',\chi) &= C\,\frac{\zeta^*_{\mathrm{G}'}(1)}{\zeta^*_{\mathrm{H}'_1}(1)\zeta^*_{\mathrm{H}'_2}(1)}\prod_v \widetilde{I}_{\gamma'}(\dot{f}'_v,\chi).\end{aligned} \tag{3.3.15}$$

We will also denote by

$$\mathrm{p}_*\colon \mathscr{H}(\widetilde{G}_v) \longrightarrow \mathscr{H}(G_v) \tag{3.3.16}$$

the pushforward map of Hecke measures.

3.3.6. *Relative-trace formula for I.* We describe the spectral and geometric expansions of I. For S a finite set of places of F_0 and $? \in \{\mathrm{rs}, \mathrm{reg}^{\pm}\}$, an $f'^S \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^S))$ is said to have *?-support* if it is in the span of those $\otimes_{v\notin S} f'_v$ such that for some place v, f'_v is supported on $G'_{?,v}$. We introduce a weaker notion.

Definition 3.3.5. We say that $f'^S \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^S))$ has *weakly ?-support* if it belongs to the subspace spanned by those pure tensors $\otimes_{v\notin S} f'_v$ such that for every $\gamma' \in \mathrm{G}'(F_0) - \mathrm{G}'_?(F_0)$, there is some $v\notin S$ such that f'_v vanishes on the $H_{1,v}\times H_{2,v}$-orbit of γ'.

Proposition 3.3.6. *Let $f' \in \mathscr{H}(\mathrm{G}'(\mathbf{A}))$ be quasicuspidal with weakly plus-regular support. Then for every character $\chi \in Y(\mathbf{C})$, we have*

$$\sum_\Pi I_\Pi(f',\chi) = I(f',\chi) = \sum_{\gamma\in \mathrm{B}'(F_0)} I_\gamma(f',\chi). \tag{3.3.17}$$

where both sums are absolutely convergent, the first one running over the cuspidal hermitian automorphic representations of $\mathrm{G}'(\mathbf{A})$.

Proof. For the spectral expansion, see [BPLZZ21, Prop. 4.1] (where it is assumed that $\chi=1$, but the proof extends to the general case). The geometric expansion is [Lu, Theorems 3.1, 6.1]; the definition of the summands in *loc. cit.* contains extra terms corresponding to regular but non-plus-regular orbits, but those vanish by our assumption on f'. □

3.4. **Relative traces for unitary groups.** We review the Jacquet–Rallis RTF for unitary groups.

3.4.1. *Orbit spaces.* Let v be a finite place of F_0 or $v=\infty$, and recall from § 2.1.3 the set $\mathscr{V}_v$ of pairs of hermitian spaces. For $V\in\mathscr{V}_v$, let

$$B_{v,V} \coloneqq H_v^V \backslash G_v^V / H_v^V, \tag{3.4.1}$$

which is isomorphic to the quotient of $U(V_{n+1})$ by the adjoint action of $U(V_n)$ via the map $[(g_n, g_{n+1})] \mapsto [g_n g_{n+1}^{-1}]$. Differently from the linear case, $B_{v,V}$ is a *subset* (open for the v-adic topology if v is non-archimedean) of the set of $F_{0,v}$-points of $\mathrm{B}_v^V := \mathrm{H}_v^V \backslash \mathrm{G}_v^V / \mathrm{H}_v^V$. Similar to § 3.3.2, we say that $g \in G_v^V$ is *regular* (for the $H_v^V \times H_v^V$-action) if its stabilizer is trivial; *semisimple* if its orbit is closed. We denote by $G_{\mathrm{rs},v}^V \subset G_v^V$ the (Zariski-open) subset of regular semisimple elements and by $B_{\mathrm{rs},v}^V$ its image in B_v^V. When $v = \infty$ is archimedean and $V_\infty = V_\infty^\circ$ is the positive definite pair, every orbit is regular semisimple, and we denote $B_\infty^\circ := B_{\infty, V_\infty^\circ}$.

Consider now the global case, and let $V \in \mathscr{V}$. We similarly define $\mathrm{G}_{\mathrm{rs}}^V \subset \mathrm{G}^V$ to be the subgroup-scheme of those g with closed orbit and trivial stabilizers for the $\mathrm{H}^V \times \mathrm{H}^V$-action. For uniformity of notation, we denote by

$$\mathrm{B}_{\mathrm{rs}}(F_0)_V \subset \mathrm{B}_{\mathrm{rs}}^V(F_0) \tag{3.4.2}$$

the image of $\mathrm{G}_{\mathrm{rs}}^V(F_0)$.

3.4.2. *Local distributions.* Let $\delta \in B_{\mathrm{rs},v,V}$ and let $\delta' \in G_{\mathrm{rs},v}^V$ be a preimage of δ. We define a local orbital-integral distribution $J_{\delta,v} = J_{\delta,v}^V$ on the Hecke algebra of $G_v = G_v^V$ by

$$J_{\delta,v}(f_v) := \int_{H_v} \int_{H_v} f_v(x^{-1}\delta' y) \, \frac{d^\natural x d^\natural y}{d^\natural g}. \tag{3.4.3}$$

For π_v a representation of $G_v = G_v^V$, we define a spherical character $J_{\pi_v} = J_{\pi_v}^V$ on $\mathscr{H}(G_v)$ by

$$J_{\pi_v}(f_v) := \mathscr{L}(1/2, \mathrm{BC}(\pi_v))^{-1} \int_{H_v} \mathrm{Tr}_{\pi_v}(\pi_v(h)\pi_v(f)) \, d^\natural h. \tag{3.4.4}$$

By our choices of measures, for all finite v, if f_v is L-valued (for some subfield $L \subset \mathbf{C}$) then so are $J_{\delta,v}(f_v)$, $J_{\pi_v}(f_v)$.

3.4.3. *Comparison with the normalization of* [Zha14b]. Let $f_v \in \mathscr{H}(G_v)$ and $\dot{f}_v \in \mathcal{S}(G_v)$ be related by

$$f_v = \dot{f}_v d^* g. \tag{3.4.5}$$

(1) Let $\widetilde{J}_{\pi_v}$ be the spherical character on $\mathcal{S}(G_v)$ defined in [Zha14b, (1.8)] (using the measure $d^* h_v$ on H_v as in § 4 *ibid.*), and denoted there by $J_{\pi_v}^\natural$. Then

$$J_{\pi_v}(f_v) = D_v^{1/2} \Delta_{\mathrm{H}_v^{\mathrm{ad}}} \, \widetilde{J}_{\pi_v}(\dot{f}_v), \tag{3.4.6}$$

(2) Let $\widetilde{J}_\delta$ be the orbital integral distribution on $\mathcal{S}(G_v)$ defined in [Zha14b, (4.2)], and denoted there by $O(\delta, \cdot)$. Then

$$J_\delta(f_v) = \frac{(\Delta_{\mathrm{H}^{\mathrm{ad}},v})^2}{\Delta_{\mathrm{G}^{\mathrm{ad}},v}} \, \widetilde{J}_\delta(\dot{f}_v). \tag{3.4.7}$$

3.4.4. *Global relative-trace formula.* Let now $V \in \mathscr{V}$ be coherent, and let $\mathrm{G} = \mathrm{G}^V$. Let $\vartheta \colon \mathscr{A}_{\mathrm{cusp}}(\mathrm{G}) \otimes \mathscr{A}_{\mathrm{cusp}}(\mathrm{G}) \to \mathbf{C}$ be the Petersson inner product (with respect to the Tamagawa measure on $[\mathrm{G}]$), and consider the H-period

$$\begin{aligned} P = P^V : \mathscr{A}_{\mathrm{cusp}}(\mathrm{G}) &\longrightarrow \mathbf{C} \\ \phi &\longmapsto \int_{[\mathrm{H}]} \phi(h)\, dh \end{aligned} \tag{3.4.8}$$

We define the following distributions on (subspaces of) $\mathscr{H}(\mathrm{G}(\mathbf{A}))$:

- let $\mathscr{H}(\mathrm{G}(\mathbf{A}))_{\mathrm{qc}} \subset \mathscr{H}(\mathrm{G}(\mathbf{A}))$ be the quasicuspidal subspace (defined as in §3.1.2). For $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}))_{\mathrm{qc}}$, we define
$$J(f) \coloneqq \mathrm{Tr}_{\vartheta}^{P\otimes P}(R(f));$$
- let π be a cuspidal automorphic representation of $\mathrm{G}(\mathbf{A})$. For $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}))$, we define
$$J_\pi(f) \coloneqq \mathrm{Tr}_{\vartheta_\pi}^{P_\pi\otimes P_\pi}(\pi(f));$$
- let $\delta \in \mathrm{B}_{\mathrm{rs}}(F_0)$. For $f = \otimes_v f_v \in \mathscr{H}(\mathrm{G}(\mathbf{A}))$, we define
$$J_\delta(f) \coloneqq \frac{\Delta_{\mathrm{G}}}{(\Delta_{\mathrm{H}})^2} \prod_v J_{\delta,v}(f_v) = \prod_{v\nmid\infty} J_{\delta,v}(f_v)\cdot J^\circ_{\delta,\infty}(f_\infty). \tag{3.4.9}$$

Analogously to Proposition 3.3.6, we have the spectral and geometric expansions ([BP21, Proposition A.2.1])

$$\sum_\pi J_\pi(f) = J(f) = \sum_{\delta\in \mathrm{B}_{\mathrm{rs}}(F_0)} J_\delta(f),$$

valid whenever $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}))$ is quasicuspidal with weakly regular semisimple support (in the analogous sense as to Definition 3.3.5), where the second sum runs over cuspidal representations of $\mathrm{G}(\mathbf{A})$.

However, unlike the factorization

$$I_\Pi(f',\chi) = \frac{1}{4}\frac{\mathscr{L}(1/2,\Pi,\chi)}{\Delta_{\mathrm{H}}\cdot \varepsilon(\frac12,\chi^2)^{\binom{n+1}{2}}} \prod_v I_{\Pi,v}(f'_v,\chi_v)$$

of Proposition 3.2.2, the analogous factorization

$$J_\pi = \frac{1}{4\Delta_{\mathrm{H}}}\mathscr{L}(1/2,\Pi,\mathbf{1}) \prod_v J_{\pi_v}$$

for a stable cuspidal tempered representation π of $\mathrm{G}(\mathbf{A})$ is highly nontrivial, and equivalent to the Ichino–Ikeda conjecture for unitary groups [Zha14b, Conjecture 1.1], whose proof is completed in [BPLZZ21]. The proof, which we briefly review in § 4.6 below (for expository purposes), goes through a comparison of local orbital integrals $I_{\gamma,v}$ and $J_{\delta,v}$ and of local spherical characters I_{Π_v} and I_π. We first review the main results on the local comparison, which are equally important in the arithmetic setting.

3.5. **Comparison of the local distributions.**

3.5.1. *Spectral matching.* Let v be a place of F_0. For $V \in \mathscr{V}_v$ and $\pi_v^V \in \mathrm{Temp}(G_v^V)$, define a spectral transfer factor

$$\kappa(\pi_v^V) = \kappa(\pi_v^V,\psi_v,\tau) \coloneqq \eta'_v((-1)^{n+1}\tau)^{\binom{n+1}{2}}\cdot \eta_v(\mathrm{disc}(V_n))^n \cdot \omega_{\pi_v^V}(-1); \tag{3.5.1}$$

this is the same as in [Zha14b, Conjecture 4.4] with the correction of [BP21a, Remark 5.52], up to a factor $\varepsilon(\frac{1}{2},\eta_v,\psi)^{\binom{n+1}{2}}$.[9]

Let S be a finite set of places of F_0. Denote $\mathscr{V}_S := \prod_{v\in S}\mathscr{V}_v$; for $V=(V_v)_{v\in S}\in\mathscr{V}_S$, denote $\mathrm{Temp}((H_S^V)\backslash G_S^V)=\prod_{v\in S}\mathrm{Temp}((H_v^{V_v})\backslash G_v^{V_v})$; for $\pi_S^V\in\mathrm{Temp}(G_S^V)$, set $\kappa(\pi_S^V):=\prod_{v\in S}\kappa(\pi_v)$ and $J_{\pi_S}:=\otimes_{v\in S}J_{\pi_v}$. For $\Pi_S\in\mathrm{Temp}(G'_S):=\prod_{v\in S}\mathrm{Temp}(G'_v)$, let $I_{\Pi_S}:=\otimes_{v\in S}I_{\Pi_v}$.

Definition 3.5.1. We say that Hecke measures $f'_S\in\mathscr{H}(G'_S)$ and $(f_S^V)_{V\in\mathscr{V}_S}\in\prod_{V\in\mathscr{V}_S}\mathscr{H}(G_S^V)$ *match spectrally* if for all $V\in\mathscr{V}_S$ and all $\pi_S^V\in\mathrm{Temp}(H_S^V\backslash G_S^V)$, we have

$$I_{\mathrm{BC}(\pi_S^V)}(f'_S,\mathbf{1})=\kappa(\pi_S^V)J_{\pi_S}(f_S^V). \tag{3.5.2}$$

3.5.2. *Geometric matching.* Let us first recall the matching of orbits for G' and G; for the details, see [Zha12, § 2.1]. Let $V\in\mathscr{V}_v$. Orbits $\gamma\in B'_{\mathrm{rs},v}$ and $\delta\in B_{\mathrm{rs},v,V}$ are said to *match* if a lift $\gamma'\in\mathrm{S}_v\subset\mathrm{GL}_{n+1}(F_v)$ of γ and a lift $\delta'\in U(V_{n+1})\subset\mathrm{GL}_{n+1}(F_v)$ of δ are conjugate for the adjoint action of $\mathrm{GL}_n(F_v)$. (This notion is independent of the choices of the lifts and of the basis of V_{n+1} giving the embedding $U(V_{n+1})\subset\mathrm{GL}_{n+1}(F)$.) The matching relation defines a bijection (an isomorphism of $F_{0,v}$-manifolds if v is non-archimedean)

$$\underline{\delta}\colon B'_{\mathrm{rs},v}\cong\bigsqcup_{V\in\mathscr{V}_v}B_{\mathrm{rs},v,V}. \tag{3.5.3}$$

If S is a finite set of places of F_0, by taking products we obtain a matching bijection

$$\underline{\delta}\colon B'_{\mathrm{rs},S}\cong\bigsqcup_{V\in\mathscr{V}_S}B_{\mathrm{rs},S,V}.$$

where $B'_{\mathrm{rs},S}:=\prod_{v\in S}B'_{\mathrm{rs},v}$, $B_{\mathrm{rs},S,V}:=\prod_{v\in S}B_{\mathrm{rs},v,V_v}$.

For the number field F_0 and for $V\in\mathscr{V}$, with the notation of (3.4.2) we have an analogous bijection

$$\underline{\delta}\colon \mathrm{B}'_{\mathrm{rs}}(F_0)\cong\bigsqcup_{V\in\mathscr{V}}\mathrm{B}_{\mathrm{rs}}(F_0)_V. \tag{3.5.4}$$

compatible with (3.5.3)

Definition 3.5.2. Let S be a finite set of places of F_0. We say that Hecke measures $f'_S\in\mathscr{H}(G'_v)$ and $(f_S^V)_V\in\prod_{V\in\mathscr{V}_S}\mathscr{H}(G_S^V)$ *match geometrically* if

$$I_{\gamma,S}(f'_S,\mathbf{1}_S)=J_{\delta,S}(f_S^V) \tag{3.5.5}$$

whenever $\gamma\in B'_{\mathrm{rs},S}$ and $\delta\in B_{\mathrm{rs},S,V}$ match.

3.5.3. *Comparison with the normalization of* [Zha14b]. Let v be a place of F_0. Suppose that f'_v is related to $\dot f'_v$ as in (3.3.14) and f_v is related to $\dot f_v$ as in (3.4.5). Let

$$c_v:=\frac{\Delta_{\mathrm{H}_1'^{\mathrm{ad}},v}\Delta_{\mathrm{H}_2'^{\mathrm{ad}},v}}{\Delta_{\mathrm{G}'^{\mathrm{ad}},v}}\cdot\frac{\Delta_{\mathrm{G}^{\mathrm{ad}},v}}{\Delta^2_{\mathrm{H}^{\mathrm{ad}},v}}. \tag{3.5.6}$$

[9] To compare the last factor in (3.5.1) with [Zha14b], recall that $\omega_{\Pi_v}(z)=\omega_{\pi_v}(z/z^c)$, so that $\omega_{\pi_v}(-1)=\omega_{\Pi_v}(\tau)$. The absence of the factor $\varepsilon(\frac{1}{2},\eta_v,\psi_v)^{\binom{n+1}{2}}$, which cancels out its presence in our local Flicker–Rallis period P_{2,Π_v}, is helpful in Lemma 4.1.1.

(1) By (3.2.5), (3.4.6), the Hecke measures f'_v and (f_v^V) match spectrally if and only if $c_v\dot f'_v$ and $(\dot f_v^V)$ match spectrally in the sense ([Zha14b, Conjecture 4.4 and last equation on p. 566]) that

$$\widetilde{I}_{\Pi_v}(c_v\dot f'_v)=\kappa(\pi_v^V)\frac{D_v^{-\dim \mathrm{G}/2}\Delta_{\mathrm{G},v}}{D_v^{-\dim \mathrm{H}/2}\zeta_{\mathrm{H},v}(1)\Delta_{\mathrm{H},v}}\cdot \widetilde{J}_{\pi_v}(\dot f_v^V).$$

(2) By (3.3.15) and (3.4.7), the Hecke measures f'_v and (f_v^V) match geometrically if and only if $c_v\dot f'$ and $(\dot f_v^V)$ match geometrically in the sense of [Zha14b, (4.14)], namely

$$\kappa_v(\gamma')^{-1}\,\widetilde{I}_{\gamma'}(c_v\dot f',\mathbf{1})=\widetilde{J}_{\delta}(\dot f_v)$$

for all matching pairs of orbits (γ,δ).

3.5.4. *Main results on the local comparisons.* Each of the following is a deep result.

Proposition 3.5.3 (Equivalence of spectral and geometric matching)**.** *Let S be a finite set of places of F_0. The pairs $f'_S\in\mathscr{H}(G'_S)$ and $(f_S^V)\in\prod_{V\in\mathscr{V}_S}\mathscr{H}(G_S^V)$ match spectrally if and only if they match geometrically.*

Proof. The proof of [BPLZZ21, Lemma 4.9], based on [BP21a, BP21], applies. (As noted in [BPLZZ21, Remark 4.10], in general this relies on [Mok15, KMSW].) Note that by § 3.5.3, the comparisons of matchings in *loc. cit.*, whose conventions are inherited from [Zha14b], are compatible with ours. □

From now on we will simply say that f'_S and (f_S^V) *match* when they match spectrally and geometrically. For a fixed $V\in\mathscr{V}_S$, we say that f'_S *purely matches* f_S^V if it matches $(f_S^V,(0^{V'})_{V'\neq V})$.

Proposition 3.5.4 (Fundamental Lemma [Yun11, BP21b])**.** *Let v be a finite place of F_0 that is unramified in F, let $V\in\mathscr{V}_v$ be the unramified pair of hermitian spaces, and recall the unit Hecke measures from* (2.3.1)*.*

The unit Hecke measure $f_v'^{\circ}$ on G'_v purely matches the unit measure f_v° on G_v^V.

Proposition 3.5.5 (Existence of matching [Zha14a, Theorem 2.6])**.** *Let v be a finite place of F_0. For every $f'\in\mathscr{H}(G'_v)$, a matching $(f^V)\in\prod_{V\in\mathscr{V}_v}$ exists; conversely, for every $(f^V)\in\prod_{V\in\mathscr{V}_v}$, a matching $f'\in\mathscr{H}(G'_v)$ exists.*

A matching result for archimedean places will be proved in § 4.3.2. We will also need to note the following (relatively easy) fact.

Lemma 3.5.6 ([Zha14a, Proposition 2.5])**.** *Let $v=w\overline{w}$ be a split place of F_v, and identify $G_v\cong G'_{n,0}\times G'_{n+1,0}$. Then $f'_v=\mathrm{p}_*(f'_w\otimes f'_{\overline{w}})\in\mathscr{H}(G'_v)$ matches $f_v:=f_w\star f_{\overline{w}}^{\vee}\in\mathscr{H}(G_v)$.*

4. Rationality

This section is dedicated to establishing the rationality of our L-values, Theorem A from the introduction (Theorem 4.2.1 below), and a rational relative-trace formula (Proposition 4.2.2). From now on, we assume that F_0 is totally real and F is CM. In §4.1 we deal with the archimedean computations using the Gaussian test function. In §4.2 we state the rationality theorem and the rational relative-trace formula, and prove some easy parts. In §4.3 we study the existence of suitable Hecke measures: the non-archimedean components rely on later results from §§ 5-6; the archimedean component is proved in §4.4 using an argument provided by Yifeng Liu, refining the technique of isolating cuspidal representations in [BPLZZ21]. In §4.5, we finish the proof of Proposition 4.2.2 and Theorem 4.2.1. In §4.6 we recall an outline of the proof of (a special case of) the Ichino–Ikeda–Harris conjecture. Logically this is not needed for this paper, but it will make the proof of our main Theorem D in § 12 easier to understand.

4.0.1. *Notation.* For a locally compact and totally disconnected group G with a fixed Haar measure dg, from now on we denote by $\mathscr{H}(G)$ the sheaf of smooth compactly supported $\mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}}$-multiples of dg; we will write $\mathscr{H}(G,R) \coloneqq \mathscr{H}(G)(R)$. (Thus the object denoted by $\mathscr{H}(G)$ in the previous section will henceforth be denoted by $\mathscr{H}(G,\mathbf{C})$).

4.1. **Archimedean theory.** We define some rational variant of the archimedean distributions of the previous section. Denote $G_\infty^\circ \coloneqq G_\infty^{V_\infty^\circ}$, $H_\infty^\circ \coloneqq H_\infty^{V_\infty^\circ}$, $B_\infty^\circ \coloneqq B_{\infty,V_\infty^\circ}$.

4.1.1. *A product of transfer factors.* Let

$$\kappa(\mathbf{1}_\infty) \coloneqq \prod_{v|\infty} \kappa(\mathbf{1}_v)$$

be the product of (3.5.1) for the trivial representation of the positive-definite group G_∞°.

Lemma 4.1.1. *For each $\gamma' \in \mathrm{G}'_{\mathrm{reg}^+}(F_{0,\infty})$, we have $\kappa_\infty(\gamma')\kappa(\mathbf{1}_\infty) \in \{\pm 1\}$.*

Proof. By Lemma 3.3.4, the first factor is a square root of $(-1)^{-\binom{n+1}{2}[F_0:\mathbf{Q}]}$; so are $\eta'_\infty(\tau)^{\binom{n+1}{2}}$ and, hence, the second factor.

□

4.1.2. *Distributions on $\mathscr{H}(G'_\infty,\mathbf{C})$.* For any tempered representation Π_∞ of G'_∞ and any $\gamma \in B'_\infty$, we define

$$\begin{aligned} I^\circ_{\Pi_\infty}(f_\infty,\chi_\infty) &\coloneqq \frac{1}{\kappa(\mathbf{1}_\infty)\Delta_{\mathrm{H}}}\mathscr{L}(1/2,\Pi_\infty,\chi_\infty)\, I_{\pi_\infty}(f_\infty,\chi_\infty), \\ I^\circ_\gamma(f'_\infty,\chi_\infty) &\coloneqq \frac{\Delta_{\mathrm{G}}}{\Delta^2_{\mathrm{H}}} L_\gamma(\chi_\infty) I_\gamma(f'_\infty,\chi_\infty). \end{aligned} \tag{4.1.1}$$

Then the factorizations of Proposition 3.2.2 and of (3.3.12) are equivalent to

$$\begin{aligned} \kappa(\mathbf{1}_\infty)^{-1} I_\Pi(f',\chi) &= \frac{1}{4}\frac{\mathscr{L}^\infty(1/2,\Pi,\chi)}{\varepsilon(\frac{1}{2},\chi^2)^{\binom{n+1}{2}}} \prod_{v\nmid\infty} I_{\Pi_v}(f'_v,\chi_v)\cdot I^\circ_{\Pi_\infty}(f_\infty,\chi_\infty), \\ I_\gamma(f',\chi) &= L^\infty_\gamma(\chi)\prod_{v\nmid\infty} I_\gamma(f'_v,\chi_v)\cdot I^\circ_\gamma(f'_\infty,\chi_\infty). \end{aligned} \tag{4.1.2}$$

4.1.3. *Distributions and special elements in $\mathscr{H}(G^\circ_\infty,\mathbf{C})$.* For any $V \in \mathscr{V}_\infty = \prod_{v|\infty}\mathscr{V}_v$, every representation π_∞^V of G_∞^V, and every $\delta \in B_{\infty,V}$ (note that G_∞^V is compact and hence every orbit is semisimple), we define variants of $J_{\pi^V_\infty}$ and $J_{\delta,\infty}$ by

$$\begin{aligned} J^\circ_{\pi_\infty}(f_\infty) &\coloneqq \int_{H^V_\infty} \mathrm{Tr}_{\pi_\infty}(\pi_\infty(h)\pi_v(f_\infty))\,dh = \frac{1}{\Delta_{\mathrm{H}}}\mathscr{L}(1/2,\mathrm{BC}(\pi_\infty))\cdot J_{\pi_\infty}(f_\infty), \\ J^\circ_{\delta,\infty}(f_\infty) &\coloneqq \int_{H^V_\infty}\int_{H^V_\infty} f_v(x^{-1}\delta' y)\,\frac{dxdy}{dg} = \frac{\Delta_{\mathrm{G}}}{\Delta^2_{\mathrm{H}}}\,J_{\delta,\infty}(f_\infty). \end{aligned} \tag{4.1.3}$$

Then the matching relations (3.5.2) and, respectively, (3.5.5) for $S=\{v|\infty\}$ are equivalent to

$$\begin{aligned} I^\circ_{\mathrm{BC}(\pi^V_\infty)}(f'_\infty) &= \frac{\kappa(\pi^V_\infty)}{\kappa(\mathbf{1}_\infty)}J^\circ_{\pi^V_\infty}(f_\infty) \\ I^\circ_{\gamma,\infty}(f'_\infty,\mathbf{1}_\infty) &= J^\circ_{\delta,\infty}(f^V_\infty). \end{aligned} \tag{4.1.4}$$

Lemma 4.1.2. *Let*

$$f^\circ_\infty \coloneqq \mathrm{vol}(G^\circ_\infty, dg)^{-1}dg \quad \in \mathscr{H}(G^\circ_\infty,\mathbf{Q}). \tag{4.1.5}$$

Then:

(1) for all $\pi_\infty\in\mathrm{Temp}(G^\circ_\infty)$, we have

$$J^\circ_{\pi_\infty}(f^\circ_\infty) = \begin{cases} \mathrm{vol}(H^\circ_\infty)\coloneqq \mathrm{vol}(H^\circ_\infty, dh_\infty) & \text{if } \pi_\infty\cong\mathbf{1} \\ 0 & \text{otherwise;} \end{cases} \tag{4.1.6}$$

(2) for all $\delta\in G^\circ_\infty$, we have

$$J^\circ_\delta(f^\circ_\infty) = \mathrm{vol}(B^\circ_\infty)^{-1} \coloneqq \frac{\mathrm{vol}(H^\circ_\infty, dh_\infty)^2}{\mathrm{vol}(G^\circ_\infty, dg_\infty)}.$$

Moreover, both of the above values are rational.

Proof. The calculation is immediate. The rationality follows from Lemma 2.2.1. □

4.1.4. *Gaussians.* Let $f^\circ_\infty=$ (4.1.5) be the standard Hecke measure on $G^\circ_\infty = G^{V_\circ}_\infty$. For each characteristic-zero field L, we put $\mathscr{H}(G^\circ_\infty,L)^\circ \coloneqq Lf^\circ_\infty$.

For L a subfield of $\mathbf{C}$, we denote by

$$\mathscr{H}(G'_\infty,L)^\bullet\subset\mathscr{H}(G'_\infty,\mathbf{C})$$

the preimage of $Lf_\infty\subset\mathscr{H}(G^\circ_\infty,L)^\circ$ under pure matching. By Proposition 4.1.3 below, the pure matching map

$$\mathrm{tr}\colon \mathscr{H}(G'_\infty,L)^\bullet\longrightarrow\mathscr{H}(G^\circ_\infty,L)^\circ$$

is surjective (here tr stands for "(smooth) transfer"). We put

$$\mathscr{H}(G'_\infty,L)^\circ\coloneqq\mathscr{H}(G'_\infty,L)^\bullet/\,\mathrm{Ker}(\mathrm{tr}),$$

we extend the definition to any characteristic-zero field L by $\mathscr{H}(G'_\infty,L)^\circ\coloneqq\mathscr{H}(G'_\infty,\mathbf{Q})^\circ\otimes_{\mathbf{Q}}L$, and we extend the notion of matching by linearity. Elements of $\mathscr{H}(G'_\infty,L)^\circ$ are called L-rational *Gaussians*. If L can be embedded into $\mathbf{C}$, we also refer to an $f'_\infty\in\mathscr{H}(G'_\infty,L)^\bullet$ as a Gaussian; we say that f'_∞ is nontrivial if its image in $\mathscr{H}(G'_\infty,L)^\circ$ is nonzero.

If S is a finite set of non-archimedean places of F_0, we put

$$\mathscr{H}(G'_{S\infty},L)^\circ\coloneqq\mathscr{H}(G'_S,L)\otimes_L\mathscr{H}(G'_\infty,L)^\circ,\qquad \mathscr{H}(\mathrm{G}'(\mathbf{A}^S),L)^\circ\coloneqq\mathscr{H}(\mathrm{G}'(\mathbf{A}^{S\infty}),L)\otimes_L\mathscr{H}(G'_\infty,L)^\circ,$$

and refer to the elements of those spaces as Gaussians too.

Proposition 4.1.3 (Existence of Gaussians)**.** *The space $\mathscr{H}(G'_\infty, \mathbf{Q})^\circ$ is nonzero.*

Proof. This follows from [BPLZZ21, Proposition 4.11]. □

Lemma 4.1.4. *Let f' be a Gaussian matching of $f^\circ_\infty = (4.1.5)$. Then for any $\gamma \in B'$ matching an element from B°_∞, we have $I^\circ_\gamma(f', \mathbf{1}) \in \mathbf{Q}$.*

Proof. We recall from (4.1.1)

$$I^\circ_\gamma(f', \mathbf{1}) \coloneqq \frac{\Delta_{\mathrm{G}}}{\Delta^2_{\mathrm{H}}} L_\gamma(\mathbf{1}) I_\gamma(f', \mathbf{1}), where$$

from (3.3.10),

$$I_\gamma(f', \mathbf{1}) = \kappa(\gamma')\, I_{\gamma'}(f', \mathbf{1}) = \kappa(\gamma') \frac{I^\natural_{\gamma'}(f', \chi)}{L_{\gamma'}(\chi)}.$$

Here γ' is any plus-regular element above $\gamma \in B'$, and $\kappa(\gamma')$ is the local transfer factor.

Recall also the orbital integral J°_δ in the unitary side (4.1.3). It follows immediately that the lemma holds If γ is regular semisimple, the lemma follows immediately from the rationality of $J^\circ_\delta(f^\circ_\infty)$ (Lemma 4.1.2 (2)) and the matching relation (4.1.4). Though the matching relation is defined only using regular semisimple orbits, the definition implies non-trivial relations for non-regular-semisimple orbital integrals. We record the result of Lu [Lu, Thm. 7.9, Remark 7.10] comparing the local orbital integrals. Let $f' \in \mathscr{H}(G'_\infty, \mathbf{C})$ purely match an $f \in \mathscr{H}(G^\circ_\infty, \mathbf{C})$. If $\gamma \in B'$, then

$$L_\gamma(\mathbf{1})^{-1} I^\circ_\gamma(f', \mathbf{1}) = \sum_\delta c_\delta J^\circ_\delta(f), \tag{4.1.7}$$

where the sum runs over all *semisimple* orbits in the compact group G°_∞ with image $\gamma \in B'$, and

$$c_\delta = \prod_{W \in \mathscr{W}(\gamma)} c_W,$$

where the set $\mathscr{W}(\gamma)$ and the constants c_W will be recalled next. The set $\mathscr{W}(\gamma)$ is a finite set of positive definite $\mathbf{C}/\mathbf{R}$-Hermitian spaces W defined in *loc. cit.*, and it can be described as follows: the stabilizer of any semisimple δ matching γ is isomorphic to the product of the compact unitary groups $U(W)$ for $W \in \mathscr{W}(\gamma)$. For W of dimension n', by [Lu, §7.4 on the Lie algebra, and (7.15) and Remark 7.10 on the group] we have

$$\begin{aligned} c_W = \mathrm{vol}^\natural(U(n', \mathbf{R}))^{-1} \prod_{i=1}^{n'} \varepsilon(1-i, \eta^i_{\mathbf{C}/\mathbf{R}}, \psi)^{-1} \cdot \varepsilon(1/2, \eta_{\mathbf{C}/\mathbf{R}}, \psi)^{n'(n'+1)/2} \\ \times \eta_{\mathbf{C}/\mathbf{R}}(\mathrm{disc}(V'))^{n'+1}. \end{aligned} \tag{4.1.8}$$

Here $\mathrm{vol}^\natural(U(n', \mathbf{R}))$ is the volume of the compact unitary group $U(n', \mathbf{R})$ for the normalized measure $d^\natural g$ of § 2.2.1, which is the measure in [Lu, §7.0.1] (for a suitable differential ω). The formula for the constant c_W differs slightly from [Lu] due to a few different conventions between ours and those in *loc. cit.*:

- the factor $\frac{\Delta_{\mathrm{G}}}{\Delta^2_{\mathrm{H}}}$ appears on both the GL and the unitary side, and hence our notion of matching is equivalent to that of [Lu];

– Theorem 7.15 in [Lu] is expressed in terms of the normalized orbital integral, and this results into the factor $L_\gamma(\mathbf{1})^{-1}$ on the left hand side of (4.1.7);

– when defining $I_\gamma(f', \mathbf{1})$, we only consider the plus-regular element γ' with image $\gamma \in B'$ and our notation has already included the transfer factor (our transfer factor is the plus-transfer factor in [Lu]);

– our orbital integral in the unitary side is taken over the full group $H^\circ_\infty \times H^\circ_\infty$, whereas in *loc. cit.* the integral is taken over the quotient of the full group by the stabilizer: this results into the volume factor in (4.1.8);

– In [Lu, (7.15)] the additional factor disappears because our choice of γ' above γ is plus-regular and the formula in [Lu, Thm. 7.9] simplifies to [Lu, Remark 7.10, (7.16)].

The factors in the second line in (4.1.8) are signs, hence lie in $\mathbf{Q}^\times$. By definition, the L-factor $L_\gamma(\mathbf{1})$ is the product

$$L_\gamma(\mathbf{1}) = \prod_{W \in \mathscr{W}(\gamma)} \prod_{i=1}^{\dim W} L(1-i, \eta^i_{\mathbf{C}/\mathbf{R}}).$$

Therefore, to show that $I^\circ_\gamma(f', \mathbf{1}) \in \mathbf{Q}$, from (4.1.7) and (4.1.8) it suffices to show that for all $n' \geq 1$, the product

$$\mathrm{vol}^\natural(U(n', \mathbf{R}))^{-1} \prod_{i=1}^{n'} L(1-i, \eta^i_{\mathbf{C}/\mathbf{R}}) \prod_{i=1}^{n'} \varepsilon(1-i, \eta^i_{\mathbf{C}/\mathbf{R}}, \psi)^{-1} \cdot \varepsilon(1/2, \eta_{\mathbf{C}/\mathbf{R}}, \psi)^{n'(n'+1)/2} \tag{4.1.9}$$

lies in $\mathbf{Q}^\times$.

By Tate's thesis (e.g. [Tat79, §3.2]), the standard choice of $\psi(x) = e^{2\pi i x}$ gives

$$\varepsilon(s, \eta^a_{\mathbf{C}/\mathbf{R}}, \psi) = i^a \in \mathbb{C}$$

for all $s \in \mathbb{C}$ and $a \in \{0, 1\}$. In particular, we have

$$\varepsilon(1/2, \eta_{\mathbf{C}/\mathbf{R}}, \psi)^{n'(n'+1)/2} = i^{n'(n'+1)/2}.$$

and

$$\prod_{i=1}^{n'} \varepsilon(1-i, \eta^i_{\mathbf{C}/\mathbf{R}}, \psi)^{-1} = i^{-\lfloor \frac{n'+1}{2} \rfloor}$$

We note that $n'(n'+1)/2 \equiv \lfloor (n'+1)/2 \rfloor \mod 2$, and hence

$$\varepsilon(1-i, \eta^i_{\mathbf{C}/\mathbf{R}}, \psi)^{-1} \cdot \varepsilon(1/2, \eta_{\mathbf{C}/\mathbf{R}}, \psi)^{n'(n'+1)/2} = \pm 1. \tag{4.1.10}$$

Next we note for $a \in \{0, 1\}$,

$$L(s, \eta^a_{\mathbf{C}/\mathbf{R}}) = L(s+a, \mathbf{1}) = \pi^{-(s+a)/2} \Gamma((s+a)/2)$$

and we have its special values

$$L(1-i, \eta^i_{\mathbf{C}/\mathbf{R}}) = \begin{cases} \pi^{-(1-i)/2} \Gamma((1-i)/2), & i \geq 0 \text{ even}, \\ \pi^{-(1-i+1)/2} \Gamma((1-i+1)/2), & i \geq 0 \text{ odd}. \end{cases}$$

In both cases we have
$$L(1-i,\eta^i_{\mathbf{C}/\mathbf{R}})\in \pi^{\lfloor i/2\rfloor}\cdot \mathbf{Q}^\times,\quad i\geq 1. \tag{4.1.11}$$
Similarly,
$$L(i,\eta^i_{\mathbf{C}/\mathbf{R}})\in \pi^{-\lfloor (i+1)/2\rfloor}\cdot \mathbf{Q}^\times,\quad i\geq 1. \tag{4.1.12}$$

We compute the volume $\mathrm{vol}^\natural(U(n',\mathbf{R}))$ of the compact unitary group. Denote by $\mathrm{vol}(U(n',\mathbf{R}))$ the volume under the unnormalized measure $d_\omega g$ of § 2.2.1, then
$$\begin{aligned}
\mathrm{vol}^\natural(U(n',\mathbf{R})) &= \prod_{i=1}^{n'} L(i,\eta^i_{\mathbf{C}/\mathbf{R}})\cdot \mathrm{vol}(U(n',\mathbf{R}))\\
&= \prod_{i=1}^{n'} L(i,\eta^i_{\mathbf{C}/\mathbf{R}})\prod_{i=1}^{n'}\mathrm{vol}(S^{2i-1})\\
\text{(by (4.1.12))}\quad &\in \prod_{i=1}^{i}\pi^{-\lfloor (i+1)/2\rfloor}\pi^i\cdot \mathbf{Q}^\times\\
&= \prod_{i=1}^{i}\pi^{\lfloor i/2\rfloor}\cdot \mathbf{Q}^\times
\end{aligned}$$
where $\mathrm{vol}(S^{2i-1})$ is the usual volume of the unit sphere of dimension $2i-1$. Combining this with (4.1.11), we have
$$\mathrm{vol}^\natural(U(n',\mathbf{R}))^{-1}\prod_{i=1}^{n'} L(1-i,\eta^i_{\mathbf{C}/\mathbf{R}})\in \mathbf{Q}^\times. \tag{4.1.13}$$
Therefore the rationality of (4.1.9) follows from (4.1.10) and (4.1.13), and the lemma follows from this and the rationality of $J^\circ_\delta(f^\circ_\infty)$ (Lemma 4.1.2 (2)). □

4.2. **Rationality statements.** We state the main results of this section, whose proofs will be completed in § 4.5.

4.2.1. *Rationality of L-values.* The following is Theorem A from the introduction.

Theorem 4.2.1. *Let L be a number field and let Π be a trivial-weight hermitian cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ defined over L. There is a function*
$$\mathscr{L}(\mathrm{M}_\Pi,\cdot)\in \mathscr{O}(Y_L)$$
such that for each $\chi\in Y_L(\mathbf{C})$ with underlying embedding $\iota\colon L\hookrightarrow \mathbf{C}$,
$$\mathscr{L}(\mathrm{M}_\Pi,\chi)=\frac{\mathscr{L}^\infty(1/2,\Pi^\iota,\chi)}{\varepsilon(\frac12,\chi^2)^{\binom{n+1}{2}}}.$$

4.2.2. *Special Hecke algebras.* Let L be a field that is embeddable in $\mathbf{C}$, let S be a finite set of finite places of F_0, and let $?\in\{\mathrm{rs},\mathrm{reg}^+,\emptyset\}$. We denote by
$$\mathscr{H}(\mathrm{G}'(\mathbf{A}^S),L)^\circ_{K_S,?,\mathrm{qc}}\subset \mathscr{H}(\mathrm{G}'(\mathbf{A}^S),L)^\circ$$

the space of Gaussian measures f'^S with weakly ?-support (Definition 3.3.5; there is no condition if $? = \emptyset$) such that for every $\iota\colon L \hookrightarrow \mathbf{C}$, some preimage $f'^{S,\iota}e_{K_S} \in \mathscr{H}(\mathrm{G}'(\mathbf{A}), \iota L)^\bullet$ of $\iota f'^S e_{K_S}$ is quasicuspidal.

If $\Pi \in \mathscr{C}(\mathrm{G}')(L)$ and $\chi \in Y(L)$, we say that a Hecke measure $f'^S \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^S), L)^\circ$ is *adapted to* (Π, χ, K_S) if $(\otimes_{v\notin S} I_{\Pi_v})(f'^S, \chi^S) \neq 0$ and for every $\iota\colon L \hookrightarrow \mathbf{C}$, some preimage $f'^{S,\iota}e_{K_S} \in \mathscr{H}(\mathrm{G}'(\mathbf{A}), \iota L)^\bullet$ of $\iota f'^S e_{K_S}$ sends $\mathscr{A}(\mathrm{G}')$ into (the image in $\mathscr{A}(\mathrm{G}')$ of) Π. We denote by

$$\mathscr{H}(\mathrm{G}'(\mathbf{A}^S), L)^\circ_{K_S, ?, \Pi, \chi}$$

the space of Gaussians with weakly ?-support that are adapted to (Π, χ, K_S). When $\chi = \mathbf{1}$ we omit it from the notation.

4.2.3. *Rational relative-trace formula.* We introduce a variant of the distribution I and its expansions. From now on, we change the notation for the distributions $I_?$ of the previous section by appending a superscript '$\mathbf{C}$', thus writing $I_?^{\mathbf{C}}$ in place of $I_?$; we also write $L_?^{\mathbf{C}}$ for the abelian complex L-functions attached to orbits.

We introduce some further notation. For a finite place v of F_0 and an ideal $m \subset \mathscr{O}_{F_{0,v}}$, let $Y_v(m) = \operatorname{Spec} \mathbf{Q}[F_{0,v}^\times/(\mathscr{O}_{F_{0,v}}^\times \cap 1 + m\mathscr{O}_{F_{0,v}})]$, viewed as the space of characters of the group within square brackets. Let $Y_v \coloneqq \varinjlim_r Y_v(v^r)$. For the sake of uniformity, we will denote $\mathscr{H}(G'_v, L)^\circ \coloneqq \mathscr{H}(G'_v, L)$ if $v \nmid \infty$, and $Y_\infty \coloneqq \operatorname{Spec} \mathbf{Q}$.

In the rest of the paper, unless otherwise noted all products '$\prod_v$' run over the union of the set of finite places v of F_0 and $\{v = \infty\}$. If $\mathscr{H}$ is a Hecke algebra over a field L and Y is an ind-scheme over L, an L-linear functional $D\colon \mathscr{H} \to \mathscr{O}(Y)$ will be called a *distribution*.

Proposition 4.2.2. *Let L be a field that can be embedded in $\mathbf{C}$. There exist:*

(1) for each finite place v of F_0 and for $v = \infty$, and for each tempered irreducible admissible representation Π_v of G'_v over L, a distribution

$$I_{\Pi_v}\colon \mathscr{H}(G'_v, L)^\circ \longrightarrow \mathscr{O}(Y_{v,L})$$

characterized by

$$I_{\Pi_v}(f'_v, \chi_v) = \begin{cases} I^{\mathbf{C}}_{\iota\Pi_v}(\iota f'_v, \chi_v) & \text{if } v \nmid \infty \\ I^{\circ,\mathbf{C}}_{\Pi_\infty}(f'_\infty, \chi_v) & \text{if } v = \infty \end{cases}$$

for each $\chi_v \in Y_{v,L}(\mathbf{C})$ with underlying embedding $\iota\colon L \hookrightarrow \mathbf{C}$;

(2) for each representation $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her}}$ over L as in Theorem 4.2.1, a distribution

$$I_\Pi\colon \mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ \longrightarrow \mathscr{O}(Y_L)$$

defined on factorizable elements $f' = \otimes_{v\nmid\infty} f'_v \otimes f'_\infty$ by

$$I_\Pi(f', \chi) = \frac{1}{4}\mathscr{L}(\mathrm{M}_\Pi, \chi) \cdot \prod_v I_{\Pi_v}(f'_v, \chi_v). \tag{4.2.1}$$

(3) for each finite place v of F_0 and for $v = \infty$, and each $\gamma \in B'_v$, a distribution

$$I_{\gamma,v}\colon \mathscr{H}(G'_v, L)^\circ \longrightarrow \mathscr{O}(Y_{v,L[\sqrt{-1}]})$$

characterized by

$$I_{\gamma,v}(f'_v, \chi_v) = \begin{cases} I^{\mathbf{C}}_{\gamma,v}(\iota f'_v, \chi_v) & \text{if } v \nmid \infty \\ I^{\circ;\mathbf{C}}_{\gamma,v}(\iota f'_v, \chi_v) & \text{if } v = \infty \end{cases}$$

for each $\chi_v \in Y_{v,L[\sqrt{-1}]}(\mathbf{C})$ *with underlying embedding* $\iota\colon L[\sqrt{-1}] \hookrightarrow \mathbf{C}$.

(4) *for each* $\gamma \in \mathrm{B}'(F_0)^\circ := \mathrm{B}'(F_0) \cap B'^{\circ}_{\infty}$,

 (a) *an element* $L_\gamma \in \mathscr{O}(Y)$, *characterized by* $L_\gamma(\chi) = L^{\infty,\mathbf{C}}_{\gamma}(\chi)$ *(the L-function without archimedean local L-factors) for every* $\chi \in Y(\mathbf{C})$;

 (b) *a distribution*

$$I_\gamma = \kappa(\mathbf{1}_\infty)^{-1} \cdot L_\gamma \cdot \prod_v I_{\gamma,v} \colon \mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ \longrightarrow \mathscr{O}(\mathrm{Y}_L)$$

 where the product is locally finite.

(5) *a distribution*

$$I\colon \mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ_{\mathrm{reg}^+,\mathrm{qc}} \longrightarrow \mathscr{O}(Y_L)$$

admitting the spectral and geometric expansions

$$\sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her}}} I_\Pi = I = \sum_{\gamma \in \mathrm{B}'(F_0)} I_\gamma$$

where both sums are locally finite.

Remark 4.2.3. It should be possible to interpret the rational distribution I as the inner product of analogues of $P_{1,\chi}$, P_2 in the rational Betti homology (in complementary degrees) of the symmetric space for G'.

We prove Proposition 4.2.2 (1)-(4); the proof of part (5) is deferred to § 4.5.

Proof of Proposition 4.2.2 (1)-(4). We need to show the existence of various distributions.

Archimedean distributions. Suppose that f'_∞ is an L-rational Gaussian matching $f_\infty = cf^\circ_\infty \in \mathscr{H}(G^\circ_\infty, L)$. Then by Lemma 4.1.2, Lemma 4.1.4, and (4.1.4), we may define

$$I_{\Pi_\infty}(f'_\infty, \mathbf{1}) := \begin{cases} c\,\mathrm{vol}(H^\circ_\infty) & \text{if } \Pi_\infty \cong \Pi^\circ_\infty \\ 0 & \text{otherwise,} \end{cases}$$

$$I_\gamma(f'_\infty) := \begin{cases} cI^\circ_\gamma(f'^\circ, \mathbf{1}) & \text{if } \gamma \in B'^\circ_\infty, \\ 0 & \text{otherwise,} \end{cases}$$

for any Gaussian f'° matching f°, where the orbital integral is rational by Lemma 4.1.4.

Orbital integrals. Suppose v is non-archimedean. Then part (3) of Proposition 4.2.2 follows from Proposition 3.3.2 (3b) together with Lemma 3.3.4.

Part (4a) is a well-known rationality theorem of Klingen and Siegel [Sie70]. Part (4b) then follows from part (3) and Proposition 3.3.2 (2), together with (3.3.9) and Lemma 4.1.1 for the elimination of $\sqrt{-1}$ from the field of rationality.

Local spherical character. It suffices to show that $P_{1,\Pi_v,\chi_v}(W_v)$, $P_{2,\Pi_v}(W'_v)$ and $\vartheta_{\Pi_v}(W_v, W'_v)$ are polynomials, with L-coefficients, in the values of W_v and χ_v. The rationality in W_v is observed in Remark 3.2.1. Then we only need to consider the function $\chi_v \mapsto P_{1,\Pi_v,\chi_v}(W_v)$. Let Y'_v be the ind-finite scheme over L of smooth characters of $\mathscr{O}_{F_{0,v}}^\times$; then $\chi \mapsto \chi_{v|\mathscr{O}_{F,0,v}^\times}$ gives an exact sequence of ind-group-schemes $1 \to Y_v^\circ \to Y_v \to Y'_v \to 1$ where $Y_v^\circ \cong \mathbf{G}_{m,L}$ parametrizes unramified characters of $F_{0,v}^\times$. Thus locally we may reduce to proving the desired result when χ_v is restricted to Y_v° at the cost of replacing Π_v by (one of locally finitely many) ramified twists. In this case, that P_{1,Π_v} is a polynomial in Y_v° and the values of W_v is one of the main results of [JPSS83], whose proof considers unramified characters of the form $|\cdot|_v^s$ but goes through in our context. □

4.3. **Test Hecke measures.** We now give some key results asserting the existence of suitable Hecke measures.

4.3.1. *Test measures at finite places.* Let v be a finite place of F_0 and let L be a field that can be embedded into $\mathbf{C}$. A character $\xi' = \xi'_1 \boxtimes \cdots \boxtimes \xi'_\nu \colon (F_w^\times)^\nu \to \mathbf{C}^\times$ is called *regular* if the characters ξ'_i are pairwise distinct. A *regular principal series* representation of G'_v is a representation $\Pi_v = \Pi_{n,v} \boxtimes \Pi_{n+1,v}$ such that for $\nu = n, n+1$, all places $w|v$, and any $\iota \colon L \hookrightarrow \mathbf{C}^\times$ the representation $\Pi_{\nu,w} \coloneqq \Pi_{\nu,v|\mathrm{GL}_\nu(F_w)}$ is unitarily induced from a regular character of the diagonal torus.

Proposition 4.3.1. *Let Π_v be a hermitian (§ 2.4.1) tempered representation of G'_v over L, and let χ_v be a smooth character of $F_{0,v}^\times$ with values in some finitely generated extension L' of L. For $? \in \{\emptyset, \mathrm{reg}^\pm\}$, denote by*

$$\mathscr{H}(G'_v, L)_{?,\Pi_v,\chi_v}$$

the set of those $f'_v \in \mathscr{H}(G'_v, L)$ that are supported in $G'_{?,v}$, and satisfy $I_{\Pi_v}(f'_v, \chi_v) \neq 0$.

(1) We have $\mathscr{H}(G'_v, L)_{\Pi_v,\chi_v} \neq \emptyset$.

(2) If Π_v and χ_v are unramified, then $f'^\circ_v \in \mathscr{H}(G'_v, L)_{\Pi_v,\chi_v}$.

(3) If v splits in F and Π_v is a regular principal series, for every choice of sign $\pm$ there exists

$$f'_\pm \in \mathscr{H}(G'_v, L)_{\mathrm{reg}^\pm,\Pi_v,\chi_v}.$$

whose matching $f_\pm \in \mathscr{H}(G_v, L)$ is invariant under a deeper Iwahori subgroup. If moreover Π_v is unramified, we can take $f'_\pm$ to match an $f_\pm \in \mathscr{H}(G_v, L)$ that is bi-invariant under an Iwahori subgroup.

The proof of part (3) relies on some explicit results from later sections. (In fact, see § 5.1.4 for a definition of Iwahori subgroups.)

Proof. We omit all subscripts v.

(1) Let $K \subset G'$ be an open compact subgroup. The restriction of $I_\Pi(\cdot, \chi)$ to $\mathscr{H}(G')_K$ is the inner product, for the natural pairing, of the elements

$$P_{1,\Pi,\chi|\Pi^K} \circ \Pi(\cdot) \in \Pi^{K,\vee} \otimes_L L', \qquad P_{2|\Pi^\vee,K} \in (\Pi^{\vee,K})^\vee \cong \Pi^K.$$

Now if K is sufficiently small, both $P_{1,\Pi,\chi|\Pi^K}$ and $P_{2|\Pi^\vee,K}$ are nonzero – the former by the theory of [JPSS83], the latter because Π, hence $\Pi^\vee$, is hermitian. Since Π^K is irreducible as an $\mathscr{H}(G', L)_K$-module, there exists an $f'_{L'} \in \mathscr{H}(G', L')_K$ such that $P_{1,\Pi,\chi|\Pi^K} \circ \Pi(f'_{L'})$ and $P_{2|\Pi^\vee,K}$ do not pair to zero. Fix an embedding $\iota\colon L' \hookrightarrow \mathbf{C}$. If $\iota(L)$ is not contained in $\mathbf{R}$, then it is dense in $\mathbf{C}$, and any $f' \in \mathscr{H}(G', L)_K$ that is sufficiently close to $f'_{L'}$ in the topology induced from $\mathbf{C}$ by ι will also have the desired nonvanishing property. If $\iota(L)$ is contained in $\mathbf{R}$, note that one of $\mathrm{Re}\,\iota f'_{L'}$, $\mathrm{Im}\,\iota f'_{L'}$ has the nonvanishing property, and then so does any sufficiently close $f' \in \mathscr{H}(G', L)_K$ (for the topology induced from $\mathbf{R}$ by ι).

(2) This follows from Remark 3.2.1.

(3) This will be proved at the end of § 6.1 based on an explicit construction from § 5.3.4.

□

4.3.2. *Test Gaussians.* For a pure tensor $f'_{S\infty} = f'_S f'_\infty \in \mathscr{H}(G'_{S\infty}, L)^\circ$, we define $f'^\iota_{S\infty} \coloneqq \iota f'_S f'^\iota_\infty$, and extend this definition to all of $\mathscr{H}(G'_{S\infty}, L)^\circ$ by linearity.

Proposition 4.3.2. *Let Π be a trivial-weight hermitian cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a field L admitting embeddings into $\mathbf{C}$, let $K = \prod_{v\nmid\infty} K_v \subset \mathrm{G}'(\mathbf{A}^\infty)$ be an open compact subgroup such that $\Pi^K \neq 0$, and let P be a finite set of non-archimedean places of F_0 containing all those for which K_v is not maximal.*

There exist a finite set S of split non-archimedean places of F_0 disjoint from P, and Gaussians

$$(f'^\iota_{S\infty})_\iota \in \prod_{\iota\colon L\hookrightarrow\mathbf{C}} \mathscr{H}(G'_{S\infty}, \iota L)^\bullet_{K_S}, \qquad f'_{S\infty} \in \mathscr{H}(G'_{S\infty}, L)^\circ_{K_S}$$

such that for every $\iota\colon L \hookrightarrow \mathbf{C}$:

(1) the image of $f'^\iota_{S\infty}$ in $\mathscr{H}(G'_{S\infty}, \iota L)^\circ_{K_S}$ equals $\iota f'_{S\infty}$;

(2) $I_{\Pi^\iota_{S\infty}}(f'^\iota_{S\infty}, \chi_{S\infty}) \neq 0$ for every unramified character $\chi_{S\infty}\colon F^\times_{0,S}F^\times_{0,\infty}/F^\times_{0,\infty} \to \mathbf{C}^\times$;

(3) $R(f'^\iota_{S\infty})$ maps $\mathscr{A}(\mathrm{G}')^K$ into $(\Pi^\iota)^K$. (In particular, for any $f'^{S\infty} \in \mathscr{H}(G', \iota L)_{K^S}$, the Hecke measure $f'^{S\infty} f'^\iota_{S\infty}$ is quasicuspidal.)

The proof will be given in § 4.4.

Lemma 4.3.3. *Let Π be a representation in $\mathscr{C}$. There exist infinitely many places v of F_0 that are split in F such that Π_v is an unramified regular principal series.*

Proof. This follows from the similar observation about Π_ν made in the proof of [CH13, Proposition 3.2.5]. □

Corollary 4.3.4. *Let Π be a trivial-weight hermitian cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over a field L admitting embeddings into $\mathbf{C}$, and let $\chi \in Y_L$. Let P be a finite set of nonarchimedean places of F_0 and let $K_P \subset G'_P$ be a compact open subgroup such that $\Pi_P^{K_P} \neq 0$.*

For $? \in \{\mathrm{reg}^+, \mathrm{reg}^-, \mathrm{rs}\}$, there exist L-rational Gaussians $f'^P_? \in \mathscr{H}(G'(\mathbf{A}^P), L)^\circ_{K_P,?,\Pi,\chi}$ with weakly ?-support that are adapted to (Π, χ, K_P) (in the sense of § 4.2.2).

Proof. In fact we construct an f'^P that has at the same time plus-regular support (at one place) and minus-regular support (at another place), hence weakly semisimple regular support since $\mathrm{G}'_{\mathrm{rs}} = \mathrm{G}'_{\mathrm{reg}^+} \cap \mathrm{G}'_{\mathrm{reg}^-}$. (The construction can of course be simplified if only one of those two properties is desired.) Let R be the set of all finite places of F_0 at which Π or χ is ramified. Let

v_+, v_- be two distinct finite places of F_0, split in F, not in $P \cup R$, such that $\Pi_{v_\pm}$ is a regular principal series and $\chi_{v_\pm}$ is unramified. Let $f'_{\pm,v_\pm}$ be as in Proposition 4.3.1 (3), and for $v \in R$ let f'_v be as in Lemma 4.3.1 (1). Let $f'_{S\infty}$ be as given by Proposition 4.3.2 for the set of places $P' = P \cup R \cup \{v_+, v_-\}$, and any level K that is maximal away from P' and sufficiently small at the places in $R \cup \{v_+, v_-\}$. Then

$$f'^P = f'_{+,v_+} f'_{-,v_-} f'_{S\infty} \prod_{v \nmid PS\infty} f'^{\circ}_v$$

is as desired. □

4.4. **Isolation of cuspidal representations via Gaussians.** In this subsection, we prove Proposition 4.3.2.

We will refine the arguments of [BPLZZ21], of which the reader is invited to open a copy. Briefly, in order to construct the desired $f'_{S\infty}$ we will start from a Gaussian $f'_{1,S'\infty}$ constructed in a simple way as a pure tensor, and then correct $f'_{1,S'\infty}$ by acting on it by a carefully chosen *multiplier* of the Hecke algebra for $\mathrm{G}'(\mathbf{A})$.

The substance of this subsection was generously provided to us by Yifeng Liu. Of course, any defects in the following pages are to be attributed to the authors only.

4.4.1. *Archimedean multipliers annihilating non-strongly typical cuspidal data.* We momentarily consider the more general situation of [BPLZZ21, § 3.2]. Consider a connected reductive algebraic group G over a number field F_0. We freely adopt notation from [BPLZZ21, § 3], up to cosmetic modifications to adapt to our conventions (for instance, in *loc. cit.* the algebraic group is denoted by G rather than G). Take a unitary automorphic character $\omega\colon \mathrm{Z}(\mathbf{A}) \to \mathbf{C}^\times$. We fix

- a subset P of primes of F_0 containing S_G, and a character

$$\xi = (\xi_\infty, \xi^{P\infty})\colon \mathcal{Z}(\mathfrak{g}) \times \mathscr{H}^{\mathrm{sph}}(\mathrm{G}(\mathbf{A}^{P\infty})) \longrightarrow \mathbf{C},$$

  where the second factor is the spherical Hecke algebra with respect to some choices of hyperspecial levels away from P (thus ξ is a P*-character* for G in the sense of [BPLZZ21, Definition 3.3]);
- a finite set S of primes of F satisfying $S_G \subseteq S \subseteq P$;
- a subgroup $K \subseteq K_0^\infty$ of finite index of the form $K = K_S \times \prod_{v \notin S} K_{0,v}$.

The following definition is modified from [BPLZZ21, Definition 3.11]; the set $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit$, consisting of classes of cuspidal automorphic representations of $\mathrm{M}(\mathbf{A})$, is defined *ibid.* p. 550.

Definition 4.4.1. Let $\mathrm{M} \subset \mathrm{G}$ be a standard Levi subgroup. We say that a $\sigma \in \mathfrak{C}(\mathrm{M}, \omega)^\heartsuit$ is *strongly* ξ_∞*-typical* if $\gamma_{\mathrm{M}}(\xi_{\sigma_\infty}) \subseteq \gamma_{\mathrm{M}}(\xi_\infty)$. We denote by $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty!}$ the subset of $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit$ consisting of strongly ξ_∞-typical elements.

It is clear that the set $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty!}$ of strongly ξ_∞-typical elements is a subset of $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty}$, the set of ξ_∞-typical elements defined in *loc. cit.* The following lemma slightly strengthens [BPLZZ21, Lemma 3.14], whose notation we simplify by putting

$$\mathcal{M}_\infty \coloneqq \mathcal{M}^\sharp_\theta(\mathfrak{h}'^*_{\mathbf{C}})^{\mathsf{W}}$$

for the Weyl-fixed elements of the space of holomorphic functions from [BPLZZ21, Definition 2.8].

We fix an element μ_∞^0 and a finite set $\mathfrak{T}$ of $K_{0,\infty}^{\mathrm{G}}$-types as described after [BPLZZ21, Definition 3.11].

Lemma 4.4.2. *For every standard Levi subgroup* $\mathrm{M} \subset \mathrm{G}$*, there exists an element*

$$\mu_\infty^{\mathrm{M}} \in \mathcal{M}_\infty$$

satisfying:

- $\mu_\infty^{\mathrm{M}}(\xi_\infty) \neq 0$,
- *for every open compact* $K_{\mathrm{M}} \subset \mathrm{M}(\mathbf{A}^\infty)$ *and every finite set* $\mathfrak{T}_{\mathrm{M}}$ *of* $K_{0,\infty}^{\mathrm{M}}$*-types satisfying the conditions following* [BPLZZ21, Definition 3.11] *with respect to* $(\mu_\infty^0, \mathfrak{T})$*, for every non-strongly typical*
$$\sigma \in \mathfrak{C}(\mathrm{M}, \omega; K_{\mathrm{M}}, \mathfrak{T}_{\mathrm{M}})^\heartsuit - \mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty !}$$
and every $s \in \mathfrak{a}^*_{\mathrm{M},\mathbf{C}}$*, we have*
$$\mu_\infty^{\mathrm{M}}(\xi^{\mathrm{G}}_{\sigma_{s,\infty}}) = 0.$$
Here, $\xi^{\mathrm{G}}_{\sigma_{s,\infty}}$ *is the infinitesimal character of* $\mathrm{Ind}_{\mathrm{P}_{\mathrm{M}}}^{\mathrm{G}}(\sigma_{s,\infty})$*.*

Proof. By Definition 4.4.1, it is easy to see that for each element $\sigma \in \mathfrak{C}(\mathrm{M}, \omega; K_{\mathrm{M}}, \mathfrak{T}_{\mathrm{M}})^\heartsuit - \mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty !}$, there exists a W-invariant polynomial function ν_σ on $\mathfrak{h}^*_{\mathbf{C}}$ satisfying $\nu_\sigma(\xi_\infty) \neq 0$ and $\nu_\sigma(\xi^{\mathrm{G}}_{\sigma_{s,\infty}}) = 0$ for every $s \in \mathfrak{a}^*_{\mathrm{M},\mathbf{C}}$. By [BPLZZ21, Lemma 3.14], we have an element $\nu_\infty^{\mathrm{M}} \in \mathcal{M}^\sharp_\theta(\mathfrak{h}^*_{\mathbf{C}})^{\mathsf{W}}$ satisfying the similar property but with $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty !}$ replaced by $\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty}$. Now by [BPLZZ21, Lemma 3.13], the set

$$\mathfrak{C}' \coloneqq \mathfrak{C}(\mathrm{M}, \omega; K_{\mathrm{M}}, \mathfrak{T}_{\mathrm{M}})^\heartsuit \cap \left(\mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty} - \mathfrak{C}(\mathrm{M}, \omega)^\heartsuit_{\xi_\infty !}\right)$$

is finite. Thus, we may take

$$\mu_\infty^{\mathrm{M}} \coloneqq \nu_\infty^{\mathrm{M}} \cdot \prod_{\sigma \in \mathfrak{C}'} \nu_\sigma.$$

□

The following is a direct analogue of [BPLZZ21, Proposition 3.15] in terms of Lemma 4.4.2; the background is the Langlands decomposition

$$L^2(\mathrm{G}(F_0)\backslash \mathrm{G}(\mathbf{A}), \omega) = \widehat{\bigoplus}_{(\mathrm{M},\sigma) \in \mathfrak{D}(\mathrm{G},\omega)^\heartsuit} L^2_{(\mathrm{M},\sigma)}(\mathrm{G}(F_0)\backslash \mathrm{G}(\mathbf{A}), \omega)$$

of (3.1) *ibid.* in terms of a set $\mathfrak{D}(\mathrm{G}, \omega)^\heartsuit$ of classes of cuspidal data.

Proposition 4.4.3. *There exists* $\mu_\infty \in \mathcal{M}_\infty$ *such that*

- $\mu_\infty(\xi_\infty) = 1$*;*
- *for every cuspidal datum* (M, σ) *for* G' *that does not belong to* $\mathfrak{D}(\mathrm{G}, \omega, K, \mathfrak{T})^\heartsuit_{\xi_\infty !}$ *and for every* $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}), \mathbf{C})_K$*, the endomorphism* $R(\mu_\infty \star f)$ *of* $L^2(\mathrm{G}(F_0)\backslash \mathrm{G}(\mathbf{A})/K, \omega)$ *annihilates the subspace* $L^2_{(\mathrm{M},\sigma)}(\mathrm{G}(F_0)\backslash \mathrm{G}(\mathbf{A})/K, \omega)$*.*

4.4.2. *Multipliers annihilating strongly typical cuspidal data for a proper Levi subgroup.* We now specialize back to the setup of Proposition 4.3.2. We denote by ξ_∞° the infinitesimal character of

Π°_∞. We still freely use terminology and notation from [BPLZZ21, § 3] where not in conflict with ours.

Lemma 4.4.4. *Let* $(\mathrm{M}, \sigma) \in \mathfrak{C}(\mathrm{M}, 1)$ *satisfy*

$$\xi^{\mathrm{G}'}_{\sigma,\infty} = \xi^\circ_\infty.$$

Then for every finite place v at which σ_v is unramified, the Satake parameters of σ_v are algebraic.

Proof. We start with some preliminaries. If M_ν is a Levi subgroup of $\mathrm{GL}_{\nu/F}$ of type $(a_1, \dots, a_r)$, put $\psi_{\mathrm{M}_\nu} := \boxtimes_{i=1}^r |\det|^{\frac{\nu - a_i}{2}}$. If σ_ν is a representation of $\mathrm{M}_\nu(\mathbf{A})$, denote $\sigma^\natural_\nu := \sigma_\nu \otimes \psi_{\mathrm{M}_\nu}$. We extend the definitions to the case of Levi subgroups of G' in the obvious way. For M a Levi subgroup of G' and σ a cuspidal automorphic representation of $\mathrm{M}(\mathbf{A})$, let $\boxplus$ be the isobaric sum introduced (for general linear groups) in [Clo90, p. 85], and let $\boxplus^T$ be the twisted version $\boxplus^T \sigma := \boxplus \sigma^\natural$ of [Clo90, Définition 1.9]. The operation $\boxplus^T$ preserves fields of rationality and induces the direct sum operation on infinity types [Clo90, Lemme 3.9 (ii)].

Let a°_∞ be the infinity type associated with ξ°_∞, which is regular ([Clo90, Définition 3.12]). Since any direct summand of a°_∞ is also regular, it follows that $\sigma^\natural$ is regular algebraic. Thus by [Clo90, Théorème 3.13], it is defined over a number field. It follows that its algebraic twist σ has algebraic Satake parameters. □

For a characteristic-zero field L, define $\mathbb{T}^{\mathrm{spl},P}_L \subset \mathscr{H}(\mathrm{G}'(\mathbf{A}^{p\infty}), L)_{\prod_{v\nmid p} K^\circ_v}$ to be the spherical Hecke algebra of elements supported at a set of places of F_0 split in F and disjoint from P. If L is a subfield of $\mathbf{C}$, define $\mathcal{M}_{\infty,L}$ to be the L-linear subspace of $\mathcal{M}_\infty$ consisting of those μ such that $\mu(\xi^\circ_\infty) \in L$. We put

$$\mathcal{M}^{\mathrm{spl},P}_L := \mathscr{H}^{\mathrm{spl},P}_L \otimes_L \mathcal{M}_{\infty,L},$$

which is stable under multiplication and preserves $\mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ_K$. We have a surjective map

$$\begin{aligned} [-]^\circ \colon \mathcal{M}^{\mathrm{spl},P}_L &\longrightarrow \mathbb{T}'^{\mathrm{spl},P}_L \\ \mu &\longmapsto [\mu]^\circ \end{aligned}$$

given by the evaluation at ξ°_∞. It is clear that the action of $\mathcal{M}^{\mathrm{spl},P}_L$ on $\mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ$ factors through $[-]^\circ$.

We denote by $\mathscr{C}_K \subset \mathscr{C}$ the subset consisting of those Π' with $\Pi'^K \neq 0$.

Lemma 4.4.5. *Let* $\Pi' \in \mathscr{C}_K(\mathbf{C})$ *and let* (M, σ) *be a strongly* ξ°_∞*-typical cuspidal datum for* G' *with*

$$\mathrm{M} \neq \mathrm{G}'.$$

Denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$.

There exists an element $\mu \in \mathcal{M}^{\mathrm{spl},P}_{\overline{\mathbf{Q}}}$ satisfying:

- *for every $f' \in \mathcal{S}(\mathrm{G}'(\mathbf{A}), \mathbf{C})_K$, the endomorphism $R(\mu \star f')$ of $L^2(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A})/K)$ annihilates the subspace $L^2_{(M,\sigma)}(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A})/K)$;*
- *$\mu(\xi^P_{\Pi'}) = 1$.*

Proof. We refine the argument in the proof of [BPLZZ21, Proposition 3.17].[10] Denote by $L' \subset \overline{\mathbf{Q}}$ the field of definition of Π'. Note that the subspace $\mathfrak{a}_M^* \subseteq \mathfrak{h}'^*$ has a natural model $\mathfrak{a}_{M,\mathbf{Q}}^* \subseteq \mathfrak{h}'^*_{\mathbf{Q}}$ over $\mathbf{Q}$. We fix a rational splitting map $\ell\colon \mathfrak{h}'^*_{\mathbf{Q}} \to \mathfrak{a}^*_{M,\mathbf{Q}}$ and an element $\alpha \in \xi_\infty^\circ$. By Ramakrishnan's Proposition 2.5.2, for every $w \in \mathsf{W}'$, there is a finite place $v[w] \notin P$ of F_0, split in F, such that $\xi^{G'}_{\sigma_{s_w,v[w]}} \neq \xi_{v[w]}$ where $s_w \coloneqq \ell(w\alpha) - \ell(\alpha) \in \mathfrak{a}^*_{M,\mathbf{Q}}$.

This allows us to choose an element $\nu_w \in \mathscr{H}(\mathrm{G}'_{v[w]}, L')_{K'_{v[w]}}$ such that

$$\nu_w(\xi_{v[w]}) \neq \nu_w(\xi^{G'}_{\sigma_{s_w,v[w]}}).$$

By the process in the proof of [BPLZZ21, Proposition 3.17], it suffices to show that for every $w' \in \mathsf{W}'$, the value $\nu_w(\xi^{G'}_{\sigma_{s_{w'},v[w]}})$ is algebraic. By Lemma 4.4.4, the Satake parameters of $\sigma_{w',v[w]}$ are algebraic numbers. Since $s_{w'} \in \mathfrak{a}^*_{M,\mathbf{Q}}$, it follows that the Satake parameters of $\sigma_{s_{w'},v[w]}$ are all algebraic numbers as well, which implies that $\nu_w(\xi^{G'}_{\sigma_{s_{w'},v[w]}})$ is algebraic. □

We now extend the result to a finite set of cuspidal data and descend it to $\mathbf{Q}$. For $\mu \in \mathcal{M}^{\mathrm{spl},P}_{\overline{\mathbf{Q}}}$ and $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, we denote by $\tau.\mu \in \mathcal{M}^{\mathrm{spl},P}_{\overline{\mathbf{Q}}}$ a chosen lift of $\tau([\mu]^\circ)$.

Proposition 4.4.6. *Let $\mathfrak{D}$ be a finite set of strongly ξ°_∞-typical cuspidal data for G' such that*

$$\mathrm{M} \neq \mathrm{G}'$$

for every $(M,\sigma) \in \mathfrak{D}$. Then there exists a collection

$$(\mu^\iota_{\mathfrak{D}})_\iota \in \prod_{\iota\colon L \hookrightarrow \mathbf{C}} \mathcal{M}^{\mathrm{spl},P}_{\iota L}$$

satisfying:

(1) for every $f' \in \mathcal{S}(\mathrm{G}'(\mathbf{A}), \mathbf{C})_K$, every $\iota\colon L \hookrightarrow \mathbf{C}$, and every $(M,\sigma) \in \mathfrak{D}$, the endomorphism $R(\mu \star f'^\iota)$ of $L^2(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A})/K)$ annihilates the subspace $L^2_{(M,\sigma)}(\mathrm{G}'(F_0)\backslash \mathrm{G}'(\mathbf{A})/K)$;

(2) there exists a $[\mu_{\mathfrak{D}}]^\circ \in \mathbb{T}'^{\mathrm{spl},P}_L$ such that $[\mu^\iota_{\mathfrak{D}}]^\circ = \iota[\mu_{\mathfrak{D}}]^\circ$ for every $\iota\colon L \hookrightarrow \mathbf{C}$;

(3) $\mu^\iota_{\mathfrak{D}}(\xi^P_{\Pi^\iota}) = 1$ for every $\iota\colon L \hookrightarrow \mathbf{C}$.

Proof. We denote the elements of $\mathfrak{D}$ simply by σ in order to lighten the notation. For each $\iota\colon L \hookrightarrow \mathbf{C}$ and each $\sigma \in \mathfrak{D}$, let $\mu_{\sigma,\iota}$ be as provided by Lemma 4.4.5 applied to σ and Π^ι. Let L' be a Galois extension of $\mathbf{Q}$ in $\mathbf{C}$ containing ιL and the fields of definition of $\mu_{\sigma,\iota}$ for every $\sigma \in \mathfrak{D}$ and every $\iota\colon L \hookrightarrow \mathbf{C}$. Now take the collection

$$\mu^\iota_{\mathfrak{D}} \coloneqq \prod_{\tau \in \mathrm{Gal}(L'/\mathbf{Q})} \prod_{\sigma \in \mathfrak{D}} \tau.\mu_{\sigma,\tau^{-1}\iota}.$$

We verify that it satisfies the desiderata. The first one is enforced by the factors with $\tau = 1$. For the second one, by Galois theory we need to check that for each $\tau' \in \mathrm{Gal}(L'/\mathbf{Q})$, we have $\tau'.\mu^\iota_{\mathfrak{D}} = \mu^{\tau'\iota}_{\mathfrak{D}}$: indeed, by a change of variables

$$\tau'([\mu^\iota_{\mathfrak{D}}]^\circ) = \prod_{\tau \in \mathrm{Gal}(L'/\mathbf{Q})} \prod_{\sigma \in \mathfrak{D}} \tau'\tau([\mu_{\sigma,\tau^{-1}\iota}]^\circ) = \prod_{\tau \in \mathrm{Gal}(L'/\mathbf{Q})} \prod_{\sigma \in \mathfrak{D}} \tau([\mu_{\sigma,\tau^{-1}\tau'\iota}]^\circ) = [\mu^{\tau'\iota}_{\mathfrak{D}}]^\circ.$$

[10] With respect to the notation of *loc. cit.*, we omit the central character ω, which in our setup is necessarily trivial.

For the third property, it suffices to note that by construction we have

$$\tau.\mu_{\sigma,\tau^{-1}\iota}(\xi^P_{\Pi^\iota}) = \tau(\mu_{\sigma,\tau^{-1}\iota}(\xi^P_{\Pi^{\tau^{-1}\iota}})) = 1$$

for each τ and σ. □

4.4.3. *Proof of Proposition 4.3.2.* Let $f'_{1,\infty} \in \mathscr{H}(G'_\infty, \mathbf{Q})^\bullet$ be a nontrivial rational Gaussian, which exists by Proposition 4.1.3. By Proposition 2.5.2, we can find a finite set S_1 of split places of F_0, disjoint from P and the ramification set of Π, and an

$$f'_{1,S_1} \in \mathscr{H}(G_{S_1}, L)_{K_{S_1}},$$

such that $\Pi(f_{1,S_1}) \neq 0$ and $\Pi'(f_{1,S_1}) = 0$ for every $\Pi' \in \mathscr{C}_{K,L} - \Pi$. For each $\iota\colon L \hookrightarrow \mathbf{C}$, let $f_1'^\iota = \iota f'_{1,S_1} \otimes f'_{1,\infty} \otimes \otimes_{v\notin S_1\infty} f_v^\circ$.

Let μ_∞ and $\mathfrak{D} = \mathfrak{D}(\mathrm{G}, \omega, K, \mathfrak{T})^\heartsuit_{\xi_\infty!}$ be as provided by Proposition 4.4.3; the set $\mathfrak{D}$ is finite and it consists of of strongly ξ°_∞-typical cuspidal data for G'. Let $(\mu^\iota_{\mathfrak{D}})_\iota$, $[\mu_{\mathfrak{D}}]^\circ$ be as provided by Proposition 4.4.6 for $\mathfrak{D}$. Let

$$(f'^\iota) \coloneqq \mu^\iota_{\mathfrak{D}} \star \mu_\infty \star f_1'^\iota, \qquad f' \coloneqq [\mu_{\mathfrak{D}}]^\circ \star [\mu_\infty]^\circ \star f'_1$$

By construction, there is a set of split places $S \supset S_1$ disjoint from P such that for $? = \iota, \emptyset$, we have $f'^? = f'^?_{S\infty} \otimes \otimes_{v\notin S} f_v^\circ$ for some

$$f'^\iota_{S\infty}, \qquad f'_{S\infty}$$

that satisfy the desired properties. The proof is complete.

4.5. **Proofs of the rationality statements.** We will prove Proposition 4.2.2 (5) (recall that the other parts were proved at the end of § 4.2) and, as an interlude, Theorem 4.2.1.

4.5.1. *Global distribution.* The global orbital-integral distributions I_γ of part (4) are well-defined and we may define the distribution I of part (5) by its asserted geometric expansion:

$$I \coloneqq \sum_{\gamma\in \mathrm{B}'(F_0)} I_\gamma.$$

We show the sum is locally finite. We may assume that f' factors as $f' = f'^\infty \otimes f'_\infty$. By definition, the sum is supported in $\mathrm{B}'(F_0) \cap B'^\circ_\infty$. The invariant map (3.3.2) sends B' isomorphically to an closed subvariety of the affine space $\mathrm{Res}_{F/F_0}\mathbb{A}^{2n+1}$. Let $\Omega^\infty \subset (\mathbf{A}^\infty_F)^{2n+1}$ be the image of the support of $f'^\infty \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^\infty))$, which is compact. Let $\Omega_\infty \subset F^{2n+1}_\infty$ be the image of B'°_∞. By definition, this is contained in the image of the positive-definite unitary group G°_∞ under the invariant map, which is compact. Therefore the support of the sum is in bijection with a subset of the set $F^{2n+1} \cap \Omega^\infty\Omega_\infty$; as the first intersecting set is discrete and the second one is compact, the intersection is finite.

By construction, I has the geometric expansion asserted in part (5); by Prop. 3.3.6, it satisfies

$$I(f', \chi) = \kappa(\mathbf{1}_\infty)^{-1} I^{\mathbf{C}}(f'^\iota, \chi) \tag{4.5.1}$$

for any $\chi \in Y_L(\mathbf{C})$ with underlying embedding $\iota\colon L \hookrightarrow \mathbf{C}$, and any $f'^\iota \in \mathscr{H}(G'(\mathbf{A}), \mathbf{C})^\bullet_{\mathrm{reg}^+,\mathrm{qc}}$ mapping to $\iota f'$.

Remark 4.5.1. By linearity, we may extend the distributions I, I_Π, I_γ to distributions (defined over $\mathbf{Q}$ or, for I_Π, the field of definition of Π) on the space of locally constant functions $\ell\colon F_0^\times\backslash\mathbf{A}^\times/F_{0,\infty}^\times\to L$, in such a way that for every $\gamma'\in \mathrm{G}'_{\mathrm{rs}}(F_0)$ with image $\gamma\in\mathrm{B}'_{\mathrm{rs}}(F_0)$ and every $f'^\infty\otimes f'_\infty\in\mathscr{H}(\mathrm{G}(\mathbf{A}),L)^\circ$, we have

$$I_\gamma(f',\ell)=\frac{I_\gamma(f'_\infty)}{\kappa(\mathbf{1}_\infty)\kappa_\infty(\gamma',\mathbf{1})}\int_{\mathrm{H}_1(\mathbf{A}^\infty)}\int_{\mathrm{H}_2(\mathbf{A}^\infty)}f'^\infty(h_1^{-1}\gamma' h_2)\ell(h_1)\eta(h_2)\,\frac{d^\natural h_1 d^\natural h_2}{d^\natural g},$$

where $d^\natural x\coloneqq\prod_{v\nmid\infty}d^\natural x_v$, and the integral reduces to a finite sum.

4.5.2. *L-function.* We are now ready to prove the rationality of $\mathscr{L}$.

Proof of Theorem 4.2.1 (= Theorem A). For $\chi\in Y$, consider the set $\mathscr{H}(\mathrm{G}'(\mathbf{A}^\infty),L)^\circ_{\mathrm{reg}^+\Pi,\chi}$ of Gaussians with weakly plus-regular support that are adapted to (Π,χ) in the sense of § 4.2.2. It is non-empty by Corollary 4.3.4. For any $\chi\in Y_L$ and $f'\in\mathscr{H}(\mathrm{G}'(\mathbf{A}^\infty),L)^\circ_{\mathrm{rs},\Pi,\chi}$, we define

$$\mathscr{L}(\mathrm{M}_\Pi,\cdot)_{f'}\coloneqq\frac{4\cdot I(f',\cdot)}{(\otimes_v I_{\Pi_v})(f'_v,\cdot)}$$

away from the zeros of the denominator. Then for any $\chi\in Y_L(\mathbf{C})$ with underlying $\iota\colon L\hookrightarrow\mathbf{C}$ and any f'^ι as in § 4.2.2, we have

$$\mathscr{L}(\mathrm{M}_\Pi,\chi)_{f'}=\frac{4\cdot I^{\mathbf{C}}_{\Pi^\iota}(f'^\iota,\chi)}{\kappa(\mathbf{1}_\infty)(\otimes_{v\nmid\infty}I^{\mathbf{C}}_{\Pi^\iota_v}\otimes I^{\circ,\mathbf{C}}_{\Pi^\circ_\infty})(f'^\iota,\chi)}=\frac{\mathscr{L}^\infty(1/2,\Pi^\iota,\chi)}{\varepsilon(\frac{1}{2},\chi^2)^{\binom{n+1}{2}}}$$

where the first equality is (4.5.1), and the second one is (4.1.2). Thus the functions $\mathscr{L}(\mathrm{M}_\Pi,\cdot)_{f'}$ glue to the desired $\mathscr{L}(\mathrm{M}_\Pi,\cdot)$.

$\square$

4.5.3. *Spectral expansion.* We define

$$I_\Pi\coloneqq\frac{1}{4}\mathscr{L}(\mathrm{M}_\Pi)\cdot\prod_v I_{\Pi_v}$$

Then the spectral expansion of part (5) of Proposition 4.2.2 follows from the definition, Proposition 3.3.6, and (4.5.1). This completes the proof of the proposition.

4.6. **On the Ichino–Ikeda conjecture.** For expository purposes, we recall an outline of the proof of the following special case of the Ichino–Ikeda–Harris conjecture (in its most general form, the conjecture is now [BPCZ22, Theorem 1.1.6.1]), paying special attention to the rationality. The basic architecture of the proof of Theorem D in § 12 will be similar.

Let $V\in\mathscr{V}^{\circ,+}$ be a coherent pair, let $\mathrm{H}=\mathrm{H}^V\subset\mathrm{G}=\mathrm{G}^V$, and let $\mathscr{A}(\mathrm{G})^\circ\coloneqq\mathbf{Q}[\mathrm{G}(F_0\backslash\mathrm{G}(\mathbf{A})/\mathrm{G}(F_{0,\infty})]$, which is equipped with the Petersson product with respect to the measure dg. Let π be a cuspidal automorphic representation of $\mathrm{G}^V(\mathbf{A})$, trivial at infinity, over a number field L. Upon choosing an embedding in $\mathrm{Hom}(\pi,\mathscr{A}(\mathrm{G})^\circ_L)$ (which is an L-line by [KMSW], see [LTX+22, Proposition C.3.1 (2)]) we have an H-period

$$P_\pi\colon\pi\longrightarrow L\tag{4.6.1}$$

defined as in (3.4.8) (where the integration reduces to a finite sum). The unique embedding $\pi^\vee \hookrightarrow \mathscr{A}(\mathrm{G})_L^\circ$ that intertwines the natural duality $\pi \times \pi^\vee \to L$ with the Petersson product gives rise to the analogous period $P_{\pi^\vee}\colon \pi^\vee \to L$.

Theorem 4.6.1. *Assume that π is stable and cuspidal, and let $\Pi \coloneqq \mathrm{BC}(\pi)$. Then for all $\phi \in \pi$, $\phi' \in \pi^\vee$, we have*

$$P_\pi(\phi)P_{\pi^\vee}(\phi') = \frac{1}{4}\mathscr{L}(\mathrm{M}_\Pi, 0)\cdot \alpha(\phi,\phi').$$

We need a lemma to isolate π within the discrete automorphic spectrum.

Lemma 4.6.2. *Let L be a characteristic-zero field and let $V \in \mathscr{V}^\circ$. Let Σ be a finite set of isomorphism classes of discrete irreducible automorphic representations of $\mathrm{G}^V(\mathbf{A})$ over $\overline{L}$, trivial at infinity (Remark 2.5.8). Let $\overline{\pi} \in \Sigma$ and assume that $\overline{\pi} = \pi \otimes_L \overline{L}$ for some representation π over L. Let P be a finite set of places of F_0 containing all places at which π is ramified. Then there is a finite set S of split places of F_0, disjoint from P, a hyperspecial subgroup $K_S \subset G_S^V$, and an $f_S \in \mathscr{H}(G_S^V, L)_{K_S}$, such that*

$$\pi(f_S) = \mathrm{id}_\pi, \qquad \pi'(f_S) = 0 \textit{ for all } \pi' \in \Sigma \textit{ with } \mathrm{BC}(\pi') \neq \mathrm{BC}(\pi).$$

Proof. We view $L \hookrightarrow \mathbf{C}$ by fixing any embedding. By Remark 2.5.8, we have a set $\Sigma' = \{\mathrm{BC}(\pi') \mid \pi' \in \Sigma\}$ of isomorphism classes of isobaric, trivial-weight automorphic representations of $\mathrm{G}'(\mathbf{A})$ over $\overline{L}$; moreover $\mathrm{BC}(\overline{\pi})$ descends to a representation $\Pi \coloneqq \mathrm{BC}(\pi)$ over L. By Proposition 2.5.2 and Remark 2.5.8, there are a finite set of split places S disjoint from P, and an $f'_S \in \mathscr{H}(\mathrm{G}'_S, L)_{K'_S}$ (for $K'_S = \mathrm{G}'(\mathscr{O}_{F_{0,S}})$), satisfying $\Pi(f'_S) = \mathrm{id}$ and $\Pi'(f'_S) = 0$ for all $\Pi' \in \Sigma' - \mathrm{BC}(\overline{\pi})$. Then the $f_S \in \mathscr{H}(G_S^V, L)$ matching f'_S satisfies the desiderata. □

Proof of Theorem 4.6.1. (For more details on the argument, see the proof of Theorem D in § 12.) The formula extends by bilinearity to any $\tau \in \pi \otimes \pi^\vee$, and by multiplicity one if suffices to prove it for any τ not annihilated by α.

By Corollary 4.3.4 (with $\chi = \mathbf{1}$ and $P = \emptyset$) and Lemma 4.6.2 (with P a set of places such that the Gaussian produced by Corollary 4.3.4 is spherical away from $P\infty$), together with the explicit matching at split place of Lemma 3.5.6, we may construct matching Gaussians $f' \in \mathscr{H}(\mathrm{G}'(\mathbf{A}), L)^\circ$ and $f \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^\circ$ with weakly regular semisimple support that are adapted to $\Pi = \mathrm{BC}(\pi)$ and, respectively, π. Then for all v and matching $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_{0,v})$, $\delta \in \mathrm{B}_{\mathrm{rs}}(F_{0,v})$,

$$I_\gamma(f'_v) = J_\delta(f_v)$$

so that by (3.3.12) and (3.4.3),

$$J_\pi(f) = J(f) = \sum_{\delta \in \mathrm{B}_{\mathrm{rs}}(F_0)^\circ} J_\delta(f) = \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I_\gamma(f') = I(f') = I_\Pi(f'),$$

where $\mathrm{B}'_{\mathrm{rs}}(F_0)^\circ = \mathrm{B}'_{\mathrm{rs}}(F_0) \cap \mathrm{B}'(F_0)^\circ$. By the factorization of I_Π in Proposition 3.2.2 and the local spectral matching (together with the fact that $\prod_v \kappa(\pi_v^V) = 1$), we have

$$J_\pi(f) = I_\Pi(f') = \frac{1}{4\Delta_{\mathrm{H}}}\mathscr{L}(1/2, \Pi)\cdot \otimes_v J_{\pi_v}^{\mathbf{C}}(f) = \frac{1}{4}\mathscr{L}(\mathrm{M}_\Pi, \mathbf{1})\cdot \otimes_{v\nmid\infty} J_{\pi_v} J_{\pi_\infty}^\circ(f),$$

where we have used the definitions of $J^{\circ}_{\pi_\infty}$ in (4.1.3)and of $\mathscr{L}(\mathrm{M}_\Pi)$ in Theorem 4.2.1. This is equivalent to the desired formula for $\tau = \pi(f)$. $\square$

5. p-adic spherical characters

This section and the next one contain the local results needed, at p-adic places, in order to develop the p-adic relative-trace formula; in particular, the construction of a suitable family of Hecke measures. Remarkably, suitable members of these families can be used at any (split) place as the regular local test measures needed to prove the results of § 4.3.

Throughout this section, we fix a non-archimedean place v of F_0 and work in a local situation, dropping all subscripts v. We denote by $\mathscr{O}$ the ring of integers of the étale F_0-algebra F, by $\mathscr{O}_0$ the ring of integers of F_0, by $\varpi \in \mathscr{O}_0$ a chosen uniformizer, and we let $q_0 := |\mathscr{O}_0/\varpi\mathscr{O}_0|$, $q := |\mathscr{O}/\varpi\mathscr{O}|$.

5.1. **Group-theoretic preliminaries.** We introduce some notation and the group-theoretic foundations for the construction of the p-adic distribution.

5.1.1. *Notation.* If v splits in F, we fix an isomorphism $F \cong F_0 \times F_0$ and we expand our list of groups to include

$$\widetilde{G}'_0 := G'_{n,0} \times G'_{n+1,0}, \qquad H'_{1,0} := G'_{n,0},$$

so that $\widetilde{G}' = \widetilde{G}'_0 \times \widetilde{G}'_0$ and $H'_1 = H'_{1,0} \times H'_{1,0}$. We may then write elements of $G' = \widetilde{G}'/(F_0^\times)^2$ as $[g_1; g_2]$ with $g_i \in \widetilde{G}'_0$.

We will denote all conjugation actions by

$$x^g := g^{-1}xg.$$

Convention. Throughout this section, for $\nu \in \{n, n+1, \emptyset\}$ and $* \in \{\emptyset, 0\}$ we will define various subgroups and elements $\square_{\nu,*}$ of $G'_{\nu,*}$ (or $\widetilde{G}'_0$ for this 'pair' of subscripts). Unless otherwise specified, *we will define $\square_{\nu,*}$ in a way that makes sense for $\nu = n, n+1$, and tacitly stipulate that $\square_*$ is the product of $\square_{n,*}$ and $\square_{n+1,*}$*, if $* = 0$, or its image via $\widetilde{G}' \to G'$ if $* = \emptyset$. For the sake of uniformity, we introduce the notation

$$\dot{G}'_0 := \widetilde{G}'_0, \qquad \dot{G}' := G'.$$

5.1.2. *Some subgroups.* The lattice $\mathscr{O}_*^\nu \subset F_*^n$ induces an integral model for $\mathrm{G}'_{\nu,*}$ over $\mathscr{O}_0$, still denoted by $\mathrm{G}'_{\nu,*}$. Let $T_{\nu,*} \subset G'_{\nu,*}$ denote the diagonal torus, and let $W_{\nu,*}$ be the associated Weyl group, identified with the permutation matrices in $G'_{\nu,*}$. We denote by

$$w_{\nu,*} \in W_{\nu,*}$$

the antidiagonal matrix $(w_{\nu,*})_{ij} = \delta_{i,\nu+1-j}$.

5.1.3. *On the torus in $G'_{\nu,*}$.* We denote by $N_{\nu,*} \subset G'_{\nu,*}$ the set of upper-triangular unipotent matrices and by

$$N^\circ_{\nu,*} := N_{\nu,*} \cap \mathrm{G}'_{\nu,*}(\mathscr{O}_0).$$

Let $T^+_{\nu,*} \subset T_{\nu,*}$ be the sub-monoid consisting of those t such that $N^{\circ,t}_{\nu,*} := (N^{\circ}_{\nu,*})^t \subset N^{\circ}_{\nu,*}$, and $T^{++}_{\nu,*} \subset T^+_{\nu,*}$ the multiplicative subset ot those t such that

$$\bigcap_{r\geq 1} N^{\circ,t^r}_{\nu,*} = \{1\}.$$

Concretely, $T^+_{\nu,*}$ (respectively $T^{++}_{\nu,*}$) consists of matrices $\mathrm{diag}(t_1,\dots,t_\nu)$ with $t_i \in F_*^\times$ and $v(t_i/t_{i+1}) \geq 0$ (respectively >0) for all $1 \leq i \leq \nu-1$.

The group $T_{\nu,*}$ is equipped with the involution

$$\iota\colon t \longmapsto w^{-1}_{\nu,*} t^{-1} w_{\nu,*},$$

which preserves $T^+_{\nu,*}$ and $T^{++}_{\nu,*}$. We still denote by ι the resulting involution on $\mathbf{Q}_p[T_{\nu,*}]$.

We identify $\mathbf{Z}^\nu$ with the space of cocharacters of $T_{\nu,*}$ via

$$\lambda \longmapsto [x \longmapsto x^\lambda := \mathrm{diag}(x^{\lambda_1},\dots,x^{\lambda_\nu})] \in T_{\nu,0} \subset T_{\nu,*},$$

where the inclusion is diagonal.

We fix the elements

$$t_{\nu,*} := \varpi^{(\nu-1,\dots,0)} \in T^{++}_{\nu,*}, \qquad z_{\nu,*} = \varpi^{\nu-1} 1_\nu \in G'_{\nu,*}. \tag{5.1.1}$$

Then

$$t^\iota_{\nu,*} = z^{-1}_{\nu,*} t_{\nu,*}, \qquad t_{\nu,*} t^\iota_{\nu,*} = \varpi^{2\rho_\nu}$$

where $\rho_\nu \in \mathbf{Z}^\nu$ denotes half the sum of positive roots (with respect to $N_{\nu,*}$); concretely,

$$\rho_\nu := \frac{1}{2}(\nu-1,\nu-3,\dots,1-\nu) \in \frac{1}{2}\mathbf{Z}^\nu.$$

5.1.4. *Iwahori and deeper Iwahori subgroups.* The standard *Iwahori subgroup*

$$\mathrm{Iw}_{\nu,*} \subset G'_{\nu,*}$$

is the set of matrices in $\mathrm{G}'_{\nu,*}(\mathscr{O}_0)$ whose reduction modulo ϖ belongs to the image of the upper-triangular matrices in $\mathrm{G}'_{\nu,*}(\mathscr{O}_0)$. An Iwahori subgroup of $G'_{\nu,*}$ is one of the form $\mathrm{Iw}^g_{\nu,*}$ for some $g \in G'_{\nu,*}$. A *deeper Iwahori subgroup* of $G'_{\nu,*}$ is an open subgroup $K \subset G'_{\nu,*}$ satisfying $K \subset \mathrm{Iw}^g_{\nu,*}$ for some $g \in G'_{\nu,*}$. It is said to be *semistandard* if $N^\circ_{\nu,*} \subset K \subset \mathrm{Iw}^g_{\nu,*}$ for some $g \in N_{G'_{\nu,*}}(T_{\nu,*})$, the normalizer of $T_{\nu,*}$ in $G'_{\nu,*}$; it is said to be *standard* if $K \subset \mathrm{Iw}_{\nu,*}$ and $K \cap N_{\nu,*} = N^\circ_{\nu,*}$.

For $r \in \mathbf{Z}-\{0\}$, we define three families of subgroups

$$K^{[r]}_{\nu,*} \subset K^{\langle r\rangle}_{\nu,*} \subset K^{(r)}_{\nu,*} \tag{5.1.2}$$

of $\mathrm{G}'_{\nu,*}(\mathscr{O}_0)$ by

$$\begin{aligned} K^{(r)}_{\nu,*} &:= \mathrm{G}'_{\nu,*}(\mathscr{O}_0) \cap t^{-r}_{\nu,*}\mathrm{G}'_{\nu,*}(\mathscr{O}_0) t^r_{\nu,*}, \\ K^{\langle r\rangle}_{\nu,*} &:= \{g \in K^{(r)}_{\nu,*} \mid g_{ii} \in 1+\varpi^{|r|-1}\mathscr{O}_*,\ 1 \leq i \leq \nu\} \\ K^{[r]}_{\nu,*} &:= \{g \in K^{(r)}_{\nu,*} \mid g_{ii} \in 1+\varpi^{|r|}\mathscr{O}_*,\ 1 \leq i \leq \nu\}. \end{aligned}$$

They are standard deeper Iwahori subgroups whenever $r \geq 1$.

For $r \geq 1$, we say that a standard deeper Iwahori $K_{\nu,*}$ has *level* $\leq r$ if

$$K_{\nu,*} \supset K_{\nu,*}^{\langle r\rangle}.$$

5.1.5. *Iwahori–Weyl symmetries.* For $c \geq 1$, define

$$w_{*,\nu,c} \coloneqq w_{\nu,*} t_{\nu,*}^{c} \in N_{G'_{\nu,*}}(T_{\nu,*}) \subset G'_{\nu,*}.$$

Let $K \subset G'_{\nu,*}$ be a semistandard deeper Iwahori subgroup. We say that K is *symmetric* if $K^{w_{\nu,*,c}} = K$ for some $c \geq 1$ such that $K_*^{\langle c\rangle} \subset K$. If v splits in F and $* = \emptyset$, we say that K is *conjugate-symmetric of depth* $c = c(K) \geq 1$ if $K = K_0 \times K_0^{w_{\nu,0,c}}$ for some standard deeper Iwahori subgroup $K_0 \subset G'_{\nu,0}$ containing $K_{\nu,0}^{\langle c\rangle}$.

Remark 5.1.1. For $r \geq 1$, the subgroups $K_{\nu,*}^{[r]} \subset K_{\nu,*}^{\langle r\rangle} \subset K_{\nu,*}^{(r)} \subset G'_{\nu,*}$ are all symmetric, whereas for $\nu \geq 3$ Iwahori subgroups are not symmetric. On the other hand, conjugate-symmetric deeper Iwahori subgroups of G'_ν are obviously abundant.

5.1.6. *Iwahori–Hecke algebras and the operators U_t.* Let $K \subset G'_{\nu,*}$ be a semistandard deeper Iwahori subgroup. Define sheaves of $\mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p}$-algebras by

$$\mathscr{H}_{K,*}^{\dagger,+} \coloneqq C_c^\infty(K\backslash KT_{\nu,*}^{+}K/K, \mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p})\,dg \quad \subset \quad \mathscr{H}_{K,*} \coloneqq C_c^\infty(K_\nu\backslash G'_{\nu,*}/K_\nu, \mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p})\,dg.$$

The involution ι extends to $\mathscr{H}_{K,*}^{\dagger,+}$ by linearity. For $x \in G'_{\nu,*}$ and a semistandard deeper Iwahori subgroup $K' \subset K$, we define

$$[KxK] \coloneqq \mathrm{vol}(K, dg)^{-1}\mathbf{1}_{KxK}\,dg$$

in $\mathscr{H}_{K,*}$. The map

$$\begin{aligned} \mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p}[T_*^{+}/T_* \cap K] &\longrightarrow \mathscr{H}_{K,*}^{\dagger,+} \\ [t] &\longmapsto U_{t,K} \coloneqq [KtK]. \end{aligned} \tag{5.1.3}$$

is an $\mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p}$-algebra isomorphism. We define

$$\mathscr{H}_{K,*}^{\dagger} \coloneqq \mathscr{H}_{K,*}^{\dagger,+}[(U_{t,K}^{-1})_{t\in T^+}] \cong \mathscr{O}_{\mathrm{Spec}\,\mathbf{Q}_p}[T_*/T_* \cap K].$$

For $? = +, \emptyset$, we define $\mathscr{H}_{\nu,*}^{\dagger,?} \coloneqq \varprojlim_K \mathscr{H}_{K,*}^{\dagger,?}$, where the limit runs over the standard deeper Iwahori subgroups and the transition maps are $\star e_K \colon \mathscr{H}_{K',*}^{\dagger,+} \to \mathscr{H}_{K,*}^{\dagger,+}$. By Lemma 5.1.2 below, the limit $U_t \coloneqq \lim U_{t,K} \in \mathscr{H}_{\nu,*}^{\dagger,?}$ is well-defined. Concretely, if we denote

$$N_{\nu*}^{\circ,(r)} \coloneqq t_{\nu,*}^{r} N_{\nu,*}^{\circ} t_{\nu,*}^{-r}$$

we have

$$U_{t_{\nu,*}} = \sum_{x \in N_{\nu,*}^{\circ}/N_{\nu,*}^{\circ,(1)}} x t_{\nu,*}$$

as operators on the $N_{\nu,*}^{\circ}$-fixed points of any smooth $G'_{\nu,*}$-module.

5.1.7. *Multiplication rules in Iwahori–Hecke algebras.* We have the following basic result.

Lemma 5.1.2. *Let $K \subset G'_{\nu,*}$ be a deeper Iwahori subgroup, and define $\ell_K \colon K\backslash G'_{\nu,*}/K \to \mathbf{N}$ by $q_*^{\ell_K(g)} \coloneqq |KgK/K| = |K/K \cap gKg^{-1}|$. Then:*

(1) *We have* $KgKg'K = Kgg'K$ *in* $\mathscr{H}_K$ *if and only if* $\ell_K(gg') = \ell_K(g) + \ell_K(g')$.

(2) *Assume that* K *is standard. Then for all* $t' \in T^+_{\nu,*}$,

$$\ell_K(t'w_{\nu,*}) = \ell_K(t') + \ell_K(w_{\nu,*}), \qquad \ell_K(w_{\nu,*}t'^{-1}) = \ell_K(w_{\nu,*}) + \ell_K(t'^{-1}).$$

(3) *Assume that* K *is standard, and let* $K' \subset K$ *be a standard deeper Iwahori subgroup. Then for all* $t' \in T^+_{\nu,*}$,

$$Kt'wK' = Kt'wK, \quad e_K \star [K't'wK'] = [Kt'wK].$$

If moreover K *is of level* $\leq c$ *and* $t't^{-c}_{\nu,*} \in T^+_{\nu,*}$, *then*

$$K't'K = Kt'K, \quad [K't'K'] \star e_K = [Kt'K].$$

(4) *For all* $g \in G'_{\nu,*}$, *we have*

$$e_K \star ge_K = q_*^{-\ell_K(g)}[KgK].$$

Proof. Part (1) is well-known, see [How85, Ch. 2]. Consider the first equality of part (2), and drop all subscripts. By part (1), it is equivalent to prove $Kt'KwK = Kt'wK$. Since the quotient $K\backslash Kt'K$ is represented by lower-triangular matrices in K, it suffices to show that for such a matrix k, we have $t'kw \in Kt'wK$; fact, since $K \supset N^\circ$ we even have $kw \in wK$. The second equation follows from taking inverses in the identity $Kt'KwK = Kt'wK$.

Consider now part (3); we only prove the equalities as sets, from which the ones in Hecke algebras can be easily obtained. For the first equality, It suffices to prove that for any lower-triangular $k \in K$ we have $t'wk \in Kt'w$, which is clear since $t'wkw^{-1}t'^{-1} \in N^\circ \subset K$. For the second one, it suffices to prove that for any lower-triangular $kt' \in Kt'$ we have $kt' \in t'K$. In fact, by the assumptions we have $t'^{-1}tk \in K^{(c)} \cap K \subset K$. Part (4) follows from the definitions. □

5.1.8. *Twisting matrices.* Let $u \in (\mathscr{O}^\times_{F,p})^n$; we will take $u = (1, \ldots, 1)^{\mathrm{t}}$ to fix ideas in computations. Then we define the *twisting matrices*[11]

$$m_{n,*} \coloneqq 1_n, \qquad m_{n+1,*} \coloneqq \begin{pmatrix} w_n & u \\ & 1 \end{pmatrix}, \tag{5.1.4}$$

and for $r \geq 1$ we let

$$m_{\nu,*,r} \coloneqq m_{\nu,*}t^r_{\nu,*}$$

5.1.9. *Subgroups of* H'_1. Recall that by the convention introduced at the beginning ot this subsection, $\square_*$ denotes the (image of the) product of $\square_{n,*}$ and $\square_{n+1,*}$ in $\dot{G}'_*$. For $r \in \mathbf{Z}_{>0}$, let

$$\begin{aligned} K^{(r)}_{H,*} &\coloneqq m_*K^{(-r)}_*m_*^{-1} \cap \mathrm{G}'_{n,*}(\mathscr{O}_0) \\ &= m_*K^{[-r]}_*m_*^{-1} \cap \mathrm{G}'_{n,*}(\mathscr{O}_0) \qquad \subset \mathrm{G}'_{n,*}(\mathscr{O}_0) \subset H'_{1,*}, \end{aligned} \tag{5.1.5}$$

where the intersections are with respect to the usual diagonal embedding $H'_{1,*} \hookrightarrow \dot{G}'_*$.

[11] For their history, see [Jan] and references therein.

Remark 5.1.3. A simple computation shows that $K_{H,*}^{(r)}$ consists of the matrices h satisfying

$$\begin{cases} h_{ij} \in \varpi^{r|i-j|}\mathscr{O}_* \\ \sum_{j=1}^n h_{ij} \in 1 + \varpi^{ir}\mathscr{O}_* \end{cases} \tag{5.1.6}$$

for all $1 \leq i, j \leq n$. This description also shows the equality in (5.1.5). We may then compute that

$$\mathrm{vol}^\circ(K_{H,*}) \coloneqq q_*^{d(n)s}\,\mathrm{vol}(K_{H,*}^{(s)}) = \prod_{i=1}^n \frac{1-q_*^{-1}}{1-q_*^{-i}} \tag{5.1.7}$$

is a rational number independent of $s \geq 1$, and a p-unit.

We record the following easily checked property, for a later use: for all $r \geq 1$, we have

$$m_{*,r}^{-1} K_{H,*}^{(r)} m_{*,r} \subset K_*^{(2r)} \cap K_*^{[r]} \subset K_*^{\langle r+1\rangle}. \tag{5.1.8}$$

5.1.10. *Twisting identity.* We come to the key result of this subsection, which refines [Jan, Lemma 5.2] in the spirit of [Loe21, Lemma 4.4.1].

Lemma 5.1.4 (Twisting identity)**.** *Let $r \geq 1$ and let $K \subset \dot{G}'_*$ be a subgroup containing $K_*^{\langle r+1\rangle}$. For all $x \in N_*^\circ$, there exists $h_x \in K_{H,*}^{(r)}$ such that*

$$m_{*,r}xtK = h_x m_{*,r+1}K \tag{5.1.9}$$

Moreover, the map

$$\begin{aligned} N_*^{\circ,(1)}\backslash N_*^\circ &\longrightarrow K_{H,*}^{(r+1)}\backslash K_{H,*}^{(r)} \\ [x] &\longmapsto [h_x] \end{aligned}$$

is well-defined and a group isomorphism.

Proof. We omit the subscript '$*$' from the notation. It suffices to take $K = K^{\langle r+1\rangle}$. Consider the diagram

$$K_H^{(r+1)}\backslash K_H^{(r)} \xrightarrow{\alpha} K^{\langle -r-1\rangle}\backslash K^{[-r]} \xleftarrow{\beta} N^{\circ,(r+1)}\backslash N^{\circ,(r)} \xleftarrow{\gamma} N^{\circ,(1)}\backslash N^\circ$$

where $\alpha\colon h \mapsto m^{-1}hm$, β is induced by the inclusion $N^{\circ,(r)} \subset K^{[-r]}$, and γ is the isomorphism $x \mapsto t^r x t^{-r}$. All four quotients have cardinality $q^{d(n)}$ where $d(n) = (5.1.11)$, and by (5.1.5), α is well-defined and injective. Hence all three maps are isomorphisms, and the second statement of the lemma is proved with $[h_x] = \alpha^{-1}\circ\beta\circ\gamma([x])$. The first statement is then easily verified using $t^{-r-1}K^{\langle -r-1\rangle}t^{r+1} = K^{\langle r+1\rangle}$. $\square$

Corollary 5.1.5. *Let $r \geq 1$, and let $K_* = K_*^{\langle r+1\rangle} \subset \dot{G}'_*$. For all $s \geq r$, we have the identities*

$$\begin{aligned} m_{*,s}U_{t_*,K_*} &= \sum_{h\in K_{H,*}^{(s+1)}\backslash K_{H,*}^{(s)}} h m_{*,s+1} e_{K_*} && \text{in } C_c(\dot{G}'_*/K_*), \\ q_*^{sd(n)}\cdot m_{*,s}U_{t_*,K_*}^{-s} &= q_*^{(s+1)d(n)}\cdot \sideset{}{^{\mathrm{avg}}}\sum_{h\in K_{H,*}^{(s+1)}\backslash K_{H,*}^{(s)}} h m_{*,s+1} U_{t_*,K_*}^{-(s+1)} && \text{in } C_c(\dot{G}'_*/K_*)\otimes_{\mathscr{H}_{K_*}^{\dagger,+}} \mathscr{H}_{K_*}^\dagger, \end{aligned} \tag{5.1.10}$$

where $\sum^{\mathrm{avg}}$ denotes average.

5.1.11. *Volumes.* The volumes of $K_{\nu,*}^{(r)}$ and $K_{H,*}^{(r)}$ are constant multiples of $q_*^{-c(\nu)r}$, respectively $q_*^{-d(n)r}$, where

$$\begin{aligned} c(\nu) &:= \frac{1}{6}(\nu-1)\nu(\nu+1) \\ d(n) &:= \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) = c(n)+c(n+1). \end{aligned} \tag{5.1.11}$$

5.2. **p-adic periods.** Let $\Pi_{\nu,*}$ be a tempered representation of $G'_{\nu,*}$ over a field L of characteristic zero. Denote by $B_{\nu,*} \subset G'_{\nu,*}$ the upper-triangular Borel and by $\delta_{B_{\nu,*}} \colon T_{\nu,*} \to \mathbf{Q}^\times$ its modulus character.

5.2.1. *Finite-slope subspace.* Let

$$\Pi_{\nu,*}^\dagger \subset \Pi_{\nu,*}^{N^\circ},$$

be the subspace where $T_{\nu,*}^+$ acts invertibly. It has a structure of $\mathscr{H}_{\nu,*}^\dagger(L)$-module, and it is isomorphic as $L[T_{\nu,*}]$-module to the twisted Jacquet module $\delta_{B_{\nu,*}} \otimes (\Pi_{\nu,*})_{N_{\nu,*}}$ of $\Pi_{\nu,*}$ (see e.g. [Eme06, Proposition 4.3.4]).

We define $c(\Pi^\dagger)$ to be the minimal $c \in \mathbf{Z}_{\geq 1}$ such that $\Pi_{\nu,*}^\dagger \subset \Pi_{\nu,*}^K$ for some deeper Iwahori subgroup K of level $\leq c$.

Denote by $\widehat{T}$ the dual torus (as a scheme over L). For a subgroup $K \subset G'$ containing N°, let $\Pi^{K,\dagger}$ be the image of Π^K in $\Pi^\dagger$ under the U_t-eigen-projection, for any sufficiently positive t. Then there are decompositions into generalized $\mathscr{H}_K^\dagger$-eigenspaces

$$\Pi_{\overline{L}}^{K,\dagger} = \bigoplus_{\xi \in \widehat{T}_L(\overline{L})} \Pi_{\overline{L}}^{K,\dagger}[\xi],$$

and similarly $\Pi_{\overline{L}}^\dagger = \bigoplus \Pi_{\overline{L}}^\dagger[\xi]$.

If Π is a subquotient of a regular principal series (as defined in § 4.3.1) and ξ is a character ot $\widehat{T}$ occurring in $\Pi_{\overline{L}}^\dagger$, then by [Jan, Proposition 1.3 (ii)] (or its proof, applied to Π_n, Π_{n+1}), any Whittaker model of $\Pi_{L(\xi)}$ contains a unique vector

$$W_\xi \tag{5.2.1}$$

satisfying $W_\xi(1) = 1$ and $U_t W = \xi(t) W$ for all $t \in T^+$.

5.2.2. *Ordinary representations.* Suppose for this paragraph only that L is a finite extension of $\mathbf{Q}_p$, with algebraic closure denoted $\overline{\mathbf{Q}}_p$.

Definition 5.2.1. Let $N^\circ \subset K \subset G'$. We say that the tempered representation Π is K-*ordinary* (with respect to Π_∞°) if there is a character $\xi^\circ \in \widehat{T}_L(\overline{\mathbf{Q}}_p)$ occurring in $\Pi^{K,\dagger}$ (that is, such that $\Pi^{K,\dagger}[\xi^\circ] \neq 0$) satisfying

$$|\xi^\circ(t')| = 1$$

for all $t' \in T^+$ and the absolute value on $\overline{\mathbf{Q}}_p$.[12] We say that Π is ordinary if it is K-ordinary for sufficiently small $K \supset N^\circ$.

[12] This definition is adapted to the local components of automorphic representations of trivial weight at infinity; in general it would need to be modified, see [Hid98].

We call a character ξ° as above an *ordinary refinement* of Π. By the following proposition, an ordinary refinement is unique and defined over the field of definition of Π. We will then denote

$$\Pi^{\rm ord} \coloneqq \Pi_{\overline{\mathbf{Q}}_p}[\xi^\circ] \cap \Pi.$$

Proposition 5.2.2. *Let Π be an ordinary tempered representation of G' over L. Then Π is a subquotient of a regular principal series, the space $\Pi^\dagger$ is T_L-semisimple, and every $\xi \in \widehat{T}_L(\overline{\mathbf{Q}}_p)$ occurring in $\Pi^\dagger_{\overline{\mathbf{Q}}_p}$ satisfies $\dim_{\overline{\mathbf{Q}}_p} \Pi^\dagger_{\overline{\mathbf{Q}}_p}[\xi] = 1$ and is defined over L. Moreover the ordinary refinement ξ° is unique.*

Proof. This is essentially [Hid98, Corollary 8.3]. We recall the argument, working over $\overline{\mathbf{Q}}_p$ without signalling this in the notation. Let $W_{G'}$ be the Weyl group of G'. Recall form § 5.2.1 that $\Pi^\dagger \cong \delta_B \otimes \Pi_N$, the δ_B-twisted Jacquet module of Π. By Frobenius reciprocity, ξ occurs in $\Pi^\dagger$ if and only if Π embeds into the normalized induction $\mathrm{Ind}_B^G(\widetilde{\xi})$ where $\widetilde{\xi} \coloneqq \delta_B^{-1/2}\xi$. Now $\mathrm{Ind}_B^{G'}(\widetilde{\xi}) \cong \mathrm{Ind}_B^{G'}(\widetilde{\xi}^w)$ for all $w \in W_{G'}$. If $\xi^\circ_{|T^+}$ is valued in units, then the stabilizer of $\widetilde{\xi}^\circ$ in $W_{G'}$ is trivial, therefore its orbit consists of $|W_{G'}|$ distinct characters $\widetilde{\xi}$, and $\mathrm{Ind}_B^{G'}(\widetilde{\xi})$ is regular. By [BZ76, Theorem 5.21], we have $\dim \Pi_N \leq |W_{G'}|$, hence all the characters ξ occur with multiplicity one. The rationality assertion follows from the fact that the $\mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$-action on the set of occurring ξ preserves valuations. $\square$

Denote by

$$\mathrm{e}^{\rm ord} \colon \Pi^{N^\circ} \longrightarrow \Pi^{\rm ord}$$

the $\mathscr{H}^\dagger$-eigenprojector, and let $e_K^{\rm ord} \coloneqq e^{\rm ord} e_K$. Thus Π is K-ordinary if $e_K^{\rm ord}\Pi = \Pi^{\rm ord}$.

Lemma 5.2.3. *Suppose that Π is ordinary and unramified, and let $K \coloneqq \mathrm{G}'(\mathscr{O}_{F_0}) \subset G'$. Then $e_K^{\rm ord}\Pi = \Pi^{\rm ord}$.*

Proof. With notation as in the proof of Proposition 5.2.2, let ϕ_w be a generator of the line $\Pi^\dagger[\xi^\circ.w]$, where we define $\xi.w$ by $\widetilde{\xi.w} = \widetilde{\xi}^w$. Write a nonzero spherical vector $\phi_K \in \Pi^K$ as

$$\phi_K = \sum_{w \in W_G} c_w \phi_w \tag{5.2.2}$$

with $c_w \in L$. Then we need to show $c_1 \neq 0$. Now by [Cas80, Lemma 3.9], the expansion of [Cas80, Lemma 3.8] (where $\chi = \widetilde{\xi}^\circ$) is of the form (5.2.2), and there one has (see Theorem 3.1 *ibid.*) that $c_1 = 1$.

$\square$

5.2.3. *p-adic Rankin–Selberg period.* Let $\chi \in Y_L$. We define a functional on $\Pi^\dagger$ by

$$P^\dagger_{1,\Pi,\chi} \coloneqq \lim_{s\to\infty} P^\dagger_{1,\Pi,\chi,s}, \qquad P^\dagger_{1,\Pi,\chi,s} \coloneqq q^{d(n)s} P_{1,\Pi,\chi} \circ m_s U_t^{-s} \colon \Pi^\dagger \longrightarrow L(\chi). \tag{5.2.3}$$

Let $c(\chi)$ to be the conductor of χ in the usual sense: $c(\chi) = 0$ if χ is unramified and otherwise $c(\chi)$ is the minimal $c \in \mathbf{Z}_{\geq 1}$ such that $\chi_{|1+\varpi^c\mathscr{O}_0} = 1$.

Lemma 5.2.4. *The sequence in the limit* (5.2.3) *stabilizes as soon as $s \geq s_0 \coloneqq \max\{1, c(\Pi^\dagger) - 1, c(\chi)\}$.*

Proof. In the definition of $P^{\dagger}_{1,\Pi,\chi,s}(W)$ in § 3.2.2, we may first integrate over $K_H^{(s_0)}$; observing that χ is $(\det K_H^{(s_0)})$-invariant, the lemma results from (5.1.10). □

5.2.4. *p-adic pairing.* We define a (non-degenerate) pairing

$$\vartheta^{\dagger}_{\Pi} := \lim_{r\to\infty} \vartheta^{\dagger}_{\Pi,r}, \qquad \vartheta^{\dagger}_{\Pi,t}(\cdot,\cdot) := q^{d(n)}\vartheta_{\Pi}(w_r U_t^{-r}\cdot,\cdot) \colon \Pi^{\dagger}\times\Pi^{\vee,\dagger} \longrightarrow L.$$

It is easy to show, using the symmetry of $K^{\langle c\rangle}$, that the sequence in the limit stabilizes as soon as $r\geq c(\Pi^{\dagger})$.

Remark 5.2.5. For all $t'\in T^{+}$ the $\vartheta^{\dagger}_{\Pi}$-adjoint of $U_{t'}$ is $U_{t'^{\iota}}$. Thus for every character ξ, the pairing $\vartheta^{\dagger}_{\Pi}$ yields a perfect pairing on $\Pi^{\dagger}[\xi]\times\Pi^{\vee,\dagger}[\xi^{\iota}]$ and moreover, for all $r\geq c(\Pi^{\dagger})$ and every semistandard deeper Iwahori subgroup $K\subset G'$, a perfect pairing on $\Pi^{K,\dagger}[\xi]\times\Pi^{\vee,K^{w_r},\dagger}[\xi^{\iota}]$.

5.2.5. *p-adic Flicker–Rallis period.* Suppose that v splits in F and that Π_{ν} is in the image of the local base change map (2.4.1); in other words, we may write $\Pi_{\nu}\cong\Pi_{\nu,0}\boxtimes\Pi^{\vee}_{\nu,0}$ for some representation Π_{ν} of $\widetilde{G}_0$. We define

$$P^{\dagger}_{2,\Pi} := \lim_{r\to\infty} P_{2,\Pi,r}, \qquad P_{2,\Pi,r} := q_0^{d(n)r} P_2\circ[1;w_{0,r}]U^{-r}_{[1;t_0]} \colon \Pi^{\dagger}\longrightarrow L.$$

The sequence in the limit stabilizes as soon as $r\geq c(\Pi^{\dagger})$.

5.2.6. *p-adic Rankin–Selberg periods at U_t-eigenvectors.* Identify Π_{n+1} (respectively Π_n) with its ψ- (respectively $\overline{\psi}$-) Whittaker model, and Π with their product. Suppose that Π is a subquotient of a regular principal series.

Let $\xi\in\widehat{T}$ be a character occurring in $\Pi^{\dagger}_{L}$; by the argument in the proof of Proposition 5.2.2, we have $\dim_{L(\xi)}\Pi^{\dagger}_{L(\xi)}\Pi^{\dagger}[\xi]=1$. We denote by $W_{\xi}\in\Pi_{L(\xi)}$ the element of (5.2.1).

Define

$$e(\Pi,\xi,\chi) := P^{\dagger}_{1,\Pi,\chi}(W_{\xi}) \in L(\xi,\chi). \tag{5.2.4}$$

Liu and Sun have recently proved an explicit formula for this term. Write $\widetilde{\xi}=\widetilde{\xi}_n\boxtimes\widetilde{\xi}_{n+1}$, and for $1\leq i\leq\nu$, let $\widetilde{\xi}_{\nu,i}\colon F^{\times}\to L(\xi)^{\times}$ be the restriction of $\widetilde{\xi}_{\nu}$ to the i^{th} component of $T_{\nu}=(F^{\times})^{\nu}/F_0^{\times}$. For any character ξ' of $F_v^{\times}$ and any place $w|v$ of F, denote by $\xi'_w := \xi_{|F_w^{\times}}$; denote by $N_w\colon F_w^{\times}\to F_0^{\times}$ the norm map. Finally, we denote by

$$\gamma(s,\xi'_{F,w},\psi_{F,w})^{-1} := L(s,\xi'_w)/\varepsilon(s,\xi'_w,\psi_{F,w})L(1-s,\xi'^{-1}_w)$$

the inverse Deligne–Langlands γ-factor of a character of $\xi'_w\colon F_w^{\times}\to\mathbf{C}^{\times}$. If

$$|\cdot|^{1/2}\xi'_k, \quad |\cdot|^{1/2}\xi''_k\colon F_w^{\times}\hookrightarrow L'^{\times}\subset\mathbf{C}$$

(for $1\leq k\leq N$) are characters with $\prod_{k=1}^{N}|\cdot|^{1/2}\xi'_k=\prod_{k=1}^{N}|\cdot|^{1/2}\xi''_k$, then it is easy to see that $\prod_{k=1}^{N}\gamma(1/2,\xi'_k,\psi_{F,w})/\gamma(1/2,\xi''_k,\psi_{F,w})$ belongs to L'. Thus the following expression gives an element of $L(\xi,\chi)$ (unless some division by zero has occurred).

Define

$$\hat{e}(\Pi,\xi,\chi) := \frac{\varepsilon(\frac{1}{2},\chi^2,\psi)^{\binom{n+1}{2}}}{L(\frac{1}{2},\Pi\otimes\chi)}\prod_{i+j\leq n}\prod_{w|v}\gamma(\frac{1}{2},\chi\circ N_w\cdot\widetilde{\xi}_{n,i,w}\widetilde{\xi}_{n+1,j,w},\psi_{F,w})^{-1}.$$

Proposition 5.2.6. *We have*

$$e(\Pi, \xi, \chi) = \pm \hat{e}(\Pi, \xi, \chi).$$

Proof. This is equivalent to the identity of [LiSu, Proposition 11.18], where the sign $\pm$ is explicit. $\square$

The key consequences for us will be Propositions 5.3.6 and Remark 5.3.3 below, both derived from the following lemma. We temporarily restore the notation of the rest of the paper.

Lemma 5.2.7. *Suppose that Π_v is a regular irreducible principal series that is the local component of a representation Π as in Theorem A. For every character ξ_v of T_v occurring in $\Pi_v^\dagger$ and every finite-order character χ_v of $F_{0,v}^\times$, we have*

$$\hat{e}(\Pi_v, \xi_v, \chi_v) \in L(\Pi, \xi, \chi)^\times.$$

Proof. By [Car14, Theorem 1.1], for each place w of F, the semisimple Weil–Deligne representation attached to $\rho_{\Pi|G_{F_w}}$ (cf. (1.2.1)) is

$$r_{\Pi,w} = \bigoplus_{1\leq i\leq n, 1\leq j\leq n+1} |\cdot|^{1/2}\widetilde{\xi}_{n,i,w}\widetilde{\xi}_{n+1,j,w},$$

and it is strictly pure of some weight that is independent of w (here we identify a character of $F_w^\times$ with its correspondent on the Weil group of F_w via class field theory). By considering $\det r_{\Pi,w}$ at an inert place w we then see that the weight must be -1. Thus for each (i, j), the character $|\cdot|^{1/2}\widetilde{\xi}_{n,i,w}\widetilde{\xi}_{n+1,j,w}$ is either ramified (so that its γ-factor is an ε-factor, hence nonzero), or it is an unramified character whose value at a uniformizer of F_w is a Weil q_w-number of weight -1, which again implies the nonvanishing of each term in the γ-factors of Hypothesis 5.2.6. $\square$

5.3. ***p*-adic spherical characters.** We go back to the notation of the rest of this section.We say that a subgroup $K \subset G'$ is *convenient* if either $K = \mathrm{G}'(\mathscr{O}_0)$, or v splits in F and K is a conjugate-symmetric deeper Iwahori as defined in § 5.1.5 (henceforth: a CSDI).

5.3.1. *Finite-slope spherical character.* Let $K \subset G'$ be a convenient subgroup. We define a distribution

$$I_{\Pi,K}^\dagger \in \mathscr{O}(\mathscr{H}_L^\dagger \times Y_v)$$

by

$$I_{\Pi,K}^\dagger(f^\dagger, \chi) \coloneqq \begin{cases} \mathrm{Tr}_{\vartheta_\Pi}^{P_{1,\Pi,\chi}^\dagger \otimes P_{2,\Pi}}(\Pi(f^\dagger e_K)) & \text{if } K = G'(\mathscr{O}_0), \\ \\ \mathrm{Tr}_{\vartheta_\Pi^\dagger}^{P_{1,\Pi,\chi}^\dagger \otimes P_{2,\Pi}^\dagger}(\Pi(f^\dagger e_K)) & \text{if } K \text{ is a CSDI}. \end{cases}$$

Remark 5.3.1. The second definition is the 'correct' one from the p-adic point of view. The first one is made because, first, in the arithmetic side the geometry will compel us to work at spherical level; and second, we have not investigated the analogue of the notion of 'conjugate-symmetric' in the nonsplit case.

5.3.2. *Eigen-decomposition.* Suppose that Π is a subquotient of a regular principal series, and denote by $\Xi_K(\Pi)$ the set of characters of T occurring in $\Pi^{K,\dagger}$.

– If $K = \mathrm{G}'(\mathscr{O}_{F_0})$ and Π is an unramified principal series, let $W_0 \in \Pi^K$, $W_0^{(\vee)} \in \Pi^{(\vee),K}$ be generators normalized by $W_0^{(\vee)}(1) = 1$, write $W_0 = \sum_\xi \lambda_\xi W_\xi$, and let

$$c_K(\Pi, \xi) \coloneqq \lambda_\xi P_{2,\Pi}(W_0^\vee)/\vartheta_\Pi(W_0, W_0^\vee) = \lambda_\xi. \tag{5.3.1}$$

where the second equality follows from Remark 3.2.1. By the same proof as for Lemma 5.2.3, we have $\lambda_\xi \neq 0$ for all $\xi \in \Xi_{\mathrm{Iw}}(\Pi)$. (An explicit formula for λ_ξ could be obtained from combining the formulas cited in that proof with the Casselman–Shalika formula [CS80] and the formulas [Ree93, Proposition 3.1] for Whittaker U_t-eigenfunctions.)

– If K is a conjugate-symmetric deeper Iwahori, define

$$c_K(\Pi, \xi) \coloneqq c(\Pi, \xi) \coloneqq \frac{P^\dagger_{2,\Pi}(W_{\xi^\iota})}{\vartheta^\dagger_\Pi(W_\xi, W_{\xi^\iota})} \in L(\xi). \tag{5.3.2}$$

Here, the denominator is nonvanishing since U_t is ϑ-adjoint to U_{t^ι}. Similarly, the numerator is nonvanishing if and only if Π_v is hermitian.

Then, in either case, by the definitions we have a decomposition

$$I^\dagger_{\Pi,K}(f^\dagger, \chi) = \sum_{\xi \in \Xi_K(\Pi)} I^\dagger_{\Pi,K,\xi}(f^\dagger, \chi) \tag{5.3.3}$$

where

$$I^\dagger_{\Pi,K,\xi}(f^\dagger, \chi) \coloneqq \xi(f^\dagger) c_K(\Pi, \xi) e(\Pi, \xi, \chi). \tag{5.3.4}$$

5.3.3. *Ordinary spherical character.* Suppose for this paragraph only that L is a finite extension of $\mathbf{Q}_p$ and that there is an $\mathscr{O}_L$-lattice $\Pi_{\mathscr{O}_L} \subset \Pi$ that is stable under $\mathscr{H}^\dagger$. Then we have Hida's description

$$\mathrm{e}^{\mathrm{ord}} = \lim_{N\to\infty} U_t^{N!}$$

for the action of the ordinary projector on Π.

Remark 5.3.2. The above assumption holds whenever Π is a local component of a global representation in $\mathscr{C}_L$. Indeed, representations in $\mathscr{C}_L$ can be realised in the Betti cohomology of the locally symmetric space attached to G', and the cohomology with coefficients in $\mathscr{O}_L$ gives a natural integral structure stable under the Hecke operators; see [Hid98] for more details.

For any convenient $K \subset G'$, we then define

$$I^{\mathrm{ord}}_{\Pi,K}(\chi) \coloneqq \lim_{N\to\infty} I^\dagger_{\Pi,K}(U_t^{N!}, \chi).$$

If Π is ordinary, we denote

$$\begin{aligned} e(\Pi, \chi) &\coloneqq e(\Pi, \xi^\circ, \chi) \in L(\chi), \\ c_K(\Pi) &\coloneqq c_K(\Pi, \xi^\circ) \in L^\times \end{aligned} \tag{5.3.5}$$

where the right-hand sides are defined in (5.2.4), (5.3.1), (5.3.2).

Remark 5.3.3. If Π_v, χ_v are as in Lemma 5.2.7 and moreover Π_v is ordinary, it follows from that lemma and Proposition 5.2.6 that $e(\Pi_v, \chi_v)$ and $c_{K_v}(\Pi)$ are nonzero.

Corollary 5.3.4. *Suppose that Π admits an $\mathscr{O}_L$-stable lattice. Then for every $\chi \in Y_L$ and every convenient $K \subset G'$, we have*

$$I^{\mathrm{ord}}_{\Pi,K}(\chi) = \begin{cases} c_K(\Pi)e(\Pi,\chi) & \textit{if } \Pi \textit{ is } K\textit{-ordinary} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from (5.3.4). □

5.3.4. *Relation to the character I_Π.* Let Π be a tempered irreducible representation of G', let $\chi\colon F_0^\times \to L^\times$ be a smooth character, let $K \subset G'$ be a convenient subgroup, and let $s \geq 1$. We say that s is sufficiently positive for χ (respectively for K) if $s \geq \max\{1, c(\chi)\}$ (respectively K contains a deeper Iwahori of level c with[13] $s \geq 2c$). We say that $f^\dagger \in \mathscr{H}^\dagger$ is *sufficiently positive for Π (respectively for s_0, for χ, for K)* if $f^\dagger \Pi \subset \Pi^\dagger$ (respectively if $U_t^{-s} f^\dagger$ belongs to $\mathscr{H}^{\dagger,+}$ for $s = s_0$ or some s that is sufficiently positive for χ, respectively for K).

It is clear that if $f^\dagger$ is in the span of $\{U_t \mid t \in T^{++}\}$ and s and Π are given, then some power of $f^\dagger$ is sufficiently positive for both s and Π.

Lemma 5.3.5. *For every s that is sufficiently positive for K and χ and every $f^\dagger$ that is sufficiently positive for s and Π, we have*

$$I^\dagger_{\Pi,K}(f^\dagger,\chi) = I_\Pi(f',\chi)$$

where

$$f' = f'_{K,s} := \begin{cases} q^{d(n)s} \cdot m_s U_t^{-s} f^\dagger e_K & \textit{if } K = \mathrm{G}(\mathscr{O}_0), & (5.3.6a) \\ q_0^{d(n)(2s-c)} \cdot m_s U_t^{-s} f^\dagger e_K U^c_{[t_0;1]}[w_{0,c}^{-1};1] & \textit{if } K \textit{ is a CSDI of depth } c. & (5.3.6b) \end{cases}$$

Proof. The first case is clear. Consider the second case, dropping the subscripts Π and K from the notation. Let $\Pi_{\dagger,K} := w_c \Pi^{\dagger,K}$, and let $\vartheta_| \colon \Pi_{\dagger,K} \otimes \Pi^{\vee,\dagger,K} \to L$ be the restriction of $\vartheta\colon \Pi\otimes\Pi^\vee \to L$, which is still a perfect pairing. By Lemma 2.6.3 (using, in order, part (1), part (2), and part (1) together with part (3)),

$$\begin{aligned} I^\dagger(f^\dagger) &= q_0^{d(n)c}\, \mathrm{Tr}_{\vartheta^\dagger}^{P_1^\dagger \otimes P_2[1;w_{0,c}]}(\Pi(f^\dagger e_K U^{-c}_{[1;t_0^\iota]})) \\ &= q_0^{-d(n)c}\, \mathrm{Tr}_{\vartheta_|}^{P_1^\dagger \otimes P_2[1;w_{0,c}]}(\Pi(f^\dagger e_K U^c_{[t_0;t_0/t_0^\iota]} w_c^{-1})) \\ &= q_0^{d(n)(2s-c)}\, \mathrm{Tr}_\vartheta^{P_1\otimes P_2}(\Pi(m_s U_t^{-s} f^\dagger e_K U^c_{[t_0;z_0]} w_c^{-1}[1;w_{0,c}^{-1}])) = I(f'), \end{aligned}$$

where f' is as asserted. □

5.3.5. *A non-vanishing result.* Unlike the rest of this section, the following result is not used for the p-local theory of the p-adic relative-trace formula, but rather as an input to Proposition 4.3.1 (3).

Proposition 5.3.6. *Let Π, χ, K be as in § 5.3.4. Suppose that v is split, Π is a regular principal series, and K is a conjugate-symmetric deeper Iwahori such that $\Pi^{K,\dagger} \neq 0$. Then*

[13] In fact, at least if K is an Iwahori subgroup or one of the subgroups (5.1.2) with $r = c \geq 1$, the weaker condition $s \geq c$ will suffice; this is only used in the application of Lemma 5.1.2 (3) in the proof of Lemma 6.2.1.

there exists an $f^\dagger \in \mathscr{H}^\dagger$ *that is sufficiently positive for* Π, χ, K, *such that the Hecke measure* $f' := f'_{K,s} =$ (5.3.6b) *satisfies*

$$I_\Pi(f'_{K,s}, \chi) \neq 0.$$

Proof. Let f'_N correspond to $f^\dagger = U_t^N$ for some sufficiently large integer N. We may and do extend scalars from L to $\mathbf{C}$; we do not alter the notation. By (5.3.3), we have

$$I_\Pi(f'_N, \chi) = \sum_{\xi \in \Xi_K(\Pi)} \xi(t)^N c_K(\Pi, \xi) e(\Pi, \xi, \chi).$$

Order the characters ξ occurring in $\Pi^\dagger$ as $\xi_1, \ldots, \xi_r$; then we may write $I_\Pi(f'_N, \chi) = a_N x$ where $x = (m_{\xi_i} c_K(\Pi, \xi_i) e(\Pi, \xi_i, \chi))_i \in \mathbf{C}^r$ and $a_N^{\mathrm{t}} = (\xi_i(t)^N)_i \in \mathbf{C}^r$. Now all entries of the vector x are nonzero by Proposition 5.2.6 and Lemma 5.2.7, and the Vandermonde matrix A with rows $a_N, \ldots a_{2N}, \ldots, a_{rN}$ is invertible. Hence there is some $1 \leq i \leq r$ such that $0 \neq a_{iN} x = I_\Pi(f'_{iN}, \chi)$, as desired. □

6. p-ADIC ORBITAL INTEGRALS

We define and study certain local orbital integrals matching the spherical characters just defined. After establishing their p-adic boundedness (as the character χ varies, in a suitable sense), the main result of this section, Proposition 6.1.2, says that in case K is a CSDI, our orbital integrals have plus-regular support, and it explicitly computes the values at all orbits.

We continue with the notation of the previous section.

6.1. **Definition and statement of the main result.**

Let $K \subset G'$ be a convenient subgroup.

6.1.1. *Definition.* For $f^\dagger$ sufficiently positive (depending on χ) and $\gamma \in B'$, let s and $f'_{K,s}$ be as in Lemma 5.3.5, and define

$$I^\dagger_{\gamma,K}(f^\dagger, \chi) := L_\gamma(\chi) I_\gamma(f'_{K,s}, \chi) = \iota^{-1} I_\gamma^{\natural,\mathbf{C}}(f'_{K,s}, \iota\chi) \tag{6.1.1}$$

whenever the last term (defined with respect an embedding $\iota\colon L \hookrightarrow \mathbf{C}$) is an absolutely convergent orbital integral (that is, it reduces to a finite sum); this is the case when γ is regular semisimple or $f'_{K,s}$ has plus-regular support.

Lemma 6.1.1. *Let* $\gamma \in B'$ *and let* $f \in \mathscr{H}^\dagger(L)$ *be such that the expressions* $I^\dagger_{\gamma,K}(f^\dagger, -)$ *are defined. Then:*

(1) the right hand side of (6.1.1) *is independent of the choices of an* s *that is sufficiently positive for* K *and* χ, *so long as* $f^\dagger$ *is sufficiently positive for* s.

(2) Suppose that $f^\dagger \in \mathscr{H}^\dagger(\mathscr{O}_L)$. *For any* s_0 *that is sufficiently positive for* K *such that* $f^\dagger$ *is sufficiently positive for* s_0, *the map*

$$\chi \longmapsto I^\dagger_{\gamma,K}(f^\dagger, \chi)$$

extends by linearity to a functional $C^\infty(F_0^\times/(1 + \varpi^{s_0}\mathscr{O}_0), \mathscr{O}_L) \to k\mathscr{O}_L$ *for some constant* $k \in \mathbf{Q}^\times$ *depending only on* K.

Proof. If K is a CSDI, let $c = c(K)$ be the depth of K, and let $T = U^{c}_{[t_0;1]}[w_{0,c}^{-1}; 1]$; if K is unramified, let $c = 0$ and $T = \mathrm{id}$. For a Hecke operator $T' = \sum_i \lambda_i \operatorname{vol}(A_i)^{-1}\mathbf{1}_{A_i} dg \in \mathscr{H}_K$, denote $\mathbf{1}[\gamma' \in T'] \coloneqq \sum \lambda_i \mathbf{1}_{A_i}(\gamma')$.

Let s_0 be sufficiently positive for K and χ. The integrand in the explicit expression for $I^{\dagger}_{\gamma,K}(f^{\dagger})$ equals

$$q_0^{d(n)(2s-c)}\chi(h_1)\mathbf{1}[\gamma h_2 \in h_1 m_s U_t^{-s} f^{\dagger} e_K T \tag{6.1.2}$$

and it is $K_H^{(s_0)} \times (K \cap H_2')$-invariant by (5.1.8). Integrating first over over $K_H^{(s_0)} \subset H_1'$, the relation (5.1.10) shows that (6.1.2) is independent of $s \geq s_0$. We also see that if

$$k = q_0^{-d(n)c(K)} \operatorname{vol}(K \cap H_2') \operatorname{vol}^{\circ}(K_H) \tag{6.1.3}$$

where $\operatorname{vol}^{\circ}(K_H) = (5.1.7)$. Then the functional $I^{\dagger}_{\gamma,K}(f^{\dagger}, -)$ sends $C^{\infty}(F_0^{\times}/(1 + \varpi^{s_0}\mathscr{O}_0), \mathscr{O}_L)$ to $k\mathscr{O}_L$.

$\square$

6.1.2. *Main result and application to regular test Hecke measures.* When K is a conjugate-symmetric deeper Iwahori and $f^{\dagger}$ is sufficiently positive, the following key result asserts that the associated $f'_{K,s}$ has plus-regular support and the p-adic orbital integral may be explicitly computed. A remarkable fact is that its value is independent of χ.

By linearity, it suffices to study the case $f^{\dagger} = U_{t'}$ for some $t' \in T^{++}$.

Proposition 6.1.2. *Let $K = K_0 \times K_0^{w_c} \subset G'$ be a conjugate-symmetric deeper Iwahori (in particular, v splits in F). Assume that $f^{\dagger} = U_{t'} \in \mathscr{H}^{\dagger}$ for some $t' \in T^{++}$. Then:*

(1) for every s that is sufficiently positive for K such that $f^{\dagger}$ is sufficiently positive for s, the support of

$$f'_{K,s} = (5.3.6\text{b})$$

is contained in G'_{reg^+}; moreover, $f'_{K,s}$ matches an $f_{K,s} \in \mathscr{H}(G', L)$ that is regularly supported and bi-invariant under a subgroup conjugate to K_0;

(2) there exists a compact subset

$$B_K^{\dagger}(f^{\dagger}) \subset B'$$

with the following property: for every smooth character χ of $F_0^{\times}$ such that $f^{\dagger}$ is sufficiently positive for χ and K, we have

$$I^{\dagger}_{\gamma,K}(f^{\dagger}, \chi) = \begin{cases} k' & \text{if } \gamma \in B_K^{\dagger}(f^{\dagger}) \\ 0 & \text{if } \gamma \notin B_K^{\dagger}(f^{\dagger}), \end{cases}$$

where $k' = kq_0^{-\ell_{K_0}(w_0)}$, with $k = (6.1.3)$.

The proof of Proposition 6.1.2 will occupy the rest of this section, which may be skipped on a first reading.

The first part allows to complete the proof of Proposition 4.3.1.

Proof of Proposition 4.3.1 (3). We drop the subscript v from the notation of the statement of the proposition. Recall that we need to find an $f'_\pm \in \mathscr{H}(G', L)$ that is supported in $G'_{\mathrm{reg}\pm}$ and adapted to a given pair (Π, χ).

We may take f'_+ to be the element $f'_{K,s}$ = (5.3.6b) associated to the data of: a conjugate-symmetric deeper Iwahori K such that $\Pi^K \neq 0$; an integer $s \geq 1$ that is sufficiently positive for χ and K; and an $f^\dagger$ that is sufficiently positive for Π, χ, K. Then f'_+ is adapted to (Π, χ) by Proposition 5.3.6, and it has plus-regular support by Proposition 6.1.2 (1). If Π is unramified we can take K to be an Iwahori subgroup, hence (again by Proposition 6.1.2 (1)) we have that f'_+ matches an f_+ that is biinvariant under an Iwahori subgroup.

We may take $f'_- := f'^{\diamond}_+$ for the involution $g^\diamond = g^{\mathrm{c},-1,\mathrm{t}}$ of Remark 3.3.1. By that remark, f'_- is minus-regular, and it is clear that its matching f_- is bi-invariant under an Iwahori subgroup if f_+ is. Moreover, $I_\Pi(f_-, \chi) = I_{\Pi^\diamond}(f_+, \chi^{-1})$ for $\Pi^\diamond(g) := \Pi(g^\diamond)$; since $\Pi^\diamond \cong \Pi^{\mathrm{c},\vee} \cong \Pi$, this expression is non-vanishing too. □

6.2. **Reduction to p-adic linear algebra.** We start working towards the proof of Proposition 6.1.2, of which we retain all the assumptions. The proof of part (2) relies on some reductions in the present subsection and § 6.3, and on two auxiliary inductive lemmas in § 6.4, 6.6, and it is completed in § 6.7. The proof of part (1) relies on the first auxiliary lemma, and is given in § 6.5.

We keep using the notation of § 5.1; however, at various steps of our descent into the argument, we will lighten (and sometimes recycle) the notation for the sake of readability. We start by dropping all apices from the notation, writing for instance f and G in place of f' and G'.

We define involutions w and ι on $\mathbf{Z}^\nu$ by

$$(\lambda^w)_i := \lambda_{\nu+1-i}, \qquad \lambda^\iota := -\lambda^w,$$

and a notion of positivity by declaring $\lambda \in \mathbf{Z}^{\nu,+}$ if $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq \nu-1$; thus ι preserves $\mathbf{Z}^{\nu,+}$. We also write $\lambda \succeq \lambda'$ if $\lambda - \lambda' \in \mathbf{Z}^{\nu,+}$. Then $\varpi^\lambda \in \underline{T}^+_{\nu,*}$ if and only if $\lambda \in \mathbf{Z}^{\nu,+}$, and $(\varpi^\lambda)^\iota = \varpi^{\lambda^\iota}$.

Extending the notation from (3.3.16), let $\mathrm{p}_\nu \colon G_\nu \to G_{\nu,0} \times G_{\nu,0}/F_0^\times$ be the projection, and let $\mathrm{p}_{\nu,*} \colon \mathscr{H}(G_\nu) \to \mathscr{H}(G_{\nu,0} \times G_{\nu,0}/F_0^\times)$ be the pushforward map. Thus $\mathrm{p} := \mathrm{p}_n \times \mathrm{p}_{n+1} \colon \widetilde{G} \to G$ and $\mathrm{p}_* = \mathrm{p}_{n,*} \otimes \mathrm{p}_{n+1,*}$.

Let c be the depth of K. By the positivity condition on $f^\dagger$ and linearity, we may assume that

$$f^\dagger_\nu = U_{t',K} = \mathrm{p}_{\nu,*}(f^\dagger_{\nu,1} \otimes f^\dagger_{\nu,2}), \qquad f^\dagger_{\nu,1} = [K_{\nu,0}\varpi^{\lambda_{\nu,1}}K_{\nu,0}], \quad f^\dagger_{\nu,2} = [K^{w_c}_{\nu,0}\varpi^{\lambda^w_{\nu,2}}K^{w_c}_{\nu,0}] \tag{6.2.1}$$

for some $\lambda_{\nu,i} \in \mathbf{Z}^\nu$ with $\lambda_{\nu,i}, \lambda^\iota_{\nu,i} \succeq (s+c)\rho_\nu$.

We decompose

$$f = f_{K,s} = (5.3.6\mathrm{b}) = q_0^{(2s-c)d(n)} \cdot m_s U_t^{-s} f^\dagger e_K U^c_{[t_0;1]}[w^{-1}_{0,c};1] = f_n \otimes f_{n+1}$$

where each f_ν is a Hecke measure on $G_\nu/F_0^\times$, and further decompose

$$f_\nu = q_0^{(2s-c)c(\nu)} m_{\nu,s} f^\dagger_\nu U^{-s}_{t_\nu} e_{K_\nu} U^{-c}_{[t_{\nu,0};1]}[w^{-1}_{0,\nu,c};1] = \mathrm{p}_{\nu,*}(f_{\nu,1} \otimes f_{\nu,2}),$$

where $f_{\nu,i} \in \mathscr{H}(G_{\nu,0})$ are defined up to some scalar ambiguities that we do not need to resolve. We also denote $f_i = f_{n,i} \otimes f_{n+1,i}$ for $i = 1, 2$.[14]

Fix a representative $\gamma = [\gamma_0; 1] = [(\gamma_{n,0}, \gamma_{n+1,0}); (1_n, 1_{n+1})] \in G$ under the decomposition $G = (G_{n,0} \times G_{n+1,0})^2 / F_0^{\times,2}$.

Decompose $H_1 = H_{1,0}^2$, and write $h_1 \in H_1$ as $h_1 = (h_{1,0}, h'_{1,0})$. In the orbital integral (3.3.4), we first integrate over H_2 (noting that $\eta = 1$), to find

$$\begin{aligned}\chi(\gamma_n) I_\gamma^\dagger(f^\dagger, \chi) &= \int_{H_1} \int_{H_2} f([h_{1,0}^{-1}\gamma_0 h_2; h_{1,0}'^{-1} h_2])\, dh_2\, \chi(h_1)\, dh_1 \\ &= \int_{(H_{1,0})^2} f^\star(h_{1,0}^{-1}\gamma_0 h'_{1,0})\, \chi((h_{1,0}, h'_{1,0}))\, dh_{1,0} dh'_{1,0}\end{aligned} \tag{6.2.2}$$

where

$$f^\star = f_1 \star f_2^\vee \in \mathscr{H}(G_{n,0} \times G_{n+1,0}).$$

(As part of of the proof, we will show that the above integral always reduces to a finite sum.)

Lemma 6.2.1. *Assume that $f^\dagger = (6.2.1)$, and let $\lambda_\nu := \lambda_{\nu,1} + \lambda_{\nu,2}^\iota \succeq 2(s+c)\rho_\nu$. Then $f^\star = f_n^\star \otimes f_{n+1}^\star$ for*

$$f_\nu^\star := q_0^{2sc(\nu) - \ell_{K_{\nu,0}}(w_{\nu,0})} m_s [K \varpi^{\lambda_\nu - 2s\rho_\nu} w K] m_s^{-1} \in \mathscr{H}_{G_\nu}$$

Proof. Let $\sigma_\nu := (\nu - 1, \ldots, 0) \in \mathbf{Z}^\nu$, so that $t = \varpi^{\sigma_\nu}$ and $\sigma_\nu + \sigma_\nu^\iota = 2\rho_\nu$. Abbreviate $w = w_{\nu,0}$, $w_c = w_{\nu,0,c}$; $m_s = m_{\nu,0,s}$; $t = t_{\nu,0}$; $K = K_{\nu,0}$; $K' = K^{w_c}$; $K'' = K_{\nu,0}^{\langle c \rangle}$. Then

$$\begin{aligned} f_\nu^\star = f_{\nu,1} \star f_{\nu,2}^\vee &= q_0^{(2s-c)c(\nu)} m_s U_t^{-s} f_{\nu,1}^\dagger e_K U_t^c w_c^{-1} e_{K'} (U_t^{-s} f_{\nu,2}^\dagger)^\vee m_s^{-1} \\ &= q_0^{2sc(\nu)} m_s [K'' \varpi^{\lambda_{\nu,1} - s\sigma_\nu} K''] e_K e_{K''} t^c e_{K''} w_c^{-1} [K'' \varpi^{-\lambda_{\nu,2} + (s-c)\sigma_\nu} K''] m_s^{-1} \\ &= q_0^{2sc(\nu)} m_s [K'' \varpi^{\lambda_{\nu,1} + (c-s)\sigma_\nu} K''] e_K w e_{K''} [K'' \varpi^{-\lambda_{\nu,2} + (s-c)\sigma_\nu} K''] m_s^{-1} \\ &= q_0^{2sc(\nu) - \ell_{K''}(w)} m_s [K'' \varpi^{\lambda_{\nu,1} - s\sigma_\nu} K''] e_K [K'' w K''] [K'' \varpi^{-\lambda_{\nu,2} + s\sigma_\nu} K''] m_s^{-1} \\ &= q_0^{2sc(\nu) - \ell_{K''}(w)} m_s [K'' \varpi^{\lambda_{\nu,1} - s\sigma_\nu} K''] e_K [K'' \varpi^{\lambda_{\nu,2}^\iota + s\sigma_\nu^\iota} w K''] m_s^{-1} \\ &= q_0^{2sc(\nu) - \ell_K(w)} m_s [K \varpi^{\lambda_{\nu,1} - s\sigma_\nu} K] [K \varpi^{\lambda_{\nu,2}^\iota + s\sigma_\nu^\iota} w K] m_s^{-1} \\ &= q_0^{2sc(\nu) - \ell_K(w)} m_s [K \varpi^{\lambda_\nu - 2s\rho_\nu} w K] m_s^{-1}, \end{aligned}$$

where we have used the symmetry of K'' and the algebra rules of Lemma 5.1.2. $\square$

Let

$$X_\nu^\circ := \varpi^{\lambda_\nu - 2s\rho_\nu} w_{\nu,0} \in G_{\nu,0}. \tag{6.2.3}$$

By Lemma 6.2.1, the integrand in (6.2.2) is non-vanishing at h_1 if and only if

$$h_{1,0}^{-1} \gamma_{\nu,0} h'_{1,0} \in m_{\nu,0,s} K_{\nu,0} X_\nu^\circ K_{\nu,0} m_{\nu,0,s}^{-1} \tag{6.2.4}$$

[14] The context should prevent any possible confusion from the clash of notation with $f_n \in \mathscr{H}(G_n/F_0^\times)$, since the integer in this f_n will never be specialized.

for $\nu = n, n+1$. Therefore, if the orbital integral $I_\gamma(f^\dagger, \chi)$ is non-vanishing, up to changing the representative γ_0 in its $H_{1,0}$-orbit we may and will assume that

$$\gamma_{\nu,0} \in m_{\nu,0,s} K_{\nu,0} X_\nu^\circ K_{\nu,0} m_{\nu,0,s}^{-1}. \tag{6.2.5}$$

We introduce the convenient variables

$$X_\nu \coloneqq m_{\nu,0,s}^{-1} \gamma_{\nu,0} m_{\nu,0,s}. \tag{6.2.6}$$

Then (6.2.5) is equivalent to

$$X_\nu \in K_{\nu,0} X_\nu^\circ K_{\nu,0} \tag{6.2.7}$$

and (6.2.4) is equivalent to

$$m_{\nu,0,s}^{-1} h_{1,0}^{-1} m_{\nu,0,s} \cdot X_\nu \cdot m_{\nu,0,s}^{-1} h'_{1,0} m_{\nu,0,s} \in K_{\nu,0} X_\nu^\circ K_{\nu,0}. \tag{6.2.8}$$

We will reduce Proposition 6.1.2 (2) to the following.

Proposition 6.2.2. *Let $X_\nu^\circ \coloneqq \varpi^{\lambda'_\nu} w_{\nu,0}$ for some $\lambda'_\nu \in \mathbf{Z}^{\nu,+}$, and let $(X_n, X_{n+1}, h_1) \in G_{n,0} \times G_{n+1,0} \times H_1$ satisfy* (6.2.7), (6.2.8) *for $\nu = n, n+1$. Then $h_1 = (h_{1,0}, h'_{1,0}) \in K_H^{(s)}$.*

Lemma 6.2.3. *Proposition 6.2.2 implies Proposition 6.1.2* (2).

Proof. By linearity we may assume that $f^\dagger$ is of the form (6.2.1). Let $X^\circ \coloneqq (X_n^\circ, X_{n+1}^\circ) \in G_{n,0} \times G_{n+1,0}$ (which depends on $f^\dagger$) be as in (6.2.3), and let $B_K^\dagger = B_K^\dagger(f^\dagger) \subset B'$ be the image of

$$m_{0,s}^{-1} K_0 X^\circ K_0 m_{0,s}^{-1} \times \{1\} \subset G.$$

We have already noted that if $\gamma \notin B_K^\dagger$, then $I_\gamma(f^\dagger) = 0$. Assume thus that $\gamma \in B_K^\dagger$, and pick a representative of the form $[\gamma_0; 1]$. Proposition 6.2.2 and the discussion preceding it, applied to $X_\nu = $ (6.2.6) and $\lambda'_\nu = \lambda_\nu - 2s\rho_\nu$, show that the integrand

$$f^\star_{H,\gamma_0,\chi} \colon h_1 \longmapsto \chi(h_1) f^\star(h_{1,0}^{-1} \gamma_0 h'_{1,0})$$

in (6.2.2) has support contained in $K_H^{(s)}$. Thus in order to prove Proposition 6.1.2 (2) we need to show

$$f^\star_{H,\gamma_0,\chi|K_H^{(s)}} = q_0^{2sd(n) - \ell_{K_0}(w_0)}. \tag{6.2.9}$$

Recall the observation from (5.1.8) that if $h_{1,0} \in K_{H,0}^{(s)}$, then $m_{0,s}^{-1} h_{1,0}^{-1} m_{0,s} \in K_0^{\langle s+1 \rangle} \subset K$, and similarly for $h'^{-1}_{1,0}$. Therefore the equivalent form (6.2.8) of (6.2.4) and the fact that $\chi_{|K_H^{(s)}} = 1$ imply (6.2.9). ∎

In §6.3 we reduce Proposition 6.2.2 to a simpler statement, to be proved in the remainder of this section.

6.3. **Contraction.** From now until the end of the section, we lighten the notation by: dropping all subscripts '0'; writing h in place of $h_{1,0}$, and h' in place of $h'_{1,0}$; and writing $m_s \in \mathrm{GL}_{n+1}(F)$ in place of $m_{n+1,s}$, whereas we recall that $m_{n,s} = t_n^s$.

We extract, from the pair of conditions on h, h' in (6.2.8), a single condition on h.

Let $e_{n+1,n} = \binom{1_n}{0} \in M_{n+1,n}(F)$ be the matrix with rows $(e_1, \dots, e_n, 0)$. Denote $\underline{s} \coloneqq (s, \dots, s) \in \mathbf{Z}^n$ and $\varpi_n \coloneqq \varpi^{\underline{1}} = \varpi 1_n \in \mathrm{GL}_n(F)$, and define the $(n+1) \times n$ matrices

$$X \coloneqq X_{n+1} m_s^{-1} e_{n+1,n} t_n^s X_n^{-1},$$

$$X^\circ \coloneqq X_{n+1}^\circ m_s^{-1} e_{n+1,n} t_n^s X_n^{\circ,-1} = \varpi^{\lambda'_{n+1}} w_{n+1} \begin{pmatrix} \varpi^{-\lambda'_n - 2s\rho_n - \underline{s}} \\ 0 \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & & & \varpi^{\lambda_n} \\ & 0 & \cdots & \\ 0 & \varpi^{\lambda_2} & & \\ \varpi^{\lambda_1} & & & \end{pmatrix},$$

where in the second-last matrix $0 \in (F^n)^{\mathrm{t}}$, and $\lambda_i \coloneqq (\lambda'_{n+1})_{n+2-i} - (\lambda'_n)_i - (n+2-2i)s$. Then

$$\lambda_{i+1} - \lambda_i \geq 2s$$

for all $1 \leq i \leq n-1$.

Let

$$\begin{aligned} \overline{h}_s &\coloneqq m_s^{-1} h^{-1} m_s = \begin{pmatrix} t_n^{-s} w_n^{-1} h^{-1} w_n t_n^s & \varpi_n^{-s} t_n^{-1}(w^{-1}h^{-1} - 1_n)u \\ & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(F), \\ h_s &\coloneqq t_n^{-s} h t_n^s \in \mathrm{GL}_n(F). \end{aligned} \tag{6.3.1}$$

Lemma 6.3.1. *If* (6.2.7) *and* (6.2.8) *are satisfied for* $\nu = n, n+1$, *then*

$$\begin{aligned} X &\in K_{n+1} X^\circ K_n \\ \overline{h}_s X h_s &\in K_{n+1} X^\circ K_n. \end{aligned} \tag{6.3.2}$$

Proof. Denote by Y_ν the left-hand side of (6.2.8). Then those equations imply that

$$Y_{n+1} m_s^{-1} e_{n+1,n} t_n^s Y_n^{-1} = \overline{h}_s X h_s \in K_{n+1} X_{n+1}^\circ K_{n+1} m_s^{-1} \begin{pmatrix} t_n^s K_n X_n^{\circ,-1} K_n \\ 0 \end{pmatrix}. \tag{6.3.3}$$

We simplify the right-hand side. First, we have

$$K_n X_n^{\circ,-1} K_n = K_n w_n \varpi^{-\lambda'_n} K_n = K_n^{\langle 2s \rangle} w_n \varpi^{-\lambda'_n} K_n,$$

where the group $K_n^{\langle 2s \rangle}$ is as in § 5.1.4 (the second equality can be shown by observing that the quotient $K_n^{\langle 2s \rangle} \backslash K_n$ is represented by lower-triangular matrices). By the symmetry of $K_n^{\langle 2s \rangle}$, we have

$$K_{n+1} m_s^{-1} \begin{pmatrix} t_n^s K_n w_n \varpi^{-\lambda'_n} K_n \\ 0 \end{pmatrix} = K_{n+1} \begin{pmatrix} \varpi_n^{-s} t_n^{-s} w_n t_n^s K_n^{\langle 2s \rangle} w_n^{-1} \varpi^{-\lambda'_n} K_n \\ 0 \end{pmatrix}$$

$$= K_{n+1} \begin{pmatrix} \varpi_n^{-s} \cdot \varpi^{-2s\rho_n} w_n K_n^{\langle s \rangle} w_n \varpi^{2s\rho_n} \varpi^{-\lambda'_n - 2s\rho_n} K_n \\ 0 \end{pmatrix} = K_{n+1} \begin{pmatrix} K_n^{\langle s \rangle} \varpi^{-\lambda'_n - 2s\rho_n - \underline{s}} K_n \\ 0 \end{pmatrix}$$

Therefore (6.3.3) is equivalent to

$$\overline{h}_s X h_s \in K_{n+1} \varpi^{\lambda'_{n+1}} w_{n+1} K_{n+1} \begin{pmatrix} K_n^{\langle 2s \rangle} \varpi^{-\lambda'_n - 2s\rho_n - \underline{s}} \\ 0 \end{pmatrix} K_n = K_{n+1} X^\circ K_n,$$

where the identity follows from writing

$$K_{n+1}\begin{pmatrix} K_n^{\langle 2s\rangle}\varpi^{-\lambda'_n-2s\rho_n-\underline{s}}K_n \\ 0\end{pmatrix} = \lim_{\varepsilon\to 0} K_{n+1}\begin{pmatrix} \varpi^{-\lambda'_n-2s\rho_n-\underline{s}} & \\ & \varepsilon\end{pmatrix} K_{n+1}e_{n+1,n}$$

and applying the multiplication rules of Lemma 5.1.2. We conclude that we have

$$X \in K_{n+1}X^\circ K_n$$
$$\overline{h}_s X h_s \in K_{n+1}X^\circ K_n,$$

where the first containment follows from the above calculation and (6.2.7). □

We show that the following solution to the contracted problem (6.3.2) implies Proposition 6.2.2.

Proposition 6.3.2. *Let K_ν be a deeper Iwahori of level $\leq s$. Let*

$$X^\circ = \begin{pmatrix} & & & 0 \\ & & & \varpi^{\lambda_n} \\ & 0 & \cdots & \\ 0 & \varpi^{\lambda_2} & & \\ \varpi^{\lambda_1} & & & \end{pmatrix} \in M_{(n+1)\times n}(F) \tag{6.3.4}$$

with $\lambda_{i+1} \geq \lambda_i + 2s$ for all $1 \leq i \leq n-1$, and let $X \in K_{n+1}X^\circ K_n$.

If $h \in \mathrm{GL}_n(F)$ satisfies

$$\overline{h}_s X h_s \in K_{n+1}X^\circ K_n$$

with the notation (6.3.1), then $h \in K_H^{(s)}$.

Lemma 6.3.3. *Proposition 6.3.2 implies Proposition 6.2.2.*

Proof. We revert for a moment to the notation of Proposition 6.2.2. The discussion preceding Proposition 6.3.2 shows that this proposition implies the conclusion that $h_{1,0} \in K_{H,0}^{(s)}$. Observe now that $(X_n^{\circ,-1}, X_{n+1}^{\circ,-1}; X_n^{-1}, X_{n+1}^{-1}; h'_{1,0}, h_{1,0})$ also satisfies the hypothesis of Proposition 6.2.2. Then the previous argument applied to these data shows that $h'_{1,0} \in K_{H,0}^{(s)}$ as well. □

The proof of Proposition 6.3.2 will occupy the rest of this section.

6.3.1. *Iwahori-invariants from minors.* We say that a size-r minor M of a matrix $X \in M_{m\times n}(F_0)$ is

- *Southwest principal* if it is obtained by deleting all but the last r rows and all but the first r columns of X;
- *quasi-SW-principal* if $r \geq 2$ and M contains the Southwest principal minor of size $r-1$;
- *anchored* if M contains part of the last row of X.

Definition 6.3.4. Fix integers $\lambda_1 < \cdots < \lambda_n$. We say that $X \in M_{(n+1)\times n}(F)$ satisfies the *Minor Condition* if for every $1 \leq r \leq n$, every $r \times r$-minor $M_X^{(r)}$ of X, and the Southwest-principal

$r \times r$-minor $P_X^{(r)}$, we have

$$v(\det M_X^{(r)}) \geq \sum_{i=1}^{r} \lambda_i, \qquad v(\det P_X^{(r)}) = \sum_{i=1}^{r} \lambda_i \tag{6.3.5}$$

We say that $X \in M_{(n+1)\times n}(F)$ satisfies the *Weak Minor Condition* if (6.3.5) holds for all anchored minors.

The first example of a matrix satisfying the Minor Condition is $X^\circ = (6.3.4)$.

Lemma 6.3.5. *Let $X, X' \in M_{(n+1)\times n}(F)$.*

(1) If

$$X' \in \mathrm{Iw}_{n+1} X \,\mathrm{Iw}_n,$$

then X satisfies the Minor Condition if and only if X' does;

(2) if

$$X' \in \begin{pmatrix} \mathrm{Iw}_n & \\ & 1 \end{pmatrix} X \,\mathrm{Iw}_n,$$

then X satisfies the Weak Minor Condition if and only if X' does.

Proof. This follows from the Cauchy–Binet formula for minors of products. □

The reader may wish to glance at the proof of the two parts of our Proposition in §§ 6.5, 6.7 before looking at the auxiliary lemmas that occupy §§ 6.4, 6.6.

6.4. **First auxiliary lemma.** We define some variants of the condition $h \in K_H^{(s)}$.

Definition 6.4.1. For $s \geq 1$, we say that a matrix $h \in \mathrm{GL}_n(F)$ is

- *s-small* if for all $1 \leq i, j \leq n$,

$$v(h_{ii}) = 0 \quad \text{and} \quad v(h_{ij}) \geq |j - i| s; \tag{6.4.1}$$

- *upper-s-small up to row i* if there is a decomposition

$$h = h_-^{(i)} h_+^{(i)}$$

where $h_+^{(i)}$ is s-small, and $h_-^{(i)}$ admits a block decomposition

$$h_-^{(i)} = \begin{pmatrix} \alpha & \\ * & * \end{pmatrix} \tag{6.4.2}$$

such that $\alpha \in M_i(F)$ is lower-triangular with units on the diagonal.

- *extremely s-small* if it is s-small and $(wh - 1_n)u = 0$.

Remark 6.4.2. The set of extremely s-small matrices is a subgroup of $K_H^{(s)}$, which in turn is a subgroup of the group of s-small matrices. If h is of the form $h_-^{(i-1)}$ and it satisfies (6.4.1) for all $j \leq i$, then h is upper-s-small up to row i. (In fact, there is a decomposition $h = h_-^{(i)} h_+^{(i)}$ with $h_+^{(i)}$ differing from the identity only in row i.)

From now until the rest of this section, we write t in place of t_n. We denote $h^{-w} := w_n h^{-1} w_n^{-1}$ for $h \in \mathrm{GL}_n(F)$, and we simply denote by 0 the zero row vector of length n. The following remark will often be used in conjunction with Lemma 6.3.5.

Remark 6.4.3. If h is s-small, then $t^{-s}ht$ and $t^{-s}h^{-w}t$ belong to Iw_n.

Lemma 6.4.4. *Let $1 \leq i \leq n$, and consider the equation*

$$X._s h := \begin{pmatrix} t^{-s}h^{-w}t^s & \\ & 1 \end{pmatrix} X t^{-s} h t^s = X', \tag{6.4.3}$$

subject to:

- *X, $X' \in M_{(n+1)\times n}(F)$ satisfy the Weak Minor Condition of Definition 6.3.4;*
- *the entries of the last i rows of X below the lower antidiagonal are zero, that is*

$$v(X_{n+2-i',i'}) = \lambda_{i'}, \qquad X_{n+2-i',j} = 0 \text{ for all } j > i' \leq i \tag{6.4.4}$$

(where the first equation is a consequence of the second one and (6.3.5)*);*

- *$h \in \mathrm{GL}_n(F)$.*

We have:

(1) for given X, every solution (h, X') has h upper-s-small up to row i;

(2) if h is of the form $h_-^{(i)}$ as in (6.4.2)*, then X' also satisfies* (6.4.4)*.*

(3) for given X', there exists a solution (h, X) with h extremely s-small (and in fact upper triangular).

Proof. We proceed by induction on i. Write

$$X = \begin{pmatrix} A \\ c \end{pmatrix}, \quad X' = \begin{pmatrix} A' \\ c' \end{pmatrix}$$

with $A, A' \in M_{n\times n}(F)$

Consider first $i = 1$. The last row of (6.4.3) reads

$$c'_j = c_1 h_{1j}/\varpi^{(j-1)} \tag{6.4.5}$$

for $j \leq n$. Thus if X, X' satisfy (6.3.5), then $v(h_{11}) = 0$ and $v(h_{1j}) \geq (j-1)s$, hence the first statement is proved and the second one is immediate. On the other hand, substituting $h_{11} = 1 - \sum_{k=2}^n h_{ik}$, $c_1 = c'_1 h_{11}^{-1}$ in (6.4.5) gives the integral linear system

$$\sum_{k=2}^{n} (c'_1 \delta_{jk} + \varpi^{(k-1)s} c'_j)\, \varpi^{(1-k)s} h_{1k} = c'_j$$

in the variables $\varpi^{(1-k)s}h_{1k}$. As the system is invertible, the third statement is proved too.

Now let $i \geq 2$ and suppose the first two statements known for $i-1$. By Remark 6.4.3 and Lemma 6.3.5, acting on the right by $h_+^{(i-1)}$ preserves the Weak Minor Condition on X'; hence we may and do replace h by $h_-^{(i-1)}$ in a decomposition $h = h_-^{(i-1)} h_+^{(i-1)}$. In other words, we may assume that for $j > i' \leq i-1$,

$$v(h_{i'i'}) = 0, \qquad h_{i',j} = 0.$$

The same conditions are then satisfied by h^{-1}.

For $j \geq i$, let

$$M^{n+2-i,j}$$

be the quasi-SW-principal minor of X' of size i whose upper-right corner is $X'_{n+2-i,j}$; thus by the induction hypothesis $M^{n+2-i,j}$ has zero entries below the antidiagonal, and its antidiagonal entries have valuations (in order, starting from the SW corner)

$$\lambda_1, \ldots, \lambda_{i-1}, v(X'_{n+2-i,j}).$$

In particular,

$$v(\det M^{n+2-i,j}) = \sum_{i'=1}^{i} \lambda_{i'} - \lambda_i + v(X'_{n+2-i,j}).$$

Hence the Minor Condition (6.3.5) implies

$$-\lambda_i + v(X'_{n+2-i,i}) = 0, \qquad -\lambda_i + v(X'_{n+2-i,j}) \geq 0 \text{ for all } j > i. \tag{6.4.6}$$

As $A' = t^{-s}h^{-w}t^s A t^{-s} h t^s$, we have for all $1 \leq j \leq n$:

$$\begin{aligned} \varpi^{-\lambda_i} X'_{n+2-i,j} &= \varpi^{-\lambda_i} \sum_{k=1}^{n-1} (h^{-w})_{n+2-i,n+1-k} \varpi^{(k+1-i)s} X_{n+1-k,k-1} h_{k+1,j} \varpi^{(k+1-j)s} \\ &= \sum_{k=1}^{i-1} h^{-1}_{i-1,k} \varpi^{(k+1-i)s} \varpi^{-\lambda_i} X_{n+1-k,k+1} h_{k+1,j} \varpi^{(k+1-j)s}, \end{aligned} \tag{6.4.7}$$

by our assumptions on h. Moreover, for $j \geq i$ by induction hypothesis $h_{k+1,j} = 0$ for all $k < i-1$, hence

$$\varpi^{-\lambda_i} X'_{n+2-i,j} = h^{-1}_{i-1,i-1} h_{i,j} \varpi^{(i-j)s} \varpi^{-\lambda_i} X_{n+2-i,i}. \tag{6.4.8}$$

Since $h^{-1}_{i-1,i-1}$ and $\varpi^{-\lambda_i} X_{n+2-i,i}$ are units, condition (6.4.6) is equivalent to

$$v(h_{i,i}) = 0, \qquad v(h_{i,j}) \geq (j-i)s$$

for all $j > i$, establishing the first statement. If h is of the form $h_-^{(i)}$, the second statement is immediate from (6.4.8).

Consider now the third statement. After replacing X' by $X'._s(h')^{-1}$ where h' is as given by this statement for $i-1$, we may and do assume that X' satisfies (6.4.4) for $i' < i$. We seek h extremely s-small, upper-triangular and differing from the identity only in row i; hence in particular h takes the form $h_-^{(i-1)}$, and by the second statement for $i-1$, we only need to find a solution to (6.4.8) in h (with the further simplification $h_{i-1,i-1} = 1$).

We set $h_{i,i} = 1 - \sum_{k \neq i} h_{ik}$, necessary for h to be extremely s-small, and substitute in $X_{n+2-i,i} = h^{-1}_{i,i} X'_{n+2-i,i}$. We find the linear system

$$\sum_{k=i+1}^{n} \left(\varpi^{-\lambda_i} X'_{n+2-i,i} \delta_{jk} + \varpi^{(k-i)s} \varpi^{-\lambda_i} X'_{n+2-i,j}\right) h_{ik} \varpi^{(i-k)s} = \varpi^{-\lambda_i} X'_{n+2-i,j}$$

in the variables $\varpi^{(i-k)s}h_{ik}$ for $k \geq i+1$. By our reductions, $-\lambda_i + v(X'_{n+2-i,i}) = 0$ and $-\lambda_i + v(X'_{n+2-i,j}) \geq 0$, hence the system is integral and invertible; its solvability implies the third statement. $\square$

6.5. **Plus-regularity of support: proof of Proposition 6.1.2** (1)**.** We prove part (1) of Proposition 6.1.2. It follows from Lemma 6.2.1 (via Lemma 3.5.6) that $f' =$ (5.3.6b) matches an $f \in \mathscr{H}(G, L)$ that is invariant under $K_0^{m_s}$. We now turn to proving that f' is supported in the plus-regular locus G'_{reg^+} (thus f is also regularly supported).

Recall that we defined in (3.3.3) the quasi-invariant $D^\pm$ on $\widetilde{G}'$, by pulling back the corresponding function $D^\pm$ on the symmetric space S. Now that the place v is split in the quadratic extension, we identify S with G_{n+1}. Tracking the process of contracting the test function, it suffices to show that the function $f_n^\star \star f_{n+1}^\star$ on $G_{n+1} = S$ has plus-regular support, where $f_n^\star$ and $f_{n+1}^\star$ are as in Lemma 6.2.1.

Now we note that, for $\gamma = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in G_{n+1} = S$, the invariant D^+ is equal (up to a sign) to

$$D^+(\gamma) = \det(c, cA, \cdots, cA^{n-1}).$$

Note the quasi-invariance property: for $h \in G_n$,

$$D^+\left(h^{-1}\gamma h\right) = D^+(\gamma)\det(h).$$

Then by definition, $\gamma \in G_{n+1}$ is plus-regular if and only if $D^+(\gamma) \neq 0$, or equivalently the vectors $c, cA, \cdots, cA^{n-1}$ form a basis.

We observe that the plus-regularity depends only the first n columns, so that we may talk about the plus-regularity of an element $\binom{A}{c} \in M_{(n+1)\times n}(F)$. Therefore by Lemma 6.3.1 (together with the discussion preceding Proposition 6.2.2), it suffices to show that the following subset of $M_{(n+1)\times n}(F)$ is inside the plus-regular locus:

$$m_s K_{n+1} X^\circ K_n t_s^{-1}, \tag{6.5.1}$$

where X° is as in (6.3.4).

By Lemma 6.3.5(1), any element in the set $K_{n+1}X^\circ K_n$ satisfies the Minor Condition (Definition 6.3.4). It thus suffices to show that, if X satisfies the Weak Minor Condition, then the element

$$\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} wt^s & \\ & 1 \end{pmatrix} X t^{-s} \in M_{(n+1)\times n}(F)$$

is plus-regular. Set

$$\widetilde{X} := \begin{pmatrix} wt^s & \\ & 1 \end{pmatrix} X t^{-s} \in M_{(n+1)\times n}(F).$$

Then we claim that $\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \widetilde{X}$ is plus-regular if and only if $\widetilde{X}$ is. To see this, we write $\widetilde{X} = \binom{\widetilde{A}}{\widetilde{c}}$ so that $\left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right) \widetilde{X} = \binom{\widetilde{A}+u\widetilde{c}}{\widetilde{c}}$. We see inductively that the span of $\widetilde{c}, \widetilde{c}(\widetilde{A}+u\widetilde{c}), \ldots, \widetilde{c}(\widetilde{A}+u\widetilde{c})^{i-1}$ is equal to the span of $\widetilde{c}, \widetilde{c}\widetilde{A}, \ldots, \widetilde{c}(\widetilde{A})^{i-1}$. The claim follows.

It remains to show that $\widetilde{X}$ is plus-regular. By Lemma 6.4.4 (3) (applied to the case $i = n$), there exists $h \in \mathrm{GL}_n(F)$ such that, if we set

$$h^{-1}\widetilde{X}h = \widetilde{X}'$$

where $\widetilde{X}' = \begin{pmatrix} wt^s & \\ & 1 \end{pmatrix} X' t^{-s} \in M_{(n+1)\times n}$, then the entries of X' below the lower antidiagonal are all zero. It therefore suffices to show that $\widetilde{X}'$ is plus-regular. We note that

$$\widetilde{X}' = \begin{pmatrix} * & a_2 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \cdots & a_n \\ * & * & \cdots & * \\ a_1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{A}' \\ \widetilde{c}' \end{pmatrix},$$

where $a_1, a_2, \ldots, a_n$ are all non-zero. For $1 \le i \le n$, denote by $e_i \in F_0^n$ the standard basis vector, and by $V_i \subset F_0^n$ the subspace spanned by $e_1, \ldots, e_i$. In particular, $\widetilde{c}' = a_1 e_1$. Then by induction we see that

$$e_1^{\mathrm{t}}(\widetilde{A}')^{i-1} \equiv a_i e_i^{\mathrm{t}} \mod V_{i-1}^{\mathrm{t}}.$$

It follows easily that the subspace spanned by $e_1^{\mathrm{t}}, \ldots, e_1^{\mathrm{t}}(\widetilde{A}')^{i-1}$ is exactly V_i^{t}, for all $1 \le i \le n$. The desired assertion follows.

6.6. **Second auxiliary lemma.** We continue with another lemma towards the proof of part (2) of Proposition 6.1.2.

Definition 6.6.1. We say that a lower-triangular matrix $h \in \mathrm{GL}_n(F)$ with units on the diagonal is *lower-s-small from column j* if

$$v(h_{ij'}) \ge (i - j')s \text{ for all } i > j' \ge j.$$

This is equivalent to the existence of a decomposition

$$h = {}^{(j)}h_- \cdot {}^{(j)}h_{--}$$

where ${}^{(j)}h_{--}$ is lower-triangular and s-small, and

$${}^{(j)}h_- = \begin{pmatrix} \alpha_- & \\ * & 1_{n+1-j} \end{pmatrix} \tag{6.6.1}$$

with $\alpha_- \in M_{j-1}(F)$ lower-triangular with units on the diagonal.

Remark 6.6.2. For a lower-triangular matrix h with units on the diagonal:

- h is lower-s-small from column j if and only if h^{-1} is;
- h is lower-s-small from column 1 if and only if it is s-small.

Lemma 6.6.3. *Let $1 \le j \le n$, and consider the equation*

$$X._s h = \begin{pmatrix} t^{-s}h^{-w}t^s & \\ & 1 \end{pmatrix} X t^{-s} h t^s = X', \tag{6.6.2}$$

subject to:

– X, $X' \in M_{(n+1)\times n}(F)$ *satisfy the Weak Minor Condition of Definition 6.3.4;*

– *the entries of* X, X' *below the lower antidiagonal are zero, that is*

$$v(X_{n+2-i,i}) = \lambda_i, \qquad X_{n+2-i,j'} = 0 \text{ for all } j' > i$$

(where the first equation is a consequence of the second one and (6.3.5)*), and similarly for* X'*;*

– *the entries of the last* $n-j$ *columns of* X *above the lower antidiagonal are zero, that is,*

$$X_{n+2-i,j'} = 0 \text{ for all } i > j' \geq j+1; \tag{6.6.3}$$

– $h \in \mathrm{GL}_n(F)$ *is lower-triangular with units on the diagonal.*

We have:

(1) for given X*, every solution* (h, X') *has* h *lower-s-small from column* j*;*

(2) if h *is of the form* ${}^{(j)}h_-$ *as in* (6.6.1)*, then* X' *also satisfies* (6.6.3)*;*

(3) for given X'*, there exists a solution* (h, X) *with* h *extremely s-small.*

Proof. We prove this by decreasing induction on j, the case $j = n$ being trivial. Thus let $j \leq n-1$ and assume the statements proved for $j+1$.

After replacing h by ${}^{(j-1)}h_-$ as in the decomposition (6.6.1), that is acting by $._s{}^{(j+1)}h_{--}$ on both sides of (6.6.2), by the induction hypothesis we are led to a situation that is equivalent for the purposes of the first two statements. Hence we may and do assume that h has the form ${}^{(j+1)}h_-$. For $i \geq j$, let

$$N^{n+1-i,j+1}$$

be the quasi-SW-principal minor of X' of size $j+1$ whose upper-right corner is $X'_{n+1-i,j+1}$; thus the matrix $N^{n+1-i,j+1}$ has vanishing entries below the antidiagonal, and its antidiagonal entries (in order, starting from the SW corner) have valuations

$$\lambda_1, \ldots, \lambda_j, v(X'_{n+1-i,j+1}).$$

In particular,

$$v(\det N^{n+1-i,j+1}) = \sum_{j'=0}^{j+1} \lambda_{j'} - \lambda_{j+1} + v(X'_{n+1-i,j+1}).$$

Hence (6.3.5) implies

$$-\lambda_{j+1} + v(X'_{n+1-i,j+1}) \geq 0. \tag{6.6.4}$$

The same condition holds for X by assumption.

We have

$$\begin{aligned}\lambda_{j+1}^{-1} X'_{n+1-i,j+1} &= \varpi^{-\lambda_{j+1}} \sum_{1\leq k,l\leq n} (h^{-w})_{n+1-i,n+1-k} \varpi^{(k-i)s} X_{n+1-k,l+1} h_{l+1,j+1} \varpi^{(l-j)s} \\ &= \sum_{k=j}^{i} h_{i,k}^{-1} \varpi^{(k-i)s} \varpi^{-\lambda_{j+1}} X_{n+1-k,j+1}\end{aligned} \tag{6.6.5}$$

where we have used our assumptions on h and X. All terms are integral except possibly the last one, whose valuation is $v(h^{-1}_{i,j}) - (i-j)s$. That this should be non-negative, for all $i > j$, is equivalent to h being lower-s-small from column j, proving the first statement.

If moreover h has the form ${}^{(j)}h_-$, then in (6.6.5) all terms are zero unless $i = j$, in which case we only have the term corresponding to $i = j = k$, giving $X'_{n+1-j,j+1} = X_{n+1-j,j+1} = 0$. This proves the second statement.

For the third statement, we seek an extremely s-small matrix h that differs from the identity only in column j. Then h^{-1} satisfies the same conditions, h is of the form ${}^{(j-1)}h_-$, and we need it to satisfy (6.6.5) (for some X), in whose right-hand side only the terms $k = j, i$ may be nonzero. Substituting

$$h^{-1}_{jj} \coloneqq 1 - \sum_{i>j} h^{-1}_{ij}, \qquad X_{n+1-j,j+1} = (1 - \sum_{i>j} h_{ij})^{-1} X'_{n+1-j,j+1},$$

and observing that for $i \geq j+1$ only the term $k = i$ may be nonzero in (6.6.5), we find

$$h^{-1}_{ij} \varpi^{(j-i)s} \varpi^{-\lambda_{j+1}} X_{n+1-j,j+1} = \varpi^{-\lambda_{j+1}} X'_{n+1-i,j+1}.$$

This is an invertible integral linear system

$$\sum_{k=j+1}^{i} (\varpi^{-\lambda_{j+1}} X'_{n+1-j,j+1} \delta_{kj} + \varpi^{k-j} \varpi^{-\lambda_{j+1}} X'_{n+1-i,j+1}) \varpi^{(j-k)s} h^{-1}_{kj} = \varpi^{-\lambda_{j+1}} X'_{n+1-i,j+1}.$$

in the variables $\varpi^{(j-k)s} h^{-1}_{kj}$. The solvability of the system implies our third statement. □

6.7. **Proof of Propositions 6.1.2** (2)**, 6.2.2, and 6.3.2.** By Lemmas 6.2.3, 6.3.3, it suffices to prove Proposition 6.3.2. Thus we need to show that for $X, Y \in K_{n+1} X^\circ K_n$, all the solutions in h to the equation

$$Y = \overline{h}_s X h_s \tag{6.7.1}$$

have $h \in K_H^{(s)}$. By Lemma 6.3.5, both X and Y satisfy the Minor Condition.

Write $X = \binom{A}{c} \in M_{(n+1)\times n}(F)$, with $c \in F^{n,\mathrm{t}}$. Then

$$\begin{aligned} Y &= \overline{h}_s X h_s = X' + X'', \\ X' &\coloneqq X._s h = \begin{pmatrix} t^{-s} h^{-w} t^s A t^{-s} h t^s \\ c t^{-s} h t^s \end{pmatrix}, \\ X'' &\coloneqq X.._s h \coloneqq \begin{pmatrix} \varpi^{-s} t^{-s} (w h^{-1} - 1_n) u \\ 0 \end{pmatrix} c t^{-s} h t^s, \end{aligned} \tag{6.7.2}$$

where the notation $X._s h$ is as in Lemmas 6.4.4, 6.6.3.

Note that $X'' = Y - X'$ is a rank-1 matrix whose rows are all multiples of row $n+1$ of X' (and whose last row is zero), so that the determinants of any pair of corresponding anchored minors of Y, X' are equal. In particular, X' also satisfies the Weak Minor Condition of Definition 6.3.4.

We proceed in several steps to show that $h \in K_H^{(s)}$.

(1) By applying first Lemma 6.4.4 (3) for $i = n$, then Lemma 6.6.3 (3) for $j = 1$, we find an extremely s-small h' such that, first, $X.._s h' = 0$ (which is automatic by the extreme smallness of h') and, second, $X._s h' = \overline{h}'_s X h_s$ has zero entries outside of the lower antidiagonal and of

column 1. Hence, up to changing variables by such an h', we may assume X satisfies these vanishing conditions.

(2) Apply Lemma 6.4.4 (1) to the equation

$$X' = X._s h, \tag{6.7.3}$$

to deduce that h is upper-s-small, $h = h_- h_+$ with h_- lower-triangular with units on the diagonal and h_+ s-small.

(3) Act on (6.7.3) by $._s h_+^{-1}$; by Remark 6.4.3 and Lemma 6.3.5 (2) this preserves the Weak Minor Condition. We can then apply Lemma 6.6.3 (1) (with $j = 1$) to the resulting equation, to conclude that h_- and h are s-small.

(4) By Remark 6.4.3 and Lemma 6.3.5 (1), we deduce that $X' = X._s h$ satisfies the full Minor Condition; in particular, all entries of X' have valuation no less than $v(\lambda_1)$. Since this also holds for the entries of Y, it must hold for the entires of X'' too. As $\lambda_1^{-1} ct^{-s} ht^s$ is integral with first entry a unit, the condition on X'' is satisfied if and only if $\varpi^{-s} t^{-s}(wh^{-1} - 1_n)u \in \mathscr{O}^n$; that is, $h \in K_H^{(s)}$.

The proof of Propositions 6.3.2, 6.2.2 and 6.1.2 (whose part (1) was proved in § 6.5) is now complete.

7. The p-adic relative-trace formula and p-adic L-functions

This section is dedicated to the construction of the p-adic L-function of Theorem B and the related RTF. In § 7.1 we give the statements. In § 7.2 we give the proofs: similarly to what done in § 4, we construct the p-adic relative-trace distribution from its geometric expansion, then we extract from it the p-adic L-function and deduce the spectral expansion. In § 7.3, we give a RTF for the derivative of the distribution.

Throughout this section, we fix a rational prime p.

7.1. **Statements.** Recall that we denote $\Gamma = \Gamma_{F_0} := F_0^\times \backslash \mathbf{A}^{\infty,\times} / \widehat{\mathscr{O}}_{F_0}^{p,\times}$, and $\mathscr{Y} := \operatorname{Spec} \mathbf{Z}_p[\![\Gamma_{F_0}]\!] \otimes \mathbf{Q}_p$ We say that $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her}}_{\mathbf{Q}_p}$ is *ordinary* if for all $v|p$, the representation Π_v is ordinary in the sense of Definition 5.2.1. The ordinary representations form an ind-subscheme

$$\mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}} \subset \mathscr{C}(\mathrm{G}')^{\mathrm{her}}_{\mathbf{Q}_p}.$$

For $K_p = \prod_v K_v$, we let $\mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}_{K_p}$ be the subscheme of those Π which are K_v-ordinary for all $v|p$.

7.1.1. *p-adic L-function.* The following is Theorem B from the introduction

Theorem 7.1.1. *Let L be a finite extension of $\mathbf{Q}_p$, and let Π be an ordinary hermitian trivial-weight cuspidal automorphic representation of $\mathrm{G}'(\mathbf{A})$ over L.*

Assume that for each place $v|p$ of F_0, v splits in F or Π_v is unramified. Then there exists a unique function

$$\mathscr{L}_p(\mathrm{M}_\Pi) \in \mathscr{O}(\mathscr{Y}_L)$$

whose restriction to $Y(p^\infty)_L$ *satisfies*

$$\mathscr{L}_p(\mathrm{M}_\Pi)(\chi) = e_p(\mathrm{M}_{\Pi\otimes\chi})\,\mathscr{L}(\mathrm{M}_\Pi)(\chi) \tag{7.1.1}$$

where $\mathscr{L}(\mathrm{M}_\Pi)$ *is the function in Theorem 4.2.1, and* $e_p(M_{\Pi\otimes\chi}) \coloneqq \prod_{v|p} e(\Pi_v,\chi_v)$ *for the factors of* (5.3.5).

7.1.2. *Generalized Radon measures.* We make the first of two preparations which will be relevant to the p-adic relative-trace formula.

Recall that a *Banach ring* is a topological ring equipped with a norm $|\cdot|$ for which it is complete; the relevant examples for us are the finite extensions of $\mathbf{Q}_p$ (with the p-adic norm) and $\mathscr{O}(\mathscr{Y})$ (with the Gauss norm).

Definition 7.1.2. Let X be a set and let R be a Banach ring. A generalized bounded Radon measure[15] with values in R is a pair $(\mu, L^{1,\infty}(X,\mu))$, where

- $L^{1,\infty}(X,\mu)\subset L^\infty(X)$ is a closed subspace of the R-Banach space of bounded R-valued function on X;
- $\mu\colon L^{1,\infty}(X,\mu)\to R$ is a bounded R-linear functional.

We will usually denote such measures simply by μ, and for $\Phi\in L^{1,\infty}(X,\mu)$, we will use the notation

$$\int_X \Phi(x)\,d\mu(x) \coloneqq \mu(\Phi).$$

When $R'\supset R$ is an extension of Banach rings, an R-valued generalized bounded Radon measure μ gives rise to an R'-valued generalized bounded Radon measure by extension of scalars, which we will still denote by μ. We say that a function $\Phi\in L^\infty(X)$ is μ-integrable if it belongs to $L^{1,\infty}(X,\mu)$. When we make an assertion regarding $\int_X \Phi\,d\mu$ for some $\Phi\in L^\infty(X)$, we implicitly include the assertion that Φ is μ-integrable.

7.1.3. *Local distributions at p.* Let $K_p=\prod_{v|p}K_v\subset \mathrm{G}'(F_{0,p})$ be a compact open subgroup that is *convenient* in the sense that each K_v is (as defined in § 5.3.1). We will say that K_p is a *conjugate-symmetric deeper Iwahori* (CSDI) if each K_v is (as defined in § 5.1.4).

For $\chi\in Y(p^\infty)$ and $f_p^\dagger=\otimes_{v|p}f_v^\dagger\in\mathscr{H}_p^\dagger=\bigotimes_{v|p}\mathscr{H}_v^\dagger$ that is sufficiently positive for Π_p, χ_p, and K_p (in the obvious sense derived from § 5.3.4 for each $v|p$), and for Π_p a tempered irreducible representation of G'_p and $\gamma\in B'_p$, we define

$$I^\dagger_{\Pi_p,K_p}(f_p^\dagger,\chi_p)\coloneqq\prod_{v|p}I^\dagger_{\Pi_v,K_v}(f_v^\dagger,\chi_v),\qquad I^\dagger_{\gamma,p,K_p}(f_p^\dagger,\chi_p)\coloneqq\prod_{v|p}I^\dagger_{\gamma,v,K_v}(f_v^\dagger,\chi_v),$$

where the last factors are as in (6.1.1); we impose the restriction that $\gamma\in B'_{\mathrm{rs},p}$ unless K_p is a CSDI.

7.1.4. *p-adic relative-trace formula.* For K_p as above, recall the Hecke subspace

$$\mathscr{H}(\mathrm{G}'(\mathbf{A}^p))^\circ_{K_p,\mathrm{rs},\mathrm{qc}}\subset\mathscr{H}(\mathrm{G}'(\mathbf{A}^p))^\circ_{\mathrm{rs}}$$

[15]When X is a topological space and $L^{1,\infty}(X,\mu)$ contains $C_c(X)$, the functional μ is a (bounded) Radon measure in the sense of Bourbaki.

of § 4.2.2. We denote $U_{t_p} = \otimes_{v|p} U_{t_v}$.

Theorem 7.1.3 (p-adic analytic RTF)**.** *Let $K_p = \prod_{v|p} K_v \subset \mathrm{G}'(F_{0,p})$ be a convenient subgroup, and let L be a finite extension of $\mathbf{Q}_p$.*

There exist:

(1) For each finite place $v \nmid p$ of F_0 and for $v = \infty$, for each $\gamma \in B'_v$, and for each tempered irreducible representation Π_v of G'_v over L, distributions

$$\mathscr{I}_{\Pi_v} \colon \mathscr{H}(G'_v, L)^\circ \longrightarrow \mathscr{O}(\mathscr{Y}_L),$$
$$\mathscr{I}_{\gamma,v} \colon \mathscr{H}(G'_v, L)^\circ \longrightarrow \mathscr{O}(\mathscr{Y}_{L[\sqrt{-1}]})$$

obtained from the corresponding distributions of Proposition 4.2.2 (1), (3) *by pullback via the restriction maps $\mathscr{Y} \ni \chi \mapsto \chi_v \in Y_v(1)_{\mathbf{Q}_p}$. (If $v = \infty$, $Y_v(1) \coloneqq \operatorname{Spec} \mathbf{Q}$.)*

(2) For each representation Π over L as in Theorem 7.1.1, a distribution

$$\mathscr{I}_\Pi \coloneqq \frac{1}{4} c_{K_p}(\Pi)\, \mathscr{L}_p(\mathrm{M}_\Pi) \prod_{v \nmid p} \mathscr{I}_{\Pi_v} \colon \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ \longrightarrow \mathscr{O}(\mathscr{Y}_L),$$

where the constant $c_{K_p}(\Pi) \coloneqq \prod_v c_{K_v}(\Pi_v)$ for the factors of (5.3.5)*.*

(3) For each $\gamma \in \mathrm{B}'(F_0)^\circ$, a bounded-by-1 p-adic L-function

$$L_{p,\gamma} \in \mathbf{Z}_p[\![\Gamma_{F_0}]\!] \subset \mathscr{O}(\mathscr{Y})$$

whose restriction to $Y(p^\infty)$ equals $L_\gamma^{(p)} \coloneqq L_\gamma / \prod_{v|p} L_{\gamma,v}$ (where L_γ is as in Proposition 4.2.2(4a)*).*

(4) An orbital-integral function

$$\mathscr{I}^p \colon \mathrm{B}'(F_0)^\circ \times \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ \longrightarrow \mathscr{O}(\mathscr{Y}_L) \tag{7.1.2}$$

defined by

$$(\gamma, f'^p) \longmapsto \mathscr{I}^p_\gamma(f'^p) \coloneqq \kappa(\mathbf{1}_\infty)^{-1} L_{p,\gamma} \prod_{v \nmid p} \mathscr{I}_{\gamma,v}.$$

which is bounded in the variable γ.

(5) (a) for every $\chi_p \in Y_p(p^\infty)$, a $\mathbf{Q}_p(\chi)$-valued generalized bounded Radon measure $I^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p)$ on $\mathrm{B}'_{\mathrm{rs}}(F_0)^\circ$, defined by the limit of weighted samplings

$$\int_{\mathrm{B}'_{\mathrm{rs}}(F_0)^\circ} \Phi(\gamma)\, dI^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p) \coloneqq \lim_{N\to\infty} \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)^\circ} I^\dagger_{\gamma,p,K_p}(U_{t_p}^{N!}, \chi_p) \cdot \Phi(\gamma) \tag{7.1.3}$$

on the space of bounded functions $\Phi \in L^\infty(\mathrm{B}'_{\mathrm{rs}}(F_0)^\circ)$ for which the sums over γ converge and the limit converges;

(b) if K_p is a CSDI, a $\mathbf{Q}_p$-valued generalized bounded Radon measure $I^{\mathrm{ord}}_{\gamma,p,K_p}$ on $\mathrm{B}'(F_0)^\circ$ (whose restriction to on $\mathrm{B}'_{\mathrm{rs}}(F_0)^\circ$ coincides with $I^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p)$ for every χ_p), defined by

$$\int_{\mathrm{B}(F_0)^\circ} \Phi(\gamma)\, dI^{\mathrm{ord}}_{\gamma,p,K_p} = k_p \cdot \lim_{N\to\infty} \sum_{\gamma \in \mathrm{B}'(F_0)^\circ \cap B^\dagger_{p,N}} \Phi(\gamma); \tag{7.1.4}$$

here, $B^{\dagger}_{p,N} = \prod_{v|p} B^{\dagger}_{K_v}(U^{N!}_{t_v,K_v})$ *and* $k_p = \prod_{v|p} k_v$*, with the factors as in* (6.1.3).

(6) A distribution

$$\mathscr{I}_{K_p}\colon \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^{\circ}_{K_p,\mathrm{qc}} \longrightarrow \mathscr{O}(\mathscr{Y}_L)$$

which admits the spectral expansion

$$\mathscr{I}_{K_p} = \sum_{\Pi\in\mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}_{K_p}} \mathscr{I}_{\Pi}$$

and:

(a) after restricting to $\mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^{\circ}_{K_p,\mathrm{rs},\mathrm{qc}}$ *if* K_p *is not a CSDI: for each finite-order* $\chi \in \mathscr{Y}_L$*, the geometric expansion in* $L(\chi)$

$$\mathscr{I}_{K_p}(f'^p, \chi) = \int_{\mathrm{B}'_{\mathrm{rs}}(F_0)^{\circ}} I^p_{\gamma}(f'^p, \chi^p)\, dI^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p),$$

(b) if K_p *is a CSDI, the geometric expansion in* $\mathscr{O}(\mathscr{Y}_L)$

$$\mathscr{I}_{K_p}(f'^p) = \int_{\mathrm{B}'(F_0)^{\circ}} \mathscr{I}^p_{\gamma}(f'^p)\, dI^{\mathrm{ord}}_{\gamma,p,K_p}.$$

7.2. **Proofs.**

We will prove Theorem 7.1.3 and, as an interlude, Theorem 7.1.1.

7.2.1. *Boundedness of local orbital integrals.* We consider the local distributions I_γ of Proposition 4.2.2 (3).

Lemma 7.2.1. *Let* $v \nmid p\infty$ *or, respectively,* $v = \infty$*, and let* $f'_v \in \mathscr{H}(G'_v, L)^{\circ}$*. There is a constant* $c(f'_v) \in \mathbf{Q}^{\times}$ *such that for every* $\gamma \in B'_v$ *(respectively, for every* $\gamma \in B^{\circ}_v$*), the polynomial*

$$I_\gamma(f'_v) \in \mathscr{O}(Y_v(1))_L \cong L[T]$$

(respectively the number $I_\gamma(f'_\infty, \mathbf{1}) \in L$*) belongs to* $c(f'_v)\mathscr{O}_L[T]$ *(respectively* $c(f'_v)\mathscr{O}_L$*). Moreover, for all but finitely many* v*, if* f'_v *is the unit Hecke measure then we may take* $c(f'_v) = 1$.

Proof. By the definitions and Lemma 3.3.4, it suffices to consider $I_{\gamma'}(f'_v, \chi_v)$ instead of $I_\gamma(f'_v, \chi_v)$, for any γ' in the unique plus-regular orbit above γ.

First consider the case of $v = \infty$, for which we may assume that f'_∞ matches f°_∞: then by the proof of Lemma 4.1.4, the function $\gamma \mapsto I_{\gamma'}(f'_\infty)$ takes finitely many values on B°_v, so that the boundedness is trivial.

Assume now that $v \nmid p\infty$. If γ' is regular semisimple, then the orbital integral $I_\gamma(f'_v, \chi_v)$ is equal to $I^{\sharp}_{\gamma'}(f'_v, \chi_v)$ defined by (3.3.4). The latter is a finite sum of the values of the integrand at the cosets under the maximal compact subgroup of $H'_{1,v} \times H'_{2,v}$ under which f'_v is invariant. Therefore the integral is a polynomial in $\chi_v(\varpi_v)$ and $\chi_v(\varpi_v)^{-1}$ whose coefficients' denominators are bounded by those of f'_v. If f'_v is the unit measure or more generally spherical, then the orbital integral is an integral polynomial since the volume of the compact open subgroup $\mathrm{GL}_n(\mathscr{O}_{F_{0,v}}) \times (\mathrm{GL}_n \times \mathrm{GL}_{n+1})(\mathscr{O}_{F_{0,v}})$ of $H'_{1,v} \times H'_{2,v}$ is equal to one by our choice of measures.

In general, for a plus-regular element γ', by [Lu, Lemma 5.14] the integral in (3.3.4) is absolutely convergent (in the archimedean topology) when the exponent of $|\chi_v|$ is small enough. This implies that for some large integer N, the product $\chi(\varpi_v)^{-N} I^{\sharp}_{\gamma'}(f'_v, \chi_v)$ is a power series in $\chi(\varpi_v)^{-1}$. Now the normalized orbital integral $I_{\gamma'}(f'_v, \chi_v)$ is by definition (cf. (3.3.7)) given by $\frac{I^{\sharp}_{\gamma'}(f'_v,\chi_v)}{L_{\gamma'}(\chi_v)}$. In particular, we have an equality of power series in $\chi(\varpi_v)^{-1}$:

$$\chi(\varpi_v)^{-N} I_{\gamma'}(f'_v, \chi_v) = \frac{\chi(\varpi_v)^{-N} I^{\sharp}_{\gamma'}(f'_v, \chi_v)}{L_{\gamma'}(\chi_v)}.$$

The same argument as for the regular semisimple case shows that the coefficients of the power series $\chi(\varpi_v)^{-N} I^{\sharp}_{\gamma'}(f'_v, \chi_v)$ are p-adically bounded, and integral if f'_v is spherical. Since $L_{\gamma'}(\chi_v)$ is an integral polynomial in the variable $\chi(\varpi_v)^{-1}$, it follows that the coefficients of the power series $\chi(\varpi_v)^{-N} I_{\gamma'}(f'_v, \chi_v)$ are also bounded (in the p-adic topology), as desired. □

7.2.2. *Proof of Theorem 7.1.3 / I.* Part (1) and the definitions in part (4) and (5) of Theorem 7.1.3 are self-explanatory.

For part (3), it suffices to show the existence of an integral interpolation of the functions on $Y(p^\infty)_{\mathbf{Q}_p} \subset \mathscr{Y}$ given by the abelian L-functions in (3.3.11) (with the Euler factors at $p\infty$ removed). This is a consequence of the results of Deligne–Ribet [DR80], who prove that for every totally even (respectively odd) finite Hecke character ξ of of a totally real field F'_0 and every even (respectively odd) $k \geq 0$, there is an $L_p(1-k, \xi) \in \mathbf{Z}_p(\xi)[\![\Gamma_{F'_0}]\!]$ interpolating $\chi' \mapsto L_p(1-k, \xi\chi')$.[16]

The boundedness of the measures $I^{\mathrm{ord}}_{\gamma,p,K_p}(\chi_p)$ (which is, importantly, uniform in χ_p) follows from Lemma 6.1.1 (2). Their explicit and uniform variant over $\mathrm{B}(F_0)^\circ$ when K_p is a CSDI follows from Proposition 6.1.2 (2).

The boundedness of $\gamma \mapsto \mathscr{I}^p_\gamma(f'^p)$ follows from the integrality of $L_{\gamma,p}$ and Lemma 7.2.1.

We have thus proved parts (1), (3), (4), (5) of Theorem 7.1.3. After some preliminaries, we will now prove the existence of the global distribution and the geometric expansion in part (6). The spectral expansion in part (6) (with the definitions from part (2)) will be proved in § 7.2.6.

7.2.3. *Finite-slope distributions.* For $\gamma \in \mathrm{B}'_{\mathrm{rs}}(\mathbf{A})$, and Π as in Theorem 7.1.1, we first define the following distributions on the subspace of $\mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ \otimes \mathscr{H}_p^\dagger$ of elements that are sufficiently positive for all the relevant data:

$$\begin{aligned} I^\dagger_{\Pi,K_p}(f'^p f^\dagger_p, \chi) &:= \frac{1}{4}\mathscr{L}(\mathrm{M}_\Pi, \chi) \cdot \prod_{v\nmid p} I_{\Pi_v}(f_v, \chi_v) \cdot I^\dagger_{\Pi_p,K_p}(f^\dagger_p, \chi) = I_\Pi(f'^p f'_p), \\ I^\dagger_{\gamma,K_p}(f'^p f^\dagger_p, \chi) &:= \kappa(\mathbf{1}_\infty)^{-1} L^p_\gamma(\chi) \cdot \prod_{v\nmid p} I_{\gamma,v}(f'_v, \chi_v) \cdot I^\dagger_{\gamma,p}(f^\dagger_p, \chi_v). \end{aligned} \tag{7.2.1}$$

Then we may define and expand

$$I^\dagger_{K_p}(f'^p f^\dagger_p, \chi) := \sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her}}_{\mathbf{Q}_p}} I^\dagger_{\Pi,K_p}(f'^p f^\dagger_p, \chi) \quad = \sum_{\gamma \in \mathrm{B}'(F_0)} I^\dagger_{\gamma,K_p}(f'^p f^\dagger_p, \chi), \tag{7.2.2}$$

where the geometric expansion is valid if f'^p has weakly plus-regular support or K_p is a CSDI, and is a consequence of Proposition 4.2.2 (5), the definition (6.1.1), and Proposition 6.1.2 (1).

[16] The results of Deligne–Ribet are stated for totally even characters only and include the interpolation at varying k: it is well-known that this allows to obtain the case of a totally odd character ξ by reduction to the totally even character $\xi\omega$, where ω is the Teichmüller character.

For an integer $s \geq 1$, we denote $\varpi_p := \prod_{v|p} \varpi_v \in \mathscr{O}_{F_{0,p}}$ and

$$\Gamma_{F_0,s} := \Gamma_{F_0}/(1 + \varpi_p^s \mathscr{O}_{F_{0,p}}).$$

Lemma 7.2.2. *Let $f'^p \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ$. There is a constant $c(f'^p, K_p) \in \mathbf{Q}^\times$ such that following holds. For each $\gamma \in \mathrm{B}'(F_0)^\circ$, each s that is sufficiently positive for K_v for all $v|p$ and each $(f_v^\dagger)_{v|p} \in \prod_{v|p} \mathscr{H}_v^\dagger(\mathscr{O}_L)$ that are sufficiently positive for s, if we set $f^\dagger = f'^p \otimes \otimes_{v|p} f_v^\dagger$, the maps*

$$Y(\varpi_p^s) \ni \chi \longmapsto I_{\gamma,K}^\dagger(f^\dagger, \chi)$$
$$Y(\varpi_p^s) \ni \chi \longmapsto I_{\gamma,K}^\dagger(f^\dagger, \chi) \quad \textit{(if } K_p \textit{ is a CSDI or } f^{'p} \textit{ has plus-regular support)}$$

extend by linearity to functionals $C(\Gamma_{F_0,s}, \mathscr{O}_L) \to c(f'^p, K_p)\mathscr{O}_L$.

Proof. The desired extension of $I_{\gamma,K_p}^\dagger(f^\dagger)$ is the convolution of the measure on $\Gamma_{F_0,s}$ given by $I_{\gamma,K_v}^\dagger(f_v)$, which are bounded in terms of K_p by Lemma 6.1.1 (2), and (the restriction of) $\mathscr{I}_\gamma^p(f'^p)$, which we have seen to be bounded uniformly in γ. For $I_{K_p}^\dagger(f^\dagger)$, the extension is defined via the (finite) geometric expansion. □

7.2.4. *Proof of Theorem 7.1.3 / II.* Corollary 5.3.4 shows that in the limit

$$I_{K_p}^{\mathrm{ord}}(f'^p, \chi) := \lim_{N\to\infty} I_{K_p}^\dagger(f'^p U_{t,K^p}^{N!}, \chi) = \sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}} \frac{1}{4} e_p(\mathrm{M}_\Pi, \chi) \mathscr{L}(\mathrm{M}_\Pi, \chi) \cdot (\otimes_{v\nmid p} I_{\Pi_v})(f'^p, \chi^p). \tag{7.2.3}$$

The existence of the limit and (7.2.2) prove that the orbital-integral functions

$$I_{(-)}^p(f'^p, \chi) \colon \gamma \longmapsto \kappa(\mathbf{1}_\infty)^{-1} \cdot (\otimes_{v\nmid p} I_{\gamma,v})(f'^p, \chi^p)$$

are $I_{\gamma,p,K_p}^{\mathrm{ord}}(\chi_p)$-integrable, and that

$$I_{K_p}^{\mathrm{ord}}(f'^p, \chi) = \int_{\mathrm{B}'_?(F_0)^\circ} I_\gamma^p(f'^p, \chi)\, dI_{\gamma,p,K_p}^{\mathrm{ord}}(\chi_p), \tag{7.2.4}$$

where $? = \emptyset$ if K_p is a CSDI and $? = \mathrm{rs}$ otherwise.

Now by Lemma 7.2.2, the map $\chi \mapsto I_{K_p}^{\mathrm{ord}}(f'^p, \chi)$ coincides, for each s, with the evaluation of a limit of uniformly (in both $f^\dagger = U_t^{N!}$ and s) bounded Radon measures on $\Gamma_{F_0,s}$, hence it extends uniquely to a bounded Radon measure

$$\mathscr{I}_{K_p}(f'^p) \colon C(\Gamma_{F_0}, L) \longrightarrow L \tag{7.2.5}$$

corresponding to the element $\mathscr{I}_{K_p}(f'^p) \in \mathscr{O}(\mathscr{Y}_L)$ of part (6).

The geometric expansion in part (6a) is (7.2.4). Then if K_p is a CSDI, by Lemma 7.2.3 below applied to (7.1.4), the distribution $\mathscr{I}_{K_p}$ has the geometric expansion described in part (6b).

Lemma 7.2.3. *Let $(\mathscr{I}_N)_{N\in\mathbf{N}}$, $\mathscr{I}_\infty \in \mathscr{O}(\mathscr{Y})$. Suppose that for all $\chi \in Y(p^\infty)$ we have*

$$\lim_{N\to\infty} \mathscr{I}_N(\chi) = \mathscr{I}_\infty(\chi).$$

Then $\lim_{N\to\infty} \mathscr{I}_N = \mathscr{I}_\infty$.

Proof. Recall that $Y(p^\infty) = \varinjlim Y(p^s)$; then observe that the ideals $J_s := \mathrm{Ker}[\mathscr{O}(\mathscr{Y}) \to \mathscr{O}(Y(p^s)) \subset \prod_{\chi\in Y(p^s)} \mathbf{Q}_p(\chi)]$ form a fundamental system of neighbourhoods of 0 in $\mathscr{O}(\mathscr{Y})$. □

We now turn to the p-adic L-function, then to the spectral expansion of $\mathscr{I}_{K_p}$.

7.2.5. *Proof of Theorem 7.1.1 (= Theorem B).* Let $K_p = \prod_v K_v$ be a convenient subgroup such that for every place $v|p$ of F_0, the representation Π_v is K_v-ordinary. Similarly to the proof ot Theorem 4.2.1, we use Corollary 4.3.4 asserting the existence of suitable test Gaussians. (Note that as $\chi_{|\mathbf{A}^{p,\times}}$ is smooth, the definition of 'adapted to (Π, χ, K_p)' in § 4.2.2 still makes sense, and the proof of that corollary goes through.)

For any $\chi \in Y(p^\infty)_L$ and any $f'^p \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^{p\infty}), L)^\circ_{K_p,\mathrm{rs},\Pi^p,\chi^p}$, we define

$$\mathscr{L}_p(\mathrm{M}_\Pi, \cdot)_{f'^p} := \frac{4\,\mathscr{I}_{K_p}(f'^p, \cdot)}{c_{K_p}(\Pi)\cdot(\otimes_{v\nmid p} I_{\Pi_v})(f'_v, \cdot)}$$

away from the zero set $\mathscr{Z}(f'^p)$ of the denominator. Note that we may assume that f'^p is a pure tensor with factors equal to the unit Hecke measure at places $v \nmid p\infty$ where Π_v is unramified; if so, each $\mathscr{Z}(f'^p)$ is the pullback of a closed subset $Z(f'^p) \subset Y_S(1)_L := \prod_{v\in S} Y_v(1)_L$ for some fixed set of places S. As $\mathscr{I}_{K_p}$ restricts to $I^{\mathrm{ord}}_{K_p}$, it follows from (7.2.3) that the functions $\mathscr{L}_p(\mathrm{M}_\Pi, \cdot)_{f'^p}$ glue to a function $\mathscr{L}_p(\mathrm{M}_\Pi, \cdot)$ with the desired interpolation properties, on the complement of the polar locus $\mathscr{Z} := \bigcap_{f'\in\mathscr{H}(\mathrm{G}'(\mathbf{A}^{p\infty}),L)^\circ_{K_p,\mathrm{rs},\Pi^p,\chi^p}} \mathscr{Z}(f'^p) \subset \mathscr{Y}_L$. By Corollary 4.3.4, the closed subset $\mathscr{Z}$ is empty. The function $\mathscr{L}_p(\mathrm{M}_\Pi)$ is still bounded since, by the Nullstellensatz applied to a finite subproduct of $\prod_{v\nmid p\infty} Y_v(1)_L$, finitely many f'^p suffice to construct $\mathscr{L}_p(\mathrm{M}_\Pi)$. This completes the proof of Theorem 7.1.1.

7.2.6. *Proof of Theorem 7.1.3 / III.* Part (2) of the theorem is now clear. The spectral expansion of $\mathscr{I}_{K_p}$ in part (6) then follows from the definitions and (7.2.4). This completes the proof of Theorem 7.1.3.

7.3. **Derivative of the analytic RTF.** We study the derivative of the distribution $\mathscr{I}_{K_p}$.

7.3.1. *Notation.* 'Denote by $\mathfrak{m} \subset \mathscr{O}_{\mathscr{Y}}$ the ideal of functions vanishing at $\chi = \mathbf{1}$. For a $\mathscr{Y}$-scheme $\mathscr{Y}'$ and a function $\Phi \in \mathfrak{m}\mathscr{O}(\mathscr{Y}')$, we say that Φ *vanishes at* $\chi = \mathbf{1}$ and we denote by $\partial\Phi$ be the image of Φ in $\mathfrak{m}/\mathfrak{m}^2 \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\mathscr{Y}'} = \mathscr{O}_{\mathscr{Y}'}\hat{\otimes}\Gamma_{F_0}$.

For $V \in \mathscr{V}^\circ$ a coherent or incoherent pair of definite hermitian spaces as in §2.1.3, and v a finite place of F_0, we let:

– $\mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V} \subset \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}$ be the subset of those isomorphism classes of representations Π such that for each finite place u of F_0, the space V_u is the one attached to Π_u by the local Gan–Gross–Prasad conjecture (Proposition 2.4.1).
– $\mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^{\circ,V}_{K_p,\mathrm{qc}} \subset \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ_{K_p,\mathrm{qc}}$ be the subspace of those f'^p that match (spectrally and geometrically, see Proposition 3.5.3) a function on $\mathscr{H}(\mathrm{G}^V(\mathbf{A}^p), L)^\circ := \mathscr{H}(\mathrm{G}^V(\mathbf{A}^p), L) \otimes_L L f^\circ_\infty$;
– $\mathbf{1}_V$ be the characteristic function of $\mathrm{B}'_{\mathrm{rs}}(\mathbf{A})_V := \prod'_u B'_{\mathrm{rs},u,V_u} \subset \mathrm{B}'_{\mathrm{rs}}(\mathbf{A})$;

– $V(v) \in \mathscr{V}^\circ$ be the pair such that $V(v)_u \cong V_u$ exactly for $u \neq v$; it is coherent if and only if V is incoherent.

Proposition 7.3.1. *Consider the situation of Theorem 7.1.3, and let $V \in \mathscr{V}^{\circ,-}$ be an incoherent pair. For all $f'^p \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^{\circ,V}_{K_p,\mathrm{qc}}$, the following hold.*

(1) For all $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}$ and all $\gamma \in \mathrm{B}'(F_0)$,

$$\mathscr{I}_{K_p}(f'^p, \mathbf{1}) = \mathscr{I}_\Pi(f'^p, \mathbf{1}) = \mathscr{I}^p_\gamma(f'^p, \mathbf{1}) = 0.$$

(2) There is a spectral expansion

$$\partial\mathscr{I}_{K_p}(f'^p) = \sum_{\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V}_{K_p}} \partial\mathscr{I}_{\Pi,K_p}(f'^p)$$

where

$$\partial\mathscr{I}_{\Pi,K_p}(f'^p) = \frac{1}{4}\, c_{K_p}(\Pi)\, \partial\mathscr{L}_p(\mathrm{M}_\Pi) \cdot (\otimes_{v \nmid p} \mathscr{I}_{\Pi_v})(f'^p, \mathbf{1}).$$

(3) Suppose that f'^p has weakly regular semisimple support. The function $\partial\mathscr{I}^p_{(-)}(f'^p)$ is integrable for the Radon measure $I^{\mathrm{ord}}_{\gamma,p,K_p} \coloneqq I^{\mathrm{ord}}_{\gamma,p,K_p}(\mathbf{1})$, and there is a geometric expansion

$$\begin{aligned}\partial\mathscr{I}_{K_p}(f'^p) &= \int_{\mathrm{B}'_{\mathrm{rs}}(F_0)^\circ} \partial\mathscr{I}^p_\gamma(f'^p)\, dI^{\mathrm{ord}}_{\gamma,p,K_p} \\ &= \int_{\mathrm{B}'_{\mathrm{rs}}(F_0)^\circ} \sum_{v \nmid p\infty \ \text{nonsplit}} \mathbf{1}_{V(v)}(\gamma) \mathscr{I}^{vp}_\gamma(f'^{vp}, \mathbf{1}) \cdot \partial\mathscr{I}_{\gamma,v}(f'_v)\, dI^{\mathrm{ord}}_{\gamma,p,K_p},\end{aligned} \tag{7.3.1}$$

where for any γ we put $\mathscr{I}^{vp}_\gamma \coloneqq \kappa(\mathbf{1}_\infty)^{-1} \cdot L_{p,\gamma} \otimes_{u \nmid vp} \mathscr{I}_{\gamma,u}$.

Proof. Consider the geometric terms $\mathscr{I}_\gamma(f'^p, \mathbf{1})$. For $\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0) \cap B'^\circ_\infty$, let $V_\gamma \in \mathscr{V}^{\circ,+}$ be the unique coherent pair such that γ matches an orbit in $B_{\mathrm{rs}}(F_0)_{V_\gamma}$ as in (3.5.4); let $\Sigma(\gamma, V)$ be the non-empty finite set of non-archimedean (and necessarily nonsplit) places of F_0 such that $V_{\gamma,v} \not\cong V_v$. If $v \in \Sigma(\gamma, V)$, then by the assumption on f'^p we have $I_{\gamma,v}(f'_v, \mathbf{1}) = 0$; hence $\mathscr{I}^p_\gamma(f'^p)$ vanishes at $\mathbf{1}$ to order at least $|\Sigma(\gamma, V)| \geq 1$. Moreover, if $v \in \Sigma(\gamma, V)$ then

$$\partial\mathscr{I}_\gamma(f'^p) = \mathscr{I}^{vp}_\gamma(f'^{vp}, \mathbf{1}) \cdot \partial\mathscr{I}_{\gamma,v}(f'_v), \tag{7.3.2}$$

which can be nonzero only if $\Sigma(\gamma, V) = \{v\}$, equivalently $V_\gamma = V(v)$.

Consider now a representation $\Pi = \Pi_n \boxtimes \Pi_{n+1} \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord}}$. Let $V_\Pi \in \mathscr{V}^{\circ,\epsilon(\Pi)}$ be the pair such that $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V_\Pi}$ (cf. Remark 2.5.7). If $\epsilon(\Pi) = -1$, then $\mathscr{L}(\mathrm{M}_\Pi, \mathbf{1}) = 0$ by the functional equation of Rankin–Selberg L-functions; this implies $\mathscr{I}_\Pi(f'^{p\infty}, \mathbf{1}) = 0$. If $\varepsilon(\Pi) = +1$, then for any finite place v such that $V_{\Pi,v} \not\cong V_v$, we have $I_{\Pi_v}(f'_v, \mathbf{1}) = 0$ by the assumption on f'^p. This completes the proof of part (1). More generally, we note that the last argument shows that

$$\Pi \notin \mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V} \implies I_{\Pi_v}(f'_v, \mathbf{1}) = 0 \ \text{ for some } v \nmid p\infty. \tag{7.3.3}$$

This shows that the sum in part (2) indeed runs over $\mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V}$; as above, this implies that $\varepsilon(\Pi) = -1$ and $\mathscr{L}_p(\mathrm{M}_\Pi, \mathbf{1}) = 0$, which implies the second equality in (2).

We now consider part (3). By the definition of the measure $I^{\mathrm{ord}}_{\gamma,p,K_p}$, the second equality follows from (7.3.2), whose right-hand side can be nonzero only if $V_\gamma = V(v)$. We consider the expansion in the first equality; it will hold without the condition that f'^p is quasicuspidal, and by linearity we may thus assume that $f'^p = \otimes_v f'_v$ is a pure tensor. Suppose first that K_p is a CSDI. Since $\int dI^{\mathrm{ord}}_{\gamma,p,K_p}$ is a bounded functional, we simply differentiate under the integral sign in Theorem 7.1.3 (6). We now consider the general case. Viewing $\mathscr{I}_{K_p}$ as in (7.2.5) we have

$$\partial \mathscr{I}_{K_p}(f'^p) = \mathscr{I}_{K_p}(f'^p, \ell)$$

(see for instance, [DL24, Lemma 3.42]), where the 'logarithm' $\ell\colon \Gamma_{F_0} \to \Gamma^{\mathrm{fr}}_{F_0} \subset \Gamma_{F_0} \hat{\otimes} \mathbf{Q}_p$ is the projection onto the maximal $\mathbf{Z}_p$-free quotient $\Gamma^{\mathrm{fr}}_{F_0}$ of Γ_{F_0}.

For $s \geq 1$, let $\ell_s\colon \Gamma_{F_0} \to \Gamma^{\mathrm{fr}}_{F_0}/p^s$ be the reduction map, and let $\widetilde{\ell}_s\colon \Gamma_{F_0} \to \Gamma^{\mathrm{fr}}_{F_0}/p^s \to \Gamma^{\mathrm{fr}}_{F_0}$ be any lift of ℓ_s, which is a linear combination of characters whose conductors at places $v|p$ do not exceed s. By the definition of $\mathscr{I}_{K_p}$, the expansion (7.2.3), and linearity, we have

$$\mathscr{I}_{K_p}(\widetilde{\ell}_s) = \lim_{N\to\infty} I^{\dagger}_{K_p}(f'^p U_{t_p}^{N!}, \widetilde{\ell}_s)$$

Then by Proposition 4.2.2 (5), we have

$$\mathscr{I}_{K_p}(\widetilde{\ell}_s) = \lim_{N\to\infty} \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I^{\dagger}_{\gamma}(f'^p U_{t_p}^{N!}, \widetilde{\ell}_s). \tag{7.3.4}$$

By Lemma 7.2.2, up to multiplying f'^p by a power of p independent of s we have that all terms in (7.3.4) are p-integral; hence it makes sense to consider the reduction of that identity modulo p^s,

$$\mathscr{I}_{K_p}(\ell_s) = \lim_{N\to\infty} \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I^{\dagger}_{\gamma}(f'^p U_{t_p}^{N!}, \ell_s)$$

in $\Gamma^{\mathrm{fr}}_{F_0}/p^s$. Now $\ell_s = \sum_{v\nmid\infty} \ell_{s,v}$, where $\ell_{s,v} := \ell_{s|F_{0,v}^\times}$, so from Remark 4.5.1, the γ-summand equals

$$\sum_{v\nmid\infty} \frac{I_\gamma(f'_\infty)}{\kappa(\mathbf{1}_\infty)\kappa_\infty(\gamma')} \int_{\mathrm{H}_1(\mathbf{A}^\infty)} \int_{\mathrm{H}_2(\mathbf{A}^\infty)} f'^\infty(h_1^{-1}\gamma' h_2) \ell_{s,v}(h_{1,v}) \eta(h_2)\, \frac{d^\natural h_1 d^\natural h_2}{d^\natural g} \tag{7.3.5}$$

in $\Gamma^{\mathrm{fr}}_{F_0}/p^s$; here $\gamma' \in \mathrm{G}'_{\mathrm{rs}}(F_0)$ is any preimage of γ. (Note that only finitely many v-summands are nonzero, hence it is trivial to interchange sum and integration.) For $v \nmid p$, the v-summand is

$$\begin{aligned}
&\frac{I_\gamma(f'_\infty)}{\kappa(\mathbf{1}_\infty)\kappa_\infty(\gamma')} I^{\dagger}_{\gamma,p}(U_{t_p}^{N!}, \mathbf{1}) \mathscr{I}_{\gamma}^{vp\infty}(f'^{vp\infty}, \mathbf{1}) \cdot \mathscr{I}_{\gamma,v}(f'_v, \ell_{s,v}) \\
\equiv &\frac{1}{\kappa(\mathbf{1}_\infty)\kappa_\infty(\gamma')} I^{\dagger}_{\gamma,p}(U_{t_p}^{N!}, \mathbf{1}) \mathscr{I}_{\gamma}^{vp}(f'^{vp}, \mathbf{1}) \cdot \partial \mathscr{I}_{\gamma,v}(f'_v).
\end{aligned}$$

For $v|p$, the v-summand in (7.3.5) is a multiple of $\mathscr{I}_\gamma^p(f'^p, \mathbf{1})$, which is zero by part (1). Therefore $\partial \mathscr{I}_{K_p}(f'^p)$ is congruent to

$$\mathscr{I}_{K_p}(f'^p, \ell) \equiv \lim_{N\to\infty} \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} I^{\dagger}_{\gamma,p}(U_{t_p}^{N!}, \mathbf{1}) \cdot \partial \mathscr{I}_\gamma^p(f'^p)$$

in $\Gamma^{\mathrm{fr}}_{F_0}/p^s$ for all s. We conclude that the above congruences amount to an equality in $\Gamma^{\mathrm{fr}}_{F_0}$; by definition, the right-hand side is

$$\int_{\mathrm{B}'_{\mathrm{rs}}(F_0)} \partial \mathscr{I}_\gamma^p(f'^p)\, dI^{\mathrm{ord}}_{\gamma,p,K_p},$$

as desired. □

Part 2. p-adic heights and the arithmetic relative-trace formula

We now study the p-adic heights of Gan–Gross–Prasad cycles. In §8, we recall the relevant Shimura varieties, the arithmetic diagonal cycles, and their moduli interpretations over the reflex fields. In §9, we study various integral models and prove some vanishing results for their cohomologies. In § 10 we collect the necessary definitions and results on cycles and p-adic heights. In §11, we define the arithmetic relative-trace distribution encoding the heights of GGP cycles, and prove the corresponding RTF.

In §§8-9, we use slightly different notation on unitary groups from the rest of the paper.

8. Unitary Shimura varieties and arithmetic diagonals

For this section and the next one, we largely follow [RSZ20, RSZ21].

8.1. **Unitary Shimura varieties.** We keep denoting by F a CM number field with maximal totally real subfield F_0 and nontrivial F/F_0-automorphism $\mathrm{c}\colon a \mapsto a^{\mathrm{c}}$. For an algebraic group G over F_0, we denote its restriction of scalars to $\mathbf{Q}$ by

$$\mathrm{G}^\flat := \mathrm{Res}_{F_0/\mathbf{Q}}\,\mathrm{G}.$$

8.1.1. *Unitary Shimura data and the associated varieties.* We denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. Let ν be a positive integer. Recall from [RSZ21, §2.2] that a *generalized CM type* (or a *signature type*) of rank ν is a function $r\colon\ \mathrm{Hom}_{\mathbf{Q}}(F,\overline{\mathbf{Q}}) \to \mathbb{Z}_{\geq 0}$, denoted $\varphi \mapsto r_\varphi$, such that

$$r_\varphi + r_{\varphi^{\mathrm{c}}} = \nu \quad \text{for all} \quad \varphi \in \mathrm{Hom}_{\mathbf{Q}}(F,\overline{\mathbf{Q}}); \tag{8.1.1}$$

here $\varphi^{\mathrm{c}} := \varphi \circ \mathrm{c}$. When $\nu = 1$, a generalized CM type is "the same" as a usual CM type, via

$$\Phi = \left\{\, \varphi \in \mathrm{Hom}_{\mathbf{Q}}(F,\overline{\mathbf{Q}}) \;\middle|\; r_\varphi = 1 \,\right\}.$$

Fix a CM type Φ of F, and let $(W, (\ ,\ ))$ be an F/F_0-hermitian vector space of dimension ν. The signatures of W at the archimedean places determine a generalized CM type r of rank n, by writing

$$\mathrm{sig}\, W_\varphi = (r_\varphi, r_{\varphi^{\mathrm{c}}}), \quad \varphi \in \Phi, \quad W_\varphi := W \otimes_{F,\varphi} \mathbf{C}.$$

Consider the unitary group

$$\mathrm{G} := \mathrm{U}(W). \tag{8.1.2}$$

For each $\varphi \in \Phi$, choose a $\mathbf{C}$-basis of W_φ with respect to which the matrix of the hermitian form $(\ ,\ )$ is given by $\mathrm{diag}(1_{r_\varphi}, -1_{r_{\varphi^{\mathrm{c}}}})$ We now define a Shimura datum $(\mathrm{G}^\flat, \{h^\flat_{\mathrm{G}}\})$, where $\{h^\flat_{\mathrm{G}}\}$ is a $\mathrm{G}^\flat(\mathbf{R})$-conjugacy class of homomorphisms $\mathbb{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\,\mathbf{G}_{m,\mathbf{R}} \to \mathrm{G}^\flat_{\mathbf{R}}$. With respect to the inclusion and the identification induced by the fixed CM type Φ,

$$\mathrm{G}^\flat(\mathbf{R}) \subset \mathrm{GL}_{F\otimes\mathbf{R}}(W\otimes\mathbf{R}) \xrightarrow[\sim]{\Phi} \prod_{\varphi\in\Phi} \mathrm{GL}_{\mathbf{C}}(W_\varphi),$$

we define $h_{\mathrm{G}^\flat}$ as $(h_{\mathrm{G}^\flat,\varphi})_{\varphi\in\Phi}$ where the φ-component is defined on $\mathbf{C}^\times$ by

$$h_{\mathrm{G}^\flat,\varphi}\colon z \longmapsto \mathrm{diag}\big(1_{r_\varphi}, (z^{\mathrm{c}}/z)1_{r_{\varphi^{\mathrm{c}}}}\big).$$

The reflex field $E(\mathrm{G}^\flat, h_{\mathrm{G}^\flat})$ of this Shimura datum is the reflex field $E_{r^\natural}$ of the function $r^\natural$, characterized by

$$\mathrm{Gal}(\overline{\mathbf{Q}}/E_{r^\natural}) = \big\{\,\sigma\in\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \,\big|\, \sigma^*(r^\natural) = r^\natural\,\big\}, \tag{8.1.3}$$

where we define a modified function

$$\begin{aligned} r^\natural\colon\ \ \mathrm{Hom}_{\mathbf{Q}}(F,\overline{\mathbf{Q}}) &\longrightarrow \mathbb{Z}_{\geq 0} \\ \varphi &\longmapsto \begin{cases} 0, & \varphi\in\Phi; \\ r_\varphi, & \varphi\in\Phi^{\mathrm{c}}. \end{cases} \end{aligned} \tag{8.1.4}$$

We then obtain a tower of Shimura varieties $(\mathrm{Sh}_K(\mathrm{G}^\flat,\{h_{\mathrm{G}^\flat}\}))_{K\subset\mathrm{G}(\mathbf{A}^\infty)}$ over $E_{r^\natural}$.

Remark 8.1.1. The Shimura variety $\mathrm{Sh}_K(\mathrm{G}^\flat,\{h_{\mathrm{G}^\flat}\})$ is not of PEL type, i.e., it is not related to a moduli problem of abelian varieties (this can be seen already from the fact that the restriction of $\{h_{\mathrm{G}^\flat}\}$ to $\mathbf{G}_m\subset\mathbb{S}$ is not mapped via the identity map to the center of $\mathrm{G}^\flat$). However, this Shimura variety is of abelian type.

8.1.2. *A special signature type.* If the generalized CM type r satisfies

$$r_\varphi = \begin{cases} \nu-1, & \varphi=\varphi_0, \\ \nu, & \varphi\in\Phi\smallsetminus\{\varphi_0\}, \end{cases} \tag{8.1.5}$$

for some CM type Φ and some $\varphi_0\in\Phi$, we say that the signature type of r is *strictly fake Drinfeld* (with respect to (Φ,φ_0)). In this case, we have $\varphi_0\colon F\xrightarrow{\sim}E_{r^\natural}$ for all $\nu\geq 1$; in other words the reflex field is F via the embedding $\varphi_0\colon F\to\mathbf{C}$ (cf. [RSZ21, Example 2.3 (ii)]). In this paper, we will only consider data of strict fake Drinfeld type. We will abbreviate $\mathrm{Sh}_K(\mathrm{G}) \coloneqq \mathrm{Sh}_K(\mathrm{G}^\flat,\{h_{\mathrm{G}^\flat}\})$, omitting the superscript $\flat$ and suppressing the datum $\{h_{\mathrm{G}^\flat}\}$.

8.1.3. *Hecke correspondences.* Recall that if $\mathscr{X}$ be a scheme, a *correspondence* on $\mathscr{X}$ is a diagram of finite morphisms

$$\begin{tikzcd} & \mathscr{X}' \arrow[dl] \arrow[dr] & \\ \mathscr{X} & & \mathscr{X}. \end{tikzcd}$$

It is said to be *étale* if both morphisms are étale. Correspondences on $\mathscr{X}$ form a monoidal category under composition. If L is a ring, we denote by $\mathrm{\acute{E}tCorr}(\mathscr{X})_L)$ the L-algebra generated by isomorphism classes of étale correspondences on $\mathscr{X}$. It acts (on the right) on cycles and cohomology of $\mathscr{X}$ by pullback and pushforward.

For each $K \subset \mathrm{G}(\mathbf{A}^\infty)$, and each characteristic-zero field L, we have an L-algebra homomorphism

$$T\colon \mathscr{H}(\mathrm{G}(\mathbf{A}^\infty), L)_K \longrightarrow \acute{\mathrm{E}}\mathrm{tCorr}(\mathrm{Sh}_K(\mathrm{G}))$$

$$[KgK] \longmapsto \left[\begin{tikzcd} & \mathrm{Sh}_{K'}(\mathrm{G}) \arrow[dl, "\mathrm{nat}_1"'] \arrow[dr, "\mathrm{nat}_g"] & \\ \mathrm{Sh}_K(\mathrm{G}) & & \mathrm{Sh}_K(\mathrm{G}) \end{tikzcd}\right] \tag{8.1.6}$$

where $[KgK] \coloneqq \mathrm{vol}(K)^{-1}\mathbf{1}_{KgK}\, dg$, $K' \coloneqq K \cap gKg^{-1}$, and the map nat_1 is the natural map induced by the embedding $K' \subset K$ while nat_g is induced by the composition

$$\mathrm{Sh}_{K'}(\mathrm{G}) \xrightarrow{\cdot g} \mathrm{Sh}_{g^{-1}K'g}(\mathrm{G}) \longrightarrow \mathrm{Sh}_K(\mathrm{G})\ .$$

For the other Shimura varieties in this section, we also have Hecke correspondences defined in an entirely analogous way.

8.1.4. *Product Shimura varieties and the arithmetic diagonal.* Let Φ be a CM type, let W_n be a hermitian space of dimension $n \geq 1$, and assume that the associated generalized CM type r_n is of strict fake Drinfeld type. Let $W_{n+1} = W \oplus^\perp Fe$ where e has norm 1. Let $\mathrm{G}_\nu = \mathrm{U}(W_\nu)$ for $\nu = n, n+1$, and let $(\mathrm{Sh}_{K_\nu}(\mathrm{G}_\nu))_{K_\nu}$ be the corresponding tower of Shimura varieties. We also have a product Shimura variety $\mathrm{Sh}_K(\mathrm{G}) = \mathrm{Sh}_K(\mathrm{G}) = \mathrm{Sh}_K(\mathrm{G}^\flat, h_{\mathrm{G}^\flat})$ associated with $\mathrm{G} = \mathrm{G}_n \times \mathrm{G}_{n+1}$ and $h_{\mathrm{G}^\flat} = h_{\mathrm{G}_n^\flat} \times h_{\mathrm{G}_{n+1}^\flat}$. Denote $\mathrm{H} \coloneqq \mathrm{G}_n$. The map

$$\jmath\colon \mathrm{H} \longrightarrow \mathrm{G}$$

that is the graph of the natural embedding $\mathrm{G}_n \to \mathrm{G}_{n+1}$ induces a corresponding map of Shimura varieties

$$\jmath\colon \mathrm{Sh}_{K_\mathrm{H}}(\mathrm{H}) \longrightarrow \mathrm{Sh}_{K_\mathrm{G}}(\mathrm{G}) \tag{8.1.7}$$

whenever $K_\mathrm{H} \subset \jmath^{-1}(K_\mathrm{G}) \subset \mathrm{H}(\mathbf{A}^\infty)$.

The target Shimura variety has dimension $2n-1$, and the image of $\jmath$ has codimension n, in the arithmetic middle dimension (i.e., the codimension is just more than half the dimension of the ambient variety). We thus call the map (8.1.7) the arithmetic diagonal, and the image cycle (defined in more detail in § 11.2.1) the arithmetic diagonal cycle in $\mathrm{Sh}(\mathrm{G})$.

8.2. **Incoherent Shimura varieties.** For our specific signature type, we may present the above Shimura varieties more symmetrically using incoherent hermitian spaces.

8.2.1. *Shimura varieties for incoherent unitary groups.* Let V be a totally positive definite incoherent F/F_0-hermitian space of dimension ν. The theory of conjugates of Shimura varieties ([MS82]; see also [Gro, ST]) shows that there exists a unique-up-to-isomorphism tower

$$(\mathrm{Sh}_K(\mathrm{G}))_{K \subset \mathrm{G}(\mathbf{A}^\infty)}$$

over $\mathrm{Spec}\, F$ with the following property. For any CM type Φ of F and any archimedean place v_0 of F_0, let $\varphi_0 \in \Phi$ be the unique embedding above v_0, let $\mathrm{G}^{(v_0)} = \mathrm{U}(V(v_0))$ be the unitary group associated to the nearby hermitian space $V(v_0)$, and let $(\mathrm{Sh}_K(\mathrm{G}^{(v_0)}))_K$ be the tower of Shimura varieties associated with the data $(\mathrm{G}^{(v_0)}, \Phi, \varphi_0)$ as in § 8.1.2. Then

$$\mathrm{Sh}_K(\mathrm{G}) \times_{\operatorname{Spec} F, \varphi_0} \operatorname{Spec} \varphi_0(F) \xrightarrow{\sim} \mathrm{Sh}_K(\mathrm{G}^{(v_0)}) \ ,$$

where we have an isomorphism $\mathrm{G}(\mathbf{A}^\infty) \simeq \mathrm{G}^{(v_0)}(\mathbf{A}^\infty)$ induced from a fixed isometry $V(v_0)_v \simeq V_v$ for all $v \nmid \infty$. We will call $\mathrm{Sh}_K(\mathrm{G})$ the Shimura varieties attached to the incoherent hermitian space V (even though strictly speaking they are not Shimura varieties defined by Deligne).

From now on we will also make the assumption that all our unitary groups G are *anisotropic*; in the incoherent case this means that $\mathrm{G}^{(v_0)}$ is anisotropic for any (hence every) archimedean place $v_0 \in \mathrm{Hom}(F_0, \mathbf{R})$. Then $\mathrm{Sh}_K(\mathrm{G})$ is proper for any compact open subgroup $K \subset \mathrm{G}(\mathbf{A}^\infty)$; this is guaranteed if $F_0 \neq \mathbf{Q}$.

8.2.2. *The arithmetic diagonal for incoherent Shimura varieties.* Fix now an incoherent pair $V = (V_n, V_{n+1}) \in \mathscr{V}^{\circ,-}$. We denote $\mathrm{G} = \mathrm{G}^V = \mathrm{G}_n^V \times \mathrm{G}_{n+1}^V := \mathrm{U}(V_n) \times \mathrm{U}(V_{n+1})$ (an incoherent unitary group as in § 1.3.1), and let $\mathrm{Sh}_{K_\mathrm{G}}(\mathrm{G})$ denote the product of Shimura varieties constructed in §8.1.4. Then for every place $v_0 \in \mathrm{Hom}(F_0, \mathbf{R})$, let $\mathrm{G}^{(v_0)} := \mathrm{G}_n^{(v_0)} \times \mathrm{G}_{n+1}^{(v_0)} := \mathrm{U}(V(v_0)_n) \times \mathrm{U}(V(v_0)_{n+1}))$ be the unitary group associated to the nearby hermitian space $V(v_0)$. Then there exists a projective system of varieties $(\mathrm{Sh}_K(\mathrm{G}))_{K \subset \mathrm{G}(\mathbf{A}^\infty)}$ over $\operatorname{Spec} F$ such that, for every embedding $\varphi_0 \colon F \to \mathbf{C}$ extending v_0 and every choice of CM type Φ such that $\varphi_0 \in \Phi$ we have

$$\mathrm{Sh}_K(\mathrm{G}) \times_{\operatorname{Spec} F} \operatorname{Spec} \varphi_0(F) \xrightarrow{\sim} \mathrm{Sh}_K(\mathrm{G}^{(v_0)}) \tag{8.2.1}$$

where $\mathrm{Sh}_K(\mathrm{G}^{(v_0)}) = \mathrm{Sh}_K(G^{(v_0),\flat}, h_{\mathrm{G}^{(v_0),\flat}})$ with $h_{\mathrm{G}^{(v_0),\flat}} = h_{\mathrm{G}_n^{(v_0),\flat}} \times h_{\mathrm{G}_{n+1}^{(v_0),\flat}}$ (the latter defined in § 8.1.1). Similarly, we have incoherent Shimura varieties $\mathrm{Sh}_{K_\mathrm{H}}(\mathrm{H})$ for the group $\mathrm{H} = \mathrm{H}^V = \mathrm{U}(V_n)$.

As in § 8.1.4, we have (finite) maps

$$\jmath \colon \mathrm{Sh}_{K_\mathrm{H}}(\mathrm{H}) \longrightarrow \mathrm{Sh}_{K_\mathrm{G}}(\mathrm{G}), \tag{8.2.2}$$

which are the pullbacks of (8.1.7) via (8.2.1).

8.3. **RSZ Shimura varieties.** The unitary Shimura varieties above do not admit natural moduli descriptions. Hence we will relate them to RSZ Shimura varieties, which admit a PEL type moduli definition. They will play an auxiliary role when computing local heigts. We will follow [RSZ21].

8.3.1. *Shimura varieties for unitary similitude groups.* We resume the notation from §8.1. Thus let Φ be a CM type, and let W be a hermitian space of dimension $\nu \geq 1$ whose associated generalized CM type r_ν is of strict fake Drinfeld type in the sense of §8.1.2. Recall also that $\mathbf{G}_m$ denotes the multiplicative group over $\mathbf{Q}$.

We first consider the group (over $\mathbf{Q}$)

$$\mathrm{G}^{\mathbf{Q}} := \mathrm{Res}_{F_0/\mathbf{Q}} \mathrm{GU}(W) \times_{\mathrm{Res}_{F_0/\mathbf{Q}} \mathbf{G}_{m,F_0}} \mathbf{G}_m$$

of unitary similitudes of $(W, (\ ,\ ))$ with similitude factor in $\mathbf{G}_m$.

Let $\{h_{\mathrm{G}^{\mathbf{Q}}}\}$ be the $\mathrm{G}^{\mathbf{Q}}(\mathbf{R})$-conjugacy class of the homomorphism $h_{\mathrm{G}^{\mathbf{Q}}} = (h_{\mathrm{G}^{\mathbf{Q}},\varphi})_{\varphi\in\Phi}$, where the components $h_{\mathrm{G}^{\mathbf{Q}},\varphi}$ are defined with respect to the inclusion

$$G^{\mathbf{Q}}(\mathbf{R}) \subset \mathrm{GL}_{F\otimes\mathbf{R}}(W\otimes\mathbf{R}) \xrightarrow[\sim]{\Phi} \prod_{\varphi\in\Phi} \mathrm{GL}_{\mathbf{C}}(W_\varphi),$$

and where each component is defined on $\mathbf{C}^\times$ by

$$h_{G^{\mathbf{Q}},\varphi}\colon z \longmapsto \mathrm{diag}(z\cdot 1_{r_\varphi}, z^{\mathrm{c}}\cdot 1_{r_{\varphi^{\mathrm{c}}}}).$$

We single out the special case $\nu = 1$. We let $W = W_0$ be totally definite and we write $\mathrm{Z}^{\mathbf{Q}} := \mathrm{G}^{\mathbf{Q}}$ (a torus over $\mathbf{Q}$) and $h_{\mathrm{Z}^{\mathbf{Q}}} := h_{\mathrm{G}^{\mathbf{Q}}}$. Explicitly,

$$\mathrm{Z}^{\mathbf{Q}} = \left\{\, z \in \mathrm{Res}_{F/\mathbf{Q}}\,\mathbf{G}_m \;\middle|\; \mathrm{Nm}_{F/F_0}(z) \in \mathbf{G}_m \,\right\}.$$

The reflex field of $(\mathrm{Z}^{\mathbf{Q}}, \{h_{\mathrm{Z}^{\mathbf{Q}}}\})$ is E_Φ, the reflex field of the CM type Φ.

8.3.2. *RSZ Shimura varieties.* The Shimura varieties of [RSZ20] are attached to the group

$$\widetilde{\mathrm{G}} := \mathrm{Z}^{\mathbf{Q}} \times_{\mathbf{G}_m} \mathrm{G}^{\mathbf{Q}}, \tag{8.3.1}$$

where the maps from the factors on the right-hand side to $\mathbf{G}_m$ are respectively given by Nm_{F/F_0} and the similitude character. In terms of the Shimura data already defined, we obtain a Shimura datum for $\widetilde{\mathrm{G}}$ by defining the Shimura homomorphism to be

$$h_{\widetilde{\mathrm{G}}}\colon \mathbf{C}^\times \xrightarrow{(h_{\mathrm{Z}^{\mathbf{Q}}}, h_{\mathrm{G}^{\mathbf{Q}}})} \widetilde{\mathrm{G}}(\mathbf{R}).$$

Then $(\widetilde{\mathrm{G}}, \{h_{\widetilde{\mathrm{G}}}\})$ has reflex field $E \subset \overline{\mathbf{Q}}$ characterized by

$$\begin{aligned} \mathrm{Gal}(\overline{\mathbf{Q}}/E) &= \left\{\, \sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \;\middle|\; \sigma\circ\Phi = \Phi \text{ and } \sigma^*(r) = r \,\right\} \\ &= \left\{\, \sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \;\middle|\; \sigma\circ\Phi = \Phi \text{ and } \sigma^*(r^\natural) = r^\natural \,\right\}. \end{aligned} \tag{8.3.2}$$

In other words, the reflex field is the common composite $E = E_\Phi E_r = E_\Phi E_{r^\natural} = E_\Phi$ for our signature type (8.1.5).

Remark 8.3.1. The RSZ Shimura varieties are related to the unitary Shimura varieties as follows. The torus $\mathrm{Z}^{\mathbf{Q}}$ embeds naturally as a central subgroup of $\mathrm{G}^{\mathbf{Q}}$, which gives rise to a product decomposition

$$\begin{aligned} \widetilde{\mathrm{G}} &\xrightarrow{\;\sim\;} \mathrm{Z}^{\mathbf{Q}} \times \mathrm{G}^\flat \\ (z, g) &\longmapsto (z, z^{-1}g), \end{aligned} \tag{8.3.3}$$

where $\mathrm{G}^\flat \subset \mathrm{G}^{\mathbf{Q}}$ is the restriction of scalars of the unitary group (8.1.2). The isomorphism (8.3.3) extends to a product decomposition of Shimura data,

$$\big(\widetilde{\mathrm{G}}, \{h_{\widetilde{\mathrm{G}}}\}\big) \cong \big(\mathrm{Z}^{\mathbf{Q}}, \{h_{\mathrm{Z}^{\mathbf{Q}}}\}\big) \times \big(\mathrm{G}^\flat, \{h_{\mathrm{G}}^\flat\}\big). \tag{8.3.4}$$

Hence, for a decomposable compact open subgroup $K_{\widetilde{\mathrm{G}}} = K_{\mathrm{Z}^{\mathbf{Q}}} \times K_{\mathrm{G}^\flat}$, there is a product decomposition

$$\mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}\big(\widetilde{\mathrm{G}}, \{h_{\widetilde{\mathrm{G}}}\}\big) \cong \mathrm{Sh}_{K_{\mathrm{Z}^{\mathbf{Q}}}}\big(\mathrm{Z}^{\mathbf{Q}}, \{h_{\mathrm{Z}^{\mathbf{Q}}}\}\big) \times \mathrm{Sh}_{K_{\mathrm{G}}}\big(\mathrm{G}, \{h_{\mathrm{G}}\}\big),$$

of Shimura varieties over E.

8.3.3. *Product Shimura varieties and the arithmetic diagonal.* Let now $W = (W_n, W_{n+1}) \in \mathscr{V}$ and $\mathrm{G} \coloneqq \mathrm{G}_n \times \mathrm{G}_{n+1} \coloneqq \mathrm{U}(W_n) \times \mathrm{U}(W_{n+1})$ be as in § 8.1.4. Similar to (8.3.1) we set

$$\widetilde{\mathrm{G}} \coloneqq \mathrm{Z}^{\mathbf{Q}} \times_{\mathbf{G}_m} \mathrm{G}_n^{\mathbf{Q}} \times_{\mathbf{G}_m} \mathrm{G}_{n+1}^{\mathbf{Q}}. \tag{8.3.5}$$

where $\mathrm{G}_\nu^{\mathbf{Q}}$ is the similitude unitary group attached to W_ν as in (8.3.5). We have an analogous Shimura datum with the reflex field $E = E_\Phi$, and an isomorphism induced by (8.3.3)

$$\widetilde{\mathrm{G}} \xrightarrow{\sim} \mathrm{Z}^{\mathbf{Q}} \times \mathrm{G}^\flat. \tag{8.3.6}$$

In this situation, we will always assume that the open compact $K_{\widetilde{\mathrm{G}}}$ is *decomposable* of the form $K_{\widetilde{\mathrm{G}}} = K_{\mathrm{Z}^{\mathbf{Q}}} \times K_{\mathrm{G}} = K_{\mathrm{Z}^{\mathbf{Q}}} \times K_n \times K_{n+1}$. In particular, we have a finite étale morphism $\mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}(\widetilde{\mathrm{G}}) \to \mathrm{Sh}_{K_{\mathrm{G}}}(\mathrm{G})_E$ over $\operatorname{Spec} E$.

Moreover, let $\widetilde{\mathrm{H}} \coloneqq \widetilde{\mathrm{G}}_n$. Then we have a map $\jmath \colon \widetilde{\mathrm{H}} \to \widetilde{\mathrm{G}}$ and corresponding maps

$$\mathrm{Sh}_{K_{\widetilde{\mathrm{H}}}}(\widetilde{\mathrm{H}}) \longrightarrow \mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}(\widetilde{\mathrm{G}}) \tag{8.3.7}$$

that are the pullbacks of (8.1.7) along the projection $\mathrm{Sh}_{K_{\mathrm{Z}^{\mathbf{Q}}} \times K_{\mathrm{G}}}(\widetilde{\mathrm{G}}) \to \mathrm{Sh}_{K_{\mathrm{G}}}(\mathrm{G})$ given by (8.3.6).

8.4. **Moduli functors over E.** We formulate the PEL type moduli functor for RSZ Shimura varieties, following [RSZ21, §3]. Denote by $(\mathrm{LNSch})_{/R}$ the category of locally noetherian schemes over a ring R, and by Sets the category of sets.

8.4.1. *The torus case.* First we consider the torus $\mathrm{Z}^{\mathbf{Q}}$. The construction of [RSZ21, §2.2], specialized to $n = 1$, gives a Kottwitz PEL moduli functor $(\mathrm{LNSch})_{/E} \to \mathrm{Sets}$, which is represented by a finite étale stack $M_{0,K_{\mathrm{Z}^{\mathbf{Q}}}}$ over $E = E_\Phi$. Since the precise definition of this functor plays only a minor auxiliary role in this paper, we omit it and refer the interested readers to *loc. cit.*; it suffices to recall that (among other data) one needs to fix a certain F/F_0-traceless element $\sqrt{\Delta} \in F^\times$ adapted to the CM type Φ. The stack $M_{0,K_{\mathrm{Z}^{\mathbf{Q}}}}$ is isomorphic, over E, to finitely many copies of the Shimura variety $\mathrm{Sh}_{K_{\mathrm{Z}^{\mathbf{Q}}}}(\mathrm{Z}^{\mathbf{Q}})$. For our purposes, it suffices to work with a fixed copy, which we denote by $M^\tau_{0,K_{\mathrm{Z}^{\mathbf{Q}}}}$.

8.4.2. *Definition of the moduli funtor.* Let now W be of dimension ν as in §§ 8.1.1, 8.3.1, and set

$$V = \mathrm{Hom}_F(W_0, W).$$

We now present the moduli functor $M_{K_{\widetilde{\mathrm{G}}}}$ represented by the Shimura variety $\mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}(\widetilde{\mathrm{G}})$. For simplicity, we will always assume

$$K_{\widetilde{\mathrm{G}}} = K_{\mathrm{Z}^{\mathbf{Q}}} \times K_{\mathrm{G}}$$

where $K_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}^\infty)$ is a compact open subgroup. For each scheme S in $(\mathrm{LNSch})_{/E}$, $M_{K_{\widetilde{\mathrm{G}}}}(S)$ is by definition the groupoid of tuples $(A_0, \iota_0, \lambda_0, \overline{\eta}_0, A, \iota, \lambda, \overline{\eta})$, where

- $(A_0, \iota_0, \lambda_0, \overline{\eta}_0)$ is an object of $M^\tau_{0,K_{\mathrm{Z}^{\mathbf{Q}}}}(S)$;
- A is an abelian scheme over S;

• $\iota\colon F \to \mathrm{End}^0(A) \coloneqq \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbf{Q}$ is an action of F on A up to isogeny satisfying the Kottwitz condition of signature type r given by (8.1.5),

$$\mathrm{char}\big(\iota(a) \mid \mathrm{Lie}\, A\big) = \prod_{\varphi \in \mathrm{Hom}(F, \overline{\mathbf{Q}})} \big(T - \varphi(a)\big)^{r_\varphi} \quad \text{for all} \quad a \in F. \tag{8.4.1}$$

• λ is a quasi-polarization on A whose Rosati involution satisfies condition

$$\mathrm{Ros}_\lambda\big(\iota(a)\big) = \iota(\overline{a}) \quad \text{for all} \quad a \in F, \tag{8.4.2}$$

and

• $\overline{\eta}$ is a K_{G}-orbit (equivalently, a $K_{\widetilde{\mathrm{G}}}$-orbit, where $K_{\widetilde{\mathrm{G}}}$ acts through its projection $K_{\widetilde{\mathrm{G}}} \to K_{\mathrm{G}}$) of isometries of $\mathbf{A}_F^\infty/\mathbf{A}^\infty$-hermitian modules

$$\eta\colon \widehat{\mathrm{V}}(A_0, A) \xrightarrow{\ \sim\ } V \otimes_F \mathbf{A}_F^\infty. \tag{8.4.3}$$

Here, denoting by $\widehat{\mathrm{V}}(A')$ the adelic Tate module of an abelian variety A',

$$\widehat{\mathrm{V}}(A_0, A) \coloneqq \mathrm{Hom}_{\mathbf{A}_F^\infty}\big(\widehat{\mathrm{V}}(A_0), \widehat{\mathrm{V}}(A)\big), \tag{8.4.4}$$

endowed with its natural $\mathbf{A}_F^\infty$-valued hermitian form h,

$$h(x, y) \coloneqq \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}_{\mathbf{A}_F^\infty}\big(\widehat{\mathrm{V}}(A_0)\big) = \mathbf{A}_F^\infty, \quad x, y \in \widehat{\mathrm{V}}(A_0, A). \tag{8.4.5}$$

Finally, there are natural functors interpreting Hecke correspondences $T(KgK)$ for $g \in \mathrm{G}(\mathbf{A}^\infty)$.

Proposition 8.4.1 ([RSZ21]). *The functor $M_{K_{\widetilde{\mathrm{G}}}}$ is represented by $\mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}(\widetilde{\mathrm{G}})$.*

9. Integral models

We define and study various integral models of the RSZ unitary Shimura varieties introduced in the last section.

9.1. **Integral models with parahoric levels.** We follow [RSZ21, § 4] with slightly different formulation. We continue with the notation of § 8, we we fix a rational prime ℓ, and we denote by $\mathcal{V}_\ell$ the set of places of F_0 over ℓ. If $\ell = 2$, then we assume that every $v \in \mathcal{V}_\ell$ is *split* in F.

We will assume that $K_{\mathrm{Z}^{\mathbf{Q}},\ell} \subset \mathrm{Z}^{\mathbf{Q}}(\mathbf{Q}_\ell)$ is maximal. Then the auxiliary moduli stack $M_{0,K_{\mathrm{Z}^{\mathbf{Q}}}}$ (respectively its substack $M_{0,K_{\mathrm{Z}^{\mathbf{Q}}}}$) has a natural integral model $\mathcal{M}_{0,K_{\mathrm{Z}^{\mathbf{Q}}}}$ (respectively $\mathcal{M}^\tau_{0,K_{\mathrm{Z}^{\mathbf{Q}}}}$), which is finite étale over $\mathrm{Spec}\, \mathcal{O}_{E,(\ell)}$. For each $v \in \mathcal{V}_\ell$, we endow the $F_v/F_{0,v}$-hermitian space $W_v \coloneqq W \otimes_F F_v$ with the $\mathbf{Q}_\ell$-valued alternating form $\mathrm{tr}_{F_v/\mathbf{Q}_\ell} \sqrt{\Delta}^{-1}(\ ,\ )$, and we fix a *vertex lattice* $\Lambda_v \subset W_v$ with respect to this form, i.e., Λ_v is an $\mathscr{O}_{F,v}$-lattice such that

$$\Lambda_v \subset \Lambda_v^\vee \subset \pi_v^{-1}\Lambda_v.$$

Here π_v denotes a uniformizer in F_v (if v splits in F, this means the image in F_v of a uniformizer for $F_{0,v}$), and $\Lambda_v^\vee \subset W_v$ denotes the dual lattice with respect to $\mathrm{tr}_{F_v/\mathbf{Q}_\ell} \sqrt{\Delta}^{-1}(\ ,\ )$.

We assume that $K_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}_{F_0}^\infty)$ is of the form $K_{\mathrm{G}} = K_{\mathrm{G}}^\ell \times K_{\mathrm{G},\ell}$, where $K_{\mathrm{G}}^\ell \subset \mathrm{G}(\mathbf{A}^{\ell\infty})$ is arbitrary and where

$$K_{\mathrm{G},\ell} = \prod_{v \in \mathcal{V}_\ell} K_{\mathrm{G},v} \subset \mathrm{G}(F_{0,\ell}) = \prod_{v \in \mathcal{V}_\ell} G_v,$$

with

$$K_{\mathrm{G},v} \coloneqq \mathrm{Stab}_{G_v}(\Lambda_v). \tag{9.1.1}$$

We note that if v is unramified in F, then $K_{\mathrm{G},v}$ is a maximal parahoric subgroup of $\mathrm{U}(W)(F_{0,v})$.

We then define $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ as the functor that associates to each scheme S in $(\mathrm{LNSch})_{/\mathscr{O}_{E,(\ell)}}$ the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^\ell)$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0^{\tau}(S)$;
- A is an abelian scheme over S;
- $\iota\colon \mathscr{O}_{F,(\ell)} \to \mathrm{End}_{(\ell)}(A)$ is an action up to prime-to-ℓ isogeny satisfying the Kottwitz condition (8.4.1) on $\mathscr{O}_{F,(\ell)}$;
- $\lambda \in \mathrm{Hom}(A, A^\vee)_{\mathbf{Z}_{(\ell)}}$ is a quasi-polarization on A whose Rosati involution satisfies condition (8.4.2) on $\mathscr{O}_{F,(\ell)}$; and
- $\overline{\eta}^\ell$ is a K_{G}^ℓ-orbit of isometries of $\mathbf{A}_F^{\ell\infty}/\mathbf{A}_{F_0}^{\ell\infty}$-hermitian modules

$$\eta^\ell\colon \widehat{\mathrm{V}}^\ell(A_0, A) \xrightarrow{\sim} V \otimes_F \mathbf{A}_F^{\ell\infty}, \tag{9.1.2}$$

where

$$\widehat{\mathrm{V}}^\ell(A_0, A) \coloneqq \mathrm{Hom}_{\mathbf{A}_F^{\ell\infty}}\big(\widehat{\mathrm{V}}^\ell(A_0), \widehat{\mathrm{V}}^\ell(A)\big), \tag{9.1.3}$$

and where the hermitian form on $\widehat{\mathrm{V}}^\ell(A_0, A)$ is the obvious prime-to-ℓ analog of (8.4.5).

We impose the following further conditions on the above tuples.

(i) Consider the decomposition of ℓ-divisible groups

$$A[\ell^\infty] = \prod_{v \in \mathcal{V}_p} A[v^\infty] \tag{9.1.4}$$

induced by the action of $\mathscr{O}_{F_0} \otimes \mathbb{Z}_\ell \cong \prod_{v\in\mathcal{V}_\ell} \mathscr{O}_{F_0,v}$. Since Ros_λ is trivial on $\mathscr{O}_{F_0}$, λ induces a polarization $\lambda_v\colon A[v^\infty] \to A^\vee[v^\infty] \cong A[v^\infty]^\vee$ of ℓ-divisible groups for each v. The condition we impose is that $\ker\lambda_v$ is contained in $A[\iota(\pi_v)]$ of rank $\#(\Lambda_v^\vee/\Lambda_v)$ for each $v \in \mathcal{V}_\ell$.

(ii) We require that the *sign condition*, the *Eisenstein condition* hold; we omit the definitions and refer to [RSZ21, §5].

The morphisms in the groupoid $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}(S)$ are the obvious ones.

We have the following result from [RSZ20, RSZ21].

Proposition 9.1.1. *The stack $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ is Deligne-Mumford, and regular with strictly semistable reduction at all places u of E above ℓ, provided that u is unramified over F. It is smooth over* $\mathrm{Spec}\,\mathscr{O}_{E,(\ell)}$ *if the lattices Λ_v have type* 0 *or* n *for every $v \mid \ell$. The generic fibre of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ is $M_{K_{\widetilde{\mathrm{G}}}}$.*

Finally, there are natural functors interpreting Hecke correspondences $T(f^\ell)$ for all $f^\ell \in \mathscr{H}(\mathrm{G}(\mathbf{A}^{\ell\infty}))_{K_{\mathrm{G}}}$. The correspondences $T(f^\ell)$ are all étale.

9.2. **More integral models at split places.** We need to have regular integral models for deeper levels at split places. We will consider two cases: the Iwahori case and the principal congruence subgroup case.

9.2.1. *Setup.* Fix a place $v \in \mathcal{V}_\ell$ that splits in F, say $v = w\overline{w}$. Let $u\colon E \to \overline{\mathbf{Q}}_\ell$ be a place of E above v; we will assume that E_u is unramified over $F_{0,v}$. Let $\widetilde{u}\colon \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$ be an embedding extending u. Then $\widetilde{u}$ induces a bijection $\mathrm{Hom}(F, \overline{\mathbf{Q}}) \simeq \mathrm{Hom}(F, \overline{\mathbf{Q}}_\ell)$. Let $\mathrm{Hom}_w(F, \overline{\mathbf{Q}})$ be the subset of $\mathrm{Hom}(F, \overline{\mathbf{Q}})$ consisting of $\varphi \in \mathrm{Hom}(F, \overline{\mathbf{Q}})$ such that $\widetilde{u} \circ \varphi$ induces w. The set $\mathrm{Hom}_w(F, \overline{\mathbf{Q}})$ depends only on u but not on the choice of $\widetilde{u}$. Note that the distinguished element φ_0 belongs to $\mathrm{Hom}_w(F, \overline{\mathbf{Q}})$. We will assume that the *matching condition* between the CM type Φ and the chosen place u of E is satisfied:

$$\mathrm{Hom}_w(F, \overline{\mathbf{Q}}) \subset \Phi, \tag{9.2.1}$$

cf. [RSZ20, §4.3]. Note that, for our signature type (8.1.5), this is equivalent to the condition that the restriction $r|_{\mathrm{Hom}_w(F,\overline{\mathbf{Q}})}$ of the signature function is of the form

$$r_\varphi = \begin{cases} n-1, & \varphi = \varphi_0 \in \mathrm{Hom}_w(F, \overline{\mathbf{Q}}); \\ n, & \varphi \in \mathrm{Hom}_w(F, \overline{\mathbf{Q}}) \smallsetminus \{\varphi_0\}. \end{cases} \tag{9.2.2}$$

9.2.2. *Principal congruence subgroups.* We now recall from [RSZ20, §4.3] the moduli problem in the case of principal congruence subgroups. Let m be a nonnegative integer, and define $K^m_{\mathrm{G},v}$ to be the principal congruence subgroup mod $\mathfrak{p}_v^m$ inside $K_{\mathrm{G},v}$, where $\mathfrak{p}_v$ denotes the prime ideal in $\mathscr{O}_{F_0}$ determined by v. Let

$$K^m_{\widetilde{\mathrm{G}}} \coloneqq K_{Z^{\mathbf{Q}}} \times K^\ell_{\mathrm{G}} \times K^m_{\mathrm{G},v} \times \prod_{v' \in \mathcal{V}_\ell \smallsetminus \{v\}} K_{\mathrm{G},v'} \subset K_{\widetilde{\mathrm{G}}}.$$

Then one can extend the definition of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}, \mathscr{O}_{E,u}}$ to the case of the level subgroup $K^m_{\widetilde{\mathrm{G}}}$ by adding a Drinfeld level-m structure at v. More precisely, consider the factors occurring in the decomposition (9.1.4) of the ℓ-divisible group $A[\ell^\infty]$,

$$A[v^\infty] = A[w^\infty] \times A[\overline{w}^\infty]. \tag{9.2.3}$$

The condition (9.2.2) implies that $A[\overline{w}^\infty]$ is a one-dimensional formal $\mathscr{O}_{F,w_0}$-module. We introduce $T_{\overline{w}}(A_0, A)[w_0^m] \coloneqq \underline{\mathrm{Hom}}_{\mathscr{O}_{F,\overline{w}}}(A_0[\overline{w}^m], A[\overline{w}^m])$ and $T_{\overline{w}}(A_0, A) \coloneqq \varinjlim_m T_{\overline{w}}(A_0, A)[w_0^m]$. Note that $T_{\overline{w}}(A_0, A)$ is a 1-dimensional formal $\mathscr{O}_{F,w_0}$-module. The datum we add to the moduli problem is an $\mathscr{O}_{F,\overline{w}}$-linear homomorphism of finite flat group schemes,

$$\phi\colon \pi_{\overline{w}}^{-m}\Lambda_{\overline{w}}/\Lambda_{\overline{w}} \longrightarrow T_{\overline{w}}(A_0, A)[\overline{w}^m], \tag{9.2.4}$$

which is a Drinfeld $\overline{w}^m$-structure on the target. Here $\Lambda_{\overline{w}}$ is the summand attached to $\overline{w}$ in the natural decomposition

$$\Lambda_v = \Lambda_w \oplus \Lambda_{\overline{w}} \tag{9.2.5}$$

with Λ_v the vertex lattice at v chosen in §9.1. See [RSZ20, §4.3] (which we note interchanges the roles of w and $\overline{w}$) for more details.

Then by [RSZ20, Theorem 4.7], the moduli problem $\mathcal{M}_{K^m_{\widetilde{\mathrm{G}}}}$ is relatively representable by a finite flat morphism to $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ and consequently it coincides with the normalization of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ in the generic fiber of $\mathcal{M}_{K^m_{\widetilde{\mathrm{G}}}}$. It is regular and flat over $\operatorname{Spec} \mathscr{O}_{E,(u)}$. Furthermore, the generic fiber $\mathcal{M}_{K^m_{\widetilde{\mathrm{G}}}} \times_{\operatorname{Spec} \mathscr{O}_{E,(u)}} \operatorname{Spec} E$ is canonically isomorphic to $M_{K^m_{\widetilde{\mathrm{G}}}}$.

9.2.3. *Iwahori subgroups.* We will also need the Iwahori case. For simplicity we assume that the vertex lattice Λ_v in (9.2.5) is self-dual. We now choose a chain of $\mathscr{O}_{F,w}$-lattices

$$\Lambda_{\overline{w}} = \Lambda_{\overline{w}}^{(0)} \subset \Lambda_{\overline{w}}^{(1)} \subset \cdots \subset \Lambda_{\overline{w}}^{(n)} = \pi_w^{-1}\Lambda_{\overline{w}},$$

where each inclusion has colength one. Equivalently, we choose a full flag in the k_v-vector space $\Lambda_w/\pi_w\Lambda_w$. This chain determines uniquely a chain of vertex $\mathscr{O}_{F,v} = \mathscr{O}_{F,w} \times \mathscr{O}_{F,\overline{w}}$-lattices $\Lambda_v^{(i)} := \Lambda_w \oplus \Lambda_{\overline{w}}^{(i)}, 0 \le i \le n$. The stabilizer of the chain $\Lambda_v^{(i)}$ is an Iwahori subgroup Iw_v of $\mathrm{Stab}(\Lambda_v)$. To the moduli problem $\mathcal{M}_{K_{\widetilde{\mathrm{G}}},\mathscr{O}_{E,u}}$, we add the datum of a chain of isogenies of $\mathscr{O}_{F,\overline{w}}$-divisible modules

$$\mathcal{G}_0 = T_{\overline{w}}(A_0, A) \longrightarrow \mathcal{G}_1 \longrightarrow \cdots \longrightarrow \mathcal{G}_n = \mathcal{G}_0/\mathcal{G}_0[\overline{w}] \tag{9.2.6}$$

with equal heights $\#k_v$. An equivalent datum is an Iw_v-orbit of the Drinfeld level structure

$$\phi\colon \pi_{\overline{w}}^{-1}\Lambda_{\overline{w}}/\Lambda_{\overline{w}} \longrightarrow T_{\overline{w}}(A_0, A)[\overline{w}].$$

The resulting moduli functor is then denoted by $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}}}$, where $K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}}$ denotes the compact subgroup of $K_{\widetilde{\mathrm{G}}}$ with the Iwahori factor at v. Then the moduli problem $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}}}$ is relatively representable by a finite flat morphism to $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ and consequently it coincides with the normalization of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ in the generic fiber of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}}}$. It is regular, proper and flat over $\mathrm{Spec}\,\mathscr{O}_{E,(u)}$. Moreover, by the theory of local models, the scheme $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}}}$ has strictly semistable reduction over $\mathrm{Spec}\,\mathscr{O}_{E,(u)}$ (namely, its generic fiber is smooth and every closed point of the special fiber admits an open neighborhood which is smooth over the scheme $\mathrm{Spec}\,\mathscr{O}_{E,(u)}[x_1, \cdots, x_m]/(\prod_{i=1}^m x_i - \varpi)$ for some $m \ge 1$, cf. [Har01, Prop. 1.3]). Moreover, there is a natural morphism from $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{m=1}}$ to $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}}}$, which is finite flat. There is a stratification of the special fiber $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}}} \otimes k_u$, where k_u denotes the residue field of $\mathscr{O}_{E,(u)}$:

$$\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}}} \otimes k_u = \bigcup_{1 \le i \le n} \mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}},k_u,i}, \tag{9.2.7}$$

where $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}},k_u,i}$ is the closed subscheme where the kernel of the isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ in (9.2.6) is connected, cf. [TY07, §3] for a similar case. By [TY07, Prop. 3.4] (or rather its proof), each of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}^{\mathrm{Iw_v}},k_u,i}$ is smooth over $\mathrm{Spec}\,k_u$.

9.2.4. *Hecke correspondences.* We recall from [RSZ20, §4.3] that, in each of the above two cases (principal and Iwahori level), there are natural functors interpreting Hecke correspondences attached to functions $\mathbf{1}_{KgK}$ for any $g \in \mathrm{G}(\mathbf{A}^\infty)$, where we simply denote $K = K_{\mathrm{G}}$:

$$\begin{tikzcd} & \mathcal{M}_{K'_{\widetilde{\mathrm{G}}}} \arrow[dl, "\mathrm{nat}_1"'] \arrow[dr, "\mathrm{nat}_g"] & \\ \mathcal{M}_{K_{\widetilde{\mathrm{G}}}} & & \mathcal{M}_{K_{\widetilde{\mathrm{G}}}} \end{tikzcd} \tag{9.2.8}$$

where $K'_{\widetilde{\mathrm{G}}} = K_{Z\mathbf{Q}} \times K'_{\mathrm{G}}$, and K'_{G} is a subgroup of $K_{\widetilde{\mathrm{G}}} \cap gK_{\widetilde{\mathrm{G}}}g^{-1}$. We refer to [RSZ20, §4.3] for the unexplained notation. (Note that in *loc. cit.*, the authors only consider the case of a principal congruence subgroup $K_{\mathrm{G}} = K_{\mathrm{G}}^m$. The Iwahori case is similar and may be reduced to the case

$K_{\mathrm{G}} = K_{\mathrm{G}}^m$ as follows. We can factorize $[\mathrm{Iw}g\mathrm{Iw}]$ as $e_{\mathrm{Iw}} \star [KgK] \star [\mathrm{Iw}g\mathrm{Iw}]$ for some $K = K_{\mathrm{G}}^m \subset \mathrm{Iw}$, and accordingly we define the correspondence for $[\mathrm{Iw}g\mathrm{Iw}]$ as the composition of the three factors: the middle one is as in *loc. cit.*, and the other two are given by the natural map from the principal level to the Iwahori level.)

Both maps nat_1 and nat_g are finite flat, and étale if $g_\ell = 1$. The Hecke correspondence (9.2.8) induces an endomorphism (by the usual pull-back and then push-forward maps) on the group of cycles (with L-coefficients), rather than merely cycles modulo rational equivalence. This endomorphism is independent of the choice of $K'_{\widetilde{\mathrm{G}}}$ in the diagram above. The resulting map

$$T\colon \mathscr{H}(\mathrm{G}(\mathbf{A}^{\ell\infty}), L)_K \longrightarrow \mathrm{\acute{E}tCorr}(\mathcal{M}_K)_L$$

is a ring homomorphism.[17] Moreover, in the Iwahori case, the away-from-ℓ Hecke correspondences preserve the stratification (9.2.7).

9.3. **Moduli functors for the product Shimura varieties.** It is now easy to extend the construction in §8.3 to the product unitary group $\widetilde{\mathrm{G}}$ defined in § 8.3.3. There are analogous moduli functors over E and over $\mathscr{O}_{E,(\ell)}$. For example, the ℓ-integral model may be succinctly defined as

$$\mathcal{M}_{K_{\widetilde{\mathrm{G}}}} = \mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}} \times_{\mathcal{M}_0^\tau} \mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_{n+1})}}, \tag{9.3.1}$$

where $K_{\widetilde{\mathrm{G}}(V_\nu)} = K_{\mathrm{Z}^{\mathbf{Q}}} \times K_\nu$ for $\nu \in \{n, n+1\}$.

The product $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ may no longer be regular even if both factors are regular, and we may need to resolve the product singularity. We will need to study two cases: the vertex parahoric case at an inert place, and the Iwahori case at a split place.

9.3.1. *Vertex parahoric level at an inert place.* We first consider the vertex parahoric case from §9.1. Fix a place $v \in \mathcal{V}_\ell$ that is inert in F and we let w denote the unique place of F above v. We fix a vertex lattice $\Lambda_v^\flat \subset V_{n,v}$ of type $0 \le t \le n$ and let $\Lambda_v = \Lambda_v^\flat \oplus \langle e \rangle_{\mathscr{O}_{F,v}} \subset V_{n+1,v}$ where the hermitian norm of the special vector e has valuation $\epsilon \in \{0,1\}$.[18] Then Λ_v is a vertex lattice of type $t+\epsilon$. We let $u : E \to \overline{\mathbf{Q}}_p$ be a place of E above v and we further assume that E_u is unramified over $F_{0,v}$. We let $K_{n,v}$ and $K_{n+1,v}$ be the stabilizer of $\Lambda_v^\flat$ and Λ_v respectively. We then call $K_v = K_{n,v} \times K_{n+1,v}$ a vertex parahoric subgroup of type $(t, t+\epsilon)$. The (self-dual) hyperspecial case corresponds to type $(0,0)$.

In this case, the integral models $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}}$ and $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_{n+1})}}$ have strictly semistable reduction over $\operatorname{Spec} \mathscr{O}_{E,u}$; and $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}}$ (resp. $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_{n+1})}}$) is smooth over $\operatorname{Spec} \mathscr{O}_{E,u}$ only when $t \in \{0, n\}$ (resp. $t+\epsilon \in \{0, n+1\}$); see Proposition 9.1.1. When $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}}$ or $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_{n+1})}}$ is non-smooth over $\operatorname{Spec} \mathscr{O}_{E,u}$, its special fiber admits a “balloon–ground” stratification ([LTX+22] for $t = 1$ and [ZZh] for general t): the special fiber is a union of two Weil divisors

$$\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}, k_u} = \mathcal{M}^\circ_{K_{\widetilde{\mathrm{G}}(V_n)}, k_u} \cup \mathcal{M}^\bullet_{K_{\widetilde{\mathrm{G}}(V_n)}, k_u} \tag{9.3.2}$$

[17] However, we do not know if the assertion remains true for the full Hecke algebra $\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty), L)_K$. When $m = 0$, the recent work of Li–Mihatsch [LM, Proposition 3.4] shows that the assertion holds.

[18] Note that in §2.1.3 we have assumed the special vector has norm 1. For the general discussion of the geometry of Shimura varieties with parahoric levels, it is more convenient to relax this condition.

where the first one $\mathcal{M}^{\circ}_{K_{\widetilde{\mathrm{G}}(V_n)},k_u}$ is called the balloon stratum and the second one $\mathcal{M}^{\bullet}_{K_{\widetilde{\mathrm{G}}(V_n)},k_u}$ is called the ground stratum. (When $t \in \{0, n\}$ we understand that the balloon stratum is empty.) When $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ is not regular, we let $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ be the blow up along the product of the balloon strata of the two factors, and denote the blow-up morphism

$$\pi\colon \widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}} \longrightarrow \mathcal{M}_{K_{\widetilde{\mathrm{G}}}}. \tag{9.3.3}$$

For $(?_n, ?_{n+1}) \in \{\circ, \bullet\}^2$, we denote by $\widetilde{\mathcal{M}}^{(?_n,?_{n+1})}_{K_{\widetilde{\mathrm{G}}},k_v}$ the strict transform of $\mathcal{M}^{?_n}_{K_{\widetilde{\mathrm{G}}(V_n)},k_u} \times \mathcal{M}^{?_{n+1}}_{K_{\widetilde{\mathrm{G}}(V_{n+1})},k_u}$. For later reference we record the following result from [LTX+22] for $t = 1$ and [ZZh] for general t.

Proposition 9.3.1. *The scheme $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ is regular with strictly semistable reduction*

$$\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}} \otimes k_u = \bigcup_{(?_n,?_{n+1})\in\{\circ,\bullet\}^2} \widetilde{\mathcal{M}}^{(?_n,?_{n+1})}_{K_{\widetilde{\mathrm{G}}},k_v}, \tag{9.3.4}$$

where the schemes $\widetilde{\mathcal{M}}^{(?_n,?_{n+1})}_{K_{\widetilde{\mathrm{G}}},k_v}$ are smooth of pure dimension $2n-1$.

The map π is small, i.e., a proper birational morphism with the property that

$$\operatorname{codim}\{z \in \mathcal{M}_{K_{\widetilde{\mathrm{G}}}} \mid \dim \pi^{-1}(z) \geq i\} \geq 2i+1,$$

for all $i \geq 0$.

9.3.2. *Iwahori level at a split place.* Fix as in §9.2 a place $v \in \mathcal{V}_\ell$ that splits in F into $v = w\overline{w}$ and we let $u\colon E \to \overline{\mathbf{Q}}_\ell$ be a place of E above v. We further assume that E_u is unramified over $F_{0,v}$. Then the integral model $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ over $\operatorname{Spec} \mathscr{O}_{E,(u)}$ is smooth if one of the two compact open subgroups $K_{n,v}$ and $K_{n+1,v}$ is hyperspecial. When both $K_{n,v}$ and $K_{n+1,v}$ are Iwahori, $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ is no longer regular and we need to resolve the product singularity. More precisely, we consider the fiber product of the stratifications from (9.2.7)

$$\mathcal{M}_{K_{\widetilde{\mathrm{G}}},k_u,(i,j)} \coloneqq \mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)},k_u,i} \times_{\mathcal{M}_0^\tau} \mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_{n+1})},k_u,j}. \tag{9.3.5}$$

We choose an ordering of the set $\{(i,j) \mid 1 \leq i \leq n, 1 \leq j \leq n+1\}$, and rename the component $\mathcal{M}_{K_{\widetilde{\mathrm{G}}},k_u,(i,j)}$ as $\mathcal{M}_{K_{\widetilde{\mathrm{G}}},k_u,r}$ where $1 \leq r \leq n(n+1)$. Let $\mathcal{M}^{(0)}_{K_{\widetilde{\mathrm{G}}}} \coloneqq \mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ and for $1 \leq r \leq n(n+1)$ let $\mathcal{M}^{(r)}_{K_{\widetilde{\mathrm{G}}}}$ be the blow-up of $\mathcal{M}^{(r-1)}_{K_{\widetilde{\mathrm{G}}}}$ along (the strict transforms of) $\mathcal{M}_{K_{\widetilde{\mathrm{G}}},k_u,r}$. We write $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ for $\mathcal{M}^{(n(n+1))}_{K_{\widetilde{\mathrm{G}}}}$, and $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}},k_u,(i,j)}$ for the strict transform of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}},k_u,(i,j)}$. The composition of the natural blow-up maps is denoted as

$$\pi\colon \widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}} \longrightarrow \mathcal{M}_{K_{\widetilde{\mathrm{G}}}}. \tag{9.3.6}$$

(We also note that the resolution in the inert case earlier can also be view a special case of the current procedure: one simply orders the components such that the first one is the product of the balloon strata.)

Proposition 9.3.2. *The scheme $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ is regular with strictly semistable reduction*

$$\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}},k_u} = \bigcup_{\substack{1\leq i\leq n,\\ 1\leq j\leq n+1}} \widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}},k_u,(i,j)}, \tag{9.3.7}$$

where the schemes $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}},k_u,(i,j)}$ are smooth of pure dimension $2n-1$. The map π is a small map.

Proof. The first part is well-known, for example see [Har01, Prop. 2.1] or [GS95]. For the smallness, we use the explicit description as in the proof of [Har01, Prop. 2.1]. Consider a point $P=(a,b)$ on the special fiber $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}},k_u}$ with an open neighborhood that is smooth over

$$\operatorname{Spec}\mathscr{O}_{E,(u)}[x_1,\cdots,x_r,y_1,\cdots y_s]/(\prod_{i=1}^r x_i-\varpi,\prod_{j=1}^s y_j-\varpi)$$

for some (uniquely-determined) integers $r,s\geq 1$, such that P lies over the point defined by $x_i=0,y_j=0,1\leq i\leq r,1\leq j\leq s$. Then keeping track of the steps of the blow-ups in *loc. cit.* shows that the dimension of the fiber of P is $\min\{r-1,s-1\}$. Note that the locus of P with fixed $r,s\geq 1$ is contained in the union of

$$(\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)},k_u,i_1}\cap\cdots\cap\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)},k_u,i_r})\times(\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_{n+1})},k_u,j_1}\cap\cdots\cap\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_{n+1})},k_u,j_s})$$

for all possible $1\leq i_1\leq\cdots\leq i_r\leq n,1\leq j_1\leq\cdots\leq j_s\leq n+1$. The codimension of such locus in $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ is $r+s-1\geq 2\min\{r-1,s-1\}+1$, which proves the smallness of the map π. $\square$

This procedure depends on the choice of an ordering and therefore it is not canonical. Nevertheless the smallness of π shows that the resolution has the property that $\pi_*\mathbf{Q}_p\simeq\mathrm{IC}$, the latter being the intersection complex of the $\mathbf{Q}_p$-sheaf (for $p\neq\ell$). Moreover, the resulting $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ and each of $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}},k_u,(i,j)}$ still has an action of $\mathscr{H}(\mathrm{G}(\mathbf{A}^{\ell\infty}))_{K_{\mathrm{G}}}$ by correspondences.

9.3.3. *Integral arithmetic diagonal.* We have an integral model

$$\jmath\colon\ \mathcal{M}_{K_{\widetilde{\mathrm{H}}}}\longrightarrow\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}\tag{9.3.8}$$

of the morphism (8.3.7). In the two cases discussed above, over a place u of E, we have the small resolution $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ and we denote by

$$\widetilde{\jmath}\colon\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{H}}}}\longrightarrow\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}\tag{9.3.9}$$

the strict transform of $\mathcal{M}_{K_{\widetilde{\mathrm{H}}}}$ along the resolution morphism. For uniformity of notation, we will put $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{H}}}}\coloneqq\mathcal{M}_{K_{\widetilde{\mathrm{H}}}}$, $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}\coloneqq\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$, $\widetilde{\jmath}\coloneqq\jmath$ in the cases where those schemes are already regular.

9.4. **Vanishing of absolute cohomology.** We will prove the vanishing of the top-degree absolute cohomology of the scheme $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$, for certain levels $K_{\widetilde{\mathrm{G}}}$ and after suitable localizations.

Let L be a finite extension of $\mathbf{Q}_p$.

9.4.1. *Correspondences that annihilate the cohomology.* We use some general result from [LL21, LL22], that we now recall. Following [LL21, Appendix B], we define a *commutative L-algebra of étale correspondences* on a scheme $\mathscr{X}$ to be a commutative L-algebra $\mathbb{T}$ equipped with a homomorphism $\mathbb{T}\to\text{ÉtCorr}(\mathscr{X})_L$.

Let $\mathscr{X}$ be a regular scheme, proper and flat of relative dimension $2n-1$ (not necessarily strictly semistable) over the ring of integers of a non-archimedean local field, with residue field k; we assume that the generic fiber X is smooth. Let $\mathbb{T}$ be a commutative L-algebra of étale correspondences on $\mathscr{X}$ with a maximal ideal $\mathfrak{m}$. Let Y denote the *reduced* special fiber of $\mathscr{X}$.

Assume that there is a stratification $Y = Y^{[m]} \supset \cdots \supset Y^{[0]}$ by closed subschemes and, for each $0 \leq i \leq d$, a refinement of $Y^{(i)} := Y^{[i]} \setminus Y^{[i-1]}$ as a disjoint union of open and closed subschemes of $Y^{(i)}$ of pure dimension d_i:

$$Y^{(i)} = \coprod_{M \in \mathfrak{S}^i} Y^{(M)},$$

over a finite set of indices $\mathfrak{S}^i$, such that

(1) For every i and $M \in \mathfrak{S}^i$, denoting by $Y^{[M]}$ the Zariski closure of $Y^{(M)}$, then $Y^{[M]}$ is *smooth* and is a disjoint union $\coprod_{M' \in \mathfrak{S}_M} Y^{(M')}$ where $\mathfrak{S}_M$ is a subset of $\mathfrak{S} := \coprod \mathfrak{S}^{(i)}$;

(2) For every i and $M \in \mathfrak{S}^i$, the scheme $Y^{(M)}$ is stable under the action of $\mathbb{T}$.

Proposition 9.4.1 (Li–Liu)**.** *Under the above assumptions, if we further suppose that either of the following two conditions holds:*

(1) $H^j(Y^{[M]} \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0$ whenever $j \neq \dim Y^{[M]}$ for every $M \in \mathfrak{S}$,

(2) $H^{2n}(X, L(n))_{\mathfrak{m}} = 0$ and $H^j(Y^{(i)} \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0$ whenever $j \leq \dim Y^{(i)} - \mathrm{codim}_{\mathscr{X}} Y^{(i)}$ for every i,

then $H^{2n}(\mathscr{X}, L(n))_{\mathfrak{m}} = 0$.

Proof. Case (1). The vanishing assumption $H^j(Y^{[M]} \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0$ is the assertion of [LL22, Prop. 4.25]. The proof of [LL22, Theorem 4.21] applies verbatim to show that the assumptions imply the desired vanishing $H^{2n}(\mathscr{X}, L(n))_{\mathfrak{m}} = 0$.

Case (2). This is [LL21, Corollary B.15]. We sketch their proof for the convenience of the reader.

By the assumption $H^{2n}(X, L(n))_{\mathfrak{m}} = 0$ and the exact sequence

$$H^{2n}_Y(\mathscr{X}) \longrightarrow H^{2n}(\mathscr{X}) \longrightarrow H^{2n}(X)$$

it suffices to show $H^{2n}_Y(\mathscr{X})_{\mathfrak{m}} = 0$. This follows from an induction using

- the exact sequences

$$H^{2n}_{Y_{j+1}}(\mathscr{X}) \longrightarrow H^{2n}_{Y_j}(\mathscr{X}) \longrightarrow H^{2n}_{Y_j^\circ}(\mathscr{X} \setminus Y_{j+1}),$$

- the absolute purity theorem of Gabber $H^{2n}_{Y_j^\circ}(\mathscr{X} \setminus Y_{j+1}) \simeq H^{2n-2n_j}(Y_j^\circ)$ for the regular local immersion $Y_j^\circ \hookrightarrow \mathscr{X} \setminus Y_{j+1}$ of codimension n_j,
- the Hochschild–Serre spectral sequence $H^r(k, H^s(Y_j^\circ \otimes_k \overline{k})(n)) \Longrightarrow H^{r+s}(Y_j^\circ)(n)$. In particular, it suffices to replace (3) by a weaker assumption $H^{2n-2n_j}(Y_j^\circ \otimes_k \overline{k}, L)_{\mathfrak{m}} = H^{2n-2n_j-1}(Y_j^\circ \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0$ (namely $H^{d_{Y_j} - c_{Y_j} - i}(Y_j^\circ \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0$ for $i = 0, 1$ where d_{Y_j} and c_{Y_j} denote respectively the dimension of Y_j and the codimension of Y_j in $\mathscr{X}$.

□

9.4.2. *The vanishing result.* We consider the scheme $\widetilde{\mathcal{M}}_{K_{\tilde{\mathrm{G}}}}$ over $\mathrm{Spec}\, \mathscr{O}_{E,u}$ of § 9.3 where v and $K_{\mathrm{G},v}$ are in one of the following cases:

(1) the split-(Drinfeld-level, hyperspecial) case: the place v is split in F and in the product (9.3.1) one of the two factors has Drinfeld-level for some integer m and the other has hyperspecial level;

(2) the split-(Iwahori, Iwahori) case: the place v is split in F and in the product (9.3.1) both factors have Iwahori level; in this case $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ is the small resolution in Proposition 9.3.2;

(3) the inert-vertex-parahoric case: the place v is inert in F and in the product (9.3.1) both factors have vertex-parahoric levels (of type $(t, t+\epsilon)$); in this case $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$ is the small resolution in Proposition 9.3.1.

Proposition 9.4.2. *Let S be a finite set of nonarchimedean places of F_0 containing those above ℓ and p and those where K_{G} is not maximal hyperspecial, and let*

$$\mathbb{T} = \mathbb{T}^{\mathrm{spl},S} := \bigotimes_{\substack{v \notin S \\ \text{split}}} \mathscr{H}(G_v, L)_{K_v} \subset \mathscr{H}(\widetilde{\mathrm{G}}(\mathbf{A}^S, L)^{\circ}_{K^S}.$$

Let $\mathfrak{m} \subset \mathbb{T}$ be the maximal ideal attached to a representation $\pi \in \mathscr{C}(\mathrm{G})(L)$. Suppose we are in one of the above three cases, and suppose moreover that the following hold:

(1) In the split-(Iwahori, Iwahori) case, the representation π_v is a (tempered) principal series.

(2) In the inert-vertex-parahoric case, the type (t_n, t_{n+1}) satisfies $t_n \in \{0, 1, n-1, n\}$ and $t_{n+1} \in \{0, 1, n, n+1\}$.

Then we have

$$H^{2n}(\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}, L(n))_{\mathfrak{m}} = 0. \tag{9.4.1}$$

Proof. We wish to apply the vanishing theorem of Li–Liu given in Proposition 9.4.1. For this, we need to specify a stratification of the reduced special fiber of $\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{G}}}}$. In the split-(Drinfeld-level, hyperspecial) case, for simplicity we consider the case the Drinfeld level takes place on the first factor $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}}$. Then the special fiber, denoted by Y_{n+1}, of the second factor in the product (9.3.1) is smooth. In [LL22, §4.3] the authors have defined a stratification of the reduced special fiber, denoted by Y_n, of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}}$, essentially a refinement of the Newton stratification

$$Y_n = \coprod_{i=0}^{n-1} \coprod_{M \in \mathfrak{S}_i} Y_n^{(M)},$$

where $\mathfrak{S}_i$ denotes the $\mathfrak{S}_m^i$ in *loc. cit.*. Here we simply take the stratification of $Y = Y_n \times Y_{n+1}$ as the product of the stratification of Y_n with Y_{n+1}

$$Y = \coprod_{i=0}^{n-1} \coprod_{M \in \mathfrak{S}_i} Y_n^{(M)} \times Y_{n+1}.$$

By [LL22, §4.3] this stratification of Y_n verifies the two conditions stated before Proposition 9.4.1 (for $\mathscr{X} = \mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_n)}}$). It follows easily that the above stratification of Y verifies the two conditions stated before Proposition 9.4.1 (for $\mathscr{X} = \mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$).

In the split-(Iwahori, Iwahori) case and the inert-vertex-parahoric case, the scheme $\mathscr{X} = \mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ has strictly semistable reduction. The special fiber Y is already reduced and we define the stratification induced by the union (9.3.7) and (9.3.4) respectively, as follows. Let $\mathcal{J}$ denote the set of indices in (9.3.7) and (9.3.4), and denote $Y = \cup_{j\in\mathcal{J}} Y_j$. Then we define $\mathfrak{S}^i$ to be the set of subsets M of I with $\#M = \#I - i$ such that $Y^{[M]} := \cap_{j\in M} Y_j$ is non-empty (then it has codimension $\#M+1$ in $\mathscr{X}$). Set $Y^{[i]} = \cup_{M\in\mathfrak{S}^i} Y^{[M]}$ and $Y^{(M)} = Y^{[M]} \setminus Y^{[\#M+1]}$. Then we have the resulting stratification

$$Y = \coprod_{i=0}^{\#\mathcal{J}} Y^{[i]} = \coprod_{i=0}^{\#\mathcal{J}} \coprod_{M\in\mathfrak{S}_i} Y^{(M)}. \tag{9.4.2}$$

The strict semistability of $\mathscr{X}$ implies that the stratification verifies the two conditions stated before Proposition 9.4.1. (Note that the scheme $Y^{[i]}$ is empty once $i > \dim Y$.)

We write $\mathbb{T} = \mathbb{T}_n \otimes \mathbb{T}_{n+1}$ and $\mathfrak{m}$ corresponding to $(\mathfrak{m}_n, \mathfrak{m}_{n+1})$ for maximal ideals $\mathfrak{m}_\nu$ of $\mathbb{T}_\nu, \nu \in \{n, n+1\}$. We will distinguish the three cases.

Split-(Drinfeld-level, hyperspecial) case. By Proposition 9.4.1 (1), it suffices to verify that, for every $M \in \mathfrak{S}$, we have $H^j(Y^{[M]} \otimes_k \overline{k}, L)_\mathfrak{m} = 0$ whenever $j \neq \dim Y^{[M]}$. This follows from the Künneth formula, [LL22, Prop. 4.25] for $H^j(Y_n^{[M]} \otimes_k \overline{k}, L)_{\mathfrak{m}_n} = 0, j \neq \dim Y_n^{[M]}$, and the similar vanishing result for $H^j(Y_{n+1} \otimes_k \overline{k}, L)_{\mathfrak{m}_{n+1}} = 0, j \neq \dim Y_{n+1}$.

Split-(Iwahori, Iwahori) case. We first define a stratification of the special fiber Z of $\mathcal{M}_{K_{\widetilde{\mathrm{G}}}}$ prior to the resolution, similar to (9.4.3):

$$Z = \coprod_{i=0}^{\#\mathcal{J}} Z^{[i]} = \coprod_{i=0}^{\#\mathcal{J}} \coprod_{M\in\mathfrak{S}_i} Z^{(M)}. \tag{9.4.3}$$

Then, under the condition (1), it follows from [LL21, (3), p. 859] that $H_c^i(Z^{(M)})_\mathfrak{m} = 0$ for all i and $M \in \mathfrak{S}$, unless $Z^{(M)}$ are maximal dimensional, in which case $H_c^i(Z^{(M)})_\mathfrak{m} = 0$ unless $i = \dim Z^{(M)}$. (In *loc. cit.* the authors only treated the case of Drinfeld levels; but the proof applies verbatim to the Iwahori case.) Now we return to the stratum $Y^{(M)}$ in (9.4.3). It is easy to see that the natural map $\pi_M \colon Y^{(M)} \to Z^{(M)}$ is smooth and the direct images $R\pi^j_{M,!}L$ are constant on $Z^{(M)}$. It follows that $H_c^i(Y^{(M)})_\mathfrak{m} = 0$ for all i and $M \in \mathfrak{S}$, unless $Y^{(M)}$ are maximal dimensional hence equal to $Z^{(M)}$, in which case $H_c^i(Y^{(M)})_\mathfrak{m} = 0$ unless $i = \dim Z^{(M)}$. It follows from the cohomological exact sequence associated to $Y^{[M]} = Y^{(M)} \cup (Y^{[M]} \setminus Y^{(M)})$ (see for example (9.4.5) below) and an induction that $H^i(Y^{(M)})_\mathfrak{m} = 0$ for all i and $M \in \mathfrak{S}$, unless $Y^{(M)}$ are maximal dimensional, in which case $H^i(Y^{[M]})_\mathfrak{m} = 0$ for $i > \dim Y^{[M]}$ and by Poincaré duality $H^i(Y^{[M]})_\mathfrak{m} = 0$ for $i \neq \dim Y^{[M]}$. We have thus verified the condition in case (1) of Proposition 9.4.1 and therefore we have proved $H^{2n}(\mathscr{X})_\mathfrak{m} = 0$ in this case.

Inert-vertex-parahoric case. We note that the moduli space $\mathcal{M}_{K_{\widetilde{\mathrm{G}}(V_\nu)}}$ for type t_ν (at v) is isomorphic to another similarly defined moduli space of type $\nu - t_\nu$. Therefore it suffices to consider the cases when $t_n, t_{n+1} \in \{0, 1\}$. We first recall from [LL21, §9, p. 868] that, when the type is $t_n = 1$, the cohomology of the balloon and the ground strata satisfy

$$H^i(Z \otimes_k \overline{k}, L)_\mathfrak{m} = 0, \quad i \neq \dim Z \tag{9.4.4}$$

for $Z = Y_n^\circ, Y_n^\bullet, Y_n^\dagger$ respectively, where we simplify the notation $Y_n^? = \mathcal{M}^?_{K_{\widetilde{\mathrm{G}}(V_n)},k}$ in (9.3.2) for $? \in \{\circ, \bullet\}$, and define $Y_n^\dagger = Y_n^\circ \cap Y_n^\bullet$. If one of t_n, t_{n+1} is 0, the proof is now similar to the split-(Drinfeld-level, hyperspecial) case, Proposition 9.4.1 (1). It remains to consider the type $(1,1)$ case. For $(?_n, ?_{n+1}) \in \{\circ, \bullet\}^2$, we will write $Y^{?_n, ?_{n+1}}$ for the strict transform of $Y_n^{?_n} \times Y_{n+1}^{?_{n+1}}$. Then by the formula for cohomology of blow-up, $H^i(Y^{?_n,?_{n+1}} \otimes_k \overline{k}, L)$ is isomorphic to

$$H^i((Y_n^{?_n} \times Y_{n+1}^{?_{n+1}}) \otimes_k \overline{k}, L) \oplus \begin{cases} 0, & (?_n, ?_{n+1}) = (\circ, \bullet) \text{ or } (\bullet, \circ) \\ H^{i-2}((Y_n^\dagger \times Y_{n+1}^\dagger) \otimes_k \overline{k}, L), & (?_n, ?_{n+1}) = (\circ, \circ) \text{ or } (\bullet, \bullet). \end{cases}$$

Similarly we can compute the cohomology of all of the closed strata $Y^{[M]}$ in terms of the notation in (9.4.3) using (9.4.4)

$$H^i(Y^{[M]} \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0, \quad i \neq \dim Y^{[M]},$$

for all $M \in \mathfrak{S}$ but one exception: the stratum $Y^{[M_0]} := Y^{\circ,\circ} \cap Y^{\bullet,\bullet}$, which is a $\mathbb{P}^1$-bundle over $Y^{\dagger,\dagger}$. Nonetheless the exceptional case has vanishing (localized at $\mathfrak{m}$) cohomology at all degree outside $i = \dim Y^{\dagger,\dagger}$ and $i = \dim Y^{\dagger,\dagger} + 2$. Using (9.4.4) (for both n and $n+1$) we can deduce that

$$H_c^i(Y^{(M)} \otimes_k \overline{k}, L)_{\mathfrak{m}} = 0, \quad i > \dim Y^{[M]}$$

for $M \neq M_0 \in \mathfrak{S}$. To treat the exceptional case, we use the exact sequence

$$H^{i-1}(Y^{[M_0]} \setminus Y^{(M_0)}) \longrightarrow H_c^i(Y^{(M_0)}) \longrightarrow H^i(Y^{[M_0]}). \tag{9.4.5}$$

Note that the stratum $Y^{[M_0]}$ has codimension 2 in $\mathscr{X}$, and $Y^{[M_0]} \setminus Y^{(M_0)}$ is smooth of codimension 3 in $\mathscr{X}$. Since $H^i(Y^{[M_0]})_{\mathfrak{m}} = 0$ when $i \geq \dim Y^{[M_0]} + 2$, and $H^i(Y^{[M_0]} \setminus Y^{(M_0)})_{\mathfrak{m}} = 0$ when $i \neq \dim Y^{[M_0]} \setminus Y^{(M_0)} = \dim Y^{[M_0]} - 1$, we conclude that $H_c^i(Y^{(M_0)})_{\mathfrak{m}} = 0$ when $i \geq \dim Y^{[M_0]} + 2$. By Poincaré duality we have $H^i(Y^{(M_0)})_{\mathfrak{m}} = 0$ when $i \leq \dim Y^{[M_0]} - 2$. Since $H^{2n}(X, L(n)) = 0$, we have verified the condition in case (2) of Proposition 9.4.1 and therefore we have proved $H^{2n}(\mathscr{X})_{\mathfrak{m}} = 0$. $\square$

Remark 9.4.3. The condition (1) in Proposition 9.4.2 may be unnecessary if one makes a more careful study on the stratification of the special fiber of the small resolution.

10. p-adic Abel-Jacobi maps and p-adic heights

We summarize the definitions and results we need from the theory of p-adic heights. For more details or more general setups, see Nekovář's original paper [Nek93] and [DL24, Appendix A]; our constructions follow the sign conventions of the latter reference. Nothing in this section is new.

The notation of this section is independent of that of the rest of the paper. We denote by L a finite extension of $\mathbf{Q}_p$, and by Γ a finite-dimensional L-vector space. T

10.1. **p-adic Abel–Jacobi maps and biextensions.** Let F be a field of characteristic different from p, and let X be a smooth projective scheme over F of pure dimension $m-1 \geq 1$. We denote by $\mathrm{Z}^\bullet(X)_R$ the module of $\bullet$-dimensional algebraic cycles with coefficients in a ring R (omitted from the notation when $R = \mathbf{Z}$), and by $\mathrm{Ch}^\bullet(X)_R = \mathrm{Z}^\bullet(X)_R/(\text{rational equivalence})$ the Chow R-module. We denote $H^i(F, -) = H^i_{\mathrm{cont}}(G_F, -)$.

10.1.1. *p-adic Abel–Jcobi maps.* Let $0 \leq d \leq m$ and consider the absolute étale cohomology $H^{2d}(X, L(d))$. By the Hochschild–Serre spectral sequence, it has a filtration $\mathrm{Fil}^\bullet$ with

$$0 \longrightarrow H^1(F, H^{2d-1}(X_{\overline{F}}, L(d))) \longrightarrow H^{2d}(X, L(d))/\mathrm{Fil}^2 \longrightarrow H^0(F, H^{2d}(X_{\overline{F}}, L(d))) \longrightarrow 0.$$

We denote by $\overline{\mathrm{cl}}\colon \mathrm{Z}^d(X)_L \to H^0(G_F, H^{2d}(X_{\overline{F}}, L(d))$ the geometric cycle class map, by $\mathrm{Z}^d(X)_L^0$ its kernel, and we let

$$\widetilde{\mathrm{cl}}\colon \mathrm{Z}_d(X)_L \longrightarrow H^{2d}(X, L(d)))/\mathrm{Fil}^2,$$
$$\mathrm{cl}\colon \mathrm{Z}_d(X)_L^0 \longrightarrow H^1(F, H^{2d-1}(X_{\overline{F}}, L(d)))$$

be the absolute cycle class map and the Abel–Jacobi map, respectively. The maps $\overline{\mathrm{cl}}$, $\widetilde{\mathrm{cl}}$ factor through the Chow group $\mathrm{Ch}^d(X)$, and the map cl factors through the image $\mathrm{Ch}^d(X)^0 \subset \mathrm{Ch}^d(X)$ of $\mathrm{Z}^d(X)^0$.

If $M \subset H^{2d-1}(X_{\overline{F}}, L(d))$ is a G_F-stable subspace, we denote by

$$\mathrm{Z}^d_M(X)^0_L, \quad \mathrm{Ch}^d_M(X)^0_L$$

the preimages in $\mathrm{Z}^d(X)^0_L$, $\mathrm{Ch}^d(X)^0_L$ of $H^1(F, M) \subset H^1(F, H^{2d-1}(X_{\overline{F}}, L(d)))$ under the Abel–Jacobi map.

Suppose that F is non-archimedean of residue characteristic ℓ. We will consider subspaces M satisfying the condition:

(1) if $\ell \neq p$: $H^1(F, M) = 0$;

(2) if $\ell = p$: $H^1_{\mathrm{st}}(F, M) = H^1_f(F, M)$.

Remark 10.1.1. Since by [NN' 16, Theorem B] the map cl takes values in the subspace

$$H^1_{\mathrm{st}}(F, H^{2d-1}(X_{\overline{F}}, L(d))),$$

the conditions above imply that

$$\mathrm{cl}(\mathrm{Z}^d_M(X)^0_L) \subset H^1_f(F, M).$$

If M is pure of weight -1 (as is implied for all $M \subset H^{2d-1}(X_{\overline{F}}, L(d))$ by the weight-monodromy conjecture), then the relevant one among the conditions above is satisfied.

10.1.2. *Biextensions from algebraic cycles.* Let $d_1, d_2 \geq 0$ be integers with $d_1 + d_2 = m$, and let

$$Z_1 \in \mathrm{Z}^{d_1}(X)^0_L, \quad Z_2 \in \mathrm{Z}^{d_2}(X)^0_L$$

be cycles with disjoint supports. Let $M_i := H^{2d_i-1}(X_{\overline{F}}, L(d_i))$. To each Z_i is associated an extension of $L[G_F]$-modules

$$0 \to M_i \to E_i \to L \to 0$$

whose class in $H^1(F, M_i)$ is the p-adic Abel–Jacobi image $\mathrm{cl}(Z_i)$. A further geometric construction yields the *biextension* $E_1^2 = E_{Z_1}^{Z_2}$ fitting in the following exact diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & L(1) \arrow[r] \arrow[d, equal] & E^2 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L(1) \arrow[r] & E_1^2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & 0 \\ & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\ & & 0 & 0 & \end{tikzcd} \tag{10.1.1}$$

where $M_1 = M_2^*(1)$ via Poincaré duality, and $E^2 \coloneqq E_2^*(1)$. We denote its class by $[E_1^2] \in H^1(F, E^2)$.

10.2. **Height pairings.** We collect some definitions and properties of local and global height pairings.

10.2.1. *Local height pairings of algebraic cycles.* Suppose that F is non-archimedean of residue characteristic ℓ. Let $\lambda\colon F^\times \hat{\otimes} L \to \Gamma$ be an L-linear map.

For $i = 1, 2$ let $M_i \subset H^{2d_i-1}(X_{\overline{F}}, L(d_i))$ be $L[G_F]$-submodules, and denote still by $\langle\, , \rangle\colon M_1 \otimes_L M_2 \to L(1)$ the restriction of the Poincaré pairing

$$\langle\, , \rangle\colon H^{2d_1-1}(X_{\overline{F}}, L(d_1)) \otimes_L H^{2d_2-1}(X_{\overline{F}}, L(d_2)) \xrightarrow{\cup} H^{2m-2}(X_{\overline{F}}, L(m)) \xrightarrow{\mathrm{Tr}} L(1),$$

where the map Tr is the sum of the trace maps for the connected components of X. Assume that M_1, M_2 satisfy the following conditions:

(1) $\langle\, , \rangle\colon M_1 \otimes_L M_2 \to L(1)$ is a perfect pairing;

(2) if $\ell \neq p$, we have $H^0(F, M_i) = 0$ for $i = 1, 2$; this implies condition (1) for M_1, M_2 in § 10.1.1, and is implied by the condition that M_i is pure of weight -1;

(3) if $\ell = p$:
  - M_i is crystalline with $\mathrm{D}_{\mathrm{crys}}(M_i)^{\varphi=1} = 0$ for $i = 1, 2$; this implies condition (2) for M_1, M_2 in § 10.1.1, and is implied by the condition that M_i is crystalline and pure of weight -1;
  - the *Panchishkin condition*: there is a (necessarily unique) extension of crystalline representations
  $$0 \to M_i^+ \to M_i \longrightarrow M_i^- \to 0$$
  such that $\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(M_i^+) = \mathbb{D}_{\mathrm{dR}}(M_i^-)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(M_i^-) = 0$; this implies that the natural map
  $$\mathbb{D}_{\mathrm{dR}}(M_i^+) \oplus \mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(M_i) \xrightarrow{\cong} \mathbb{D}_{\mathrm{dR}}(M_i) \tag{10.2.1}$$
  is a splitting of the Hodge filtration on $\mathbb{D}_{\mathrm{dR}}(M_i)$.

Assume that $Z_1 \in \mathrm{Z}^{d_1}_{M_1}(X)^0$, $Z_2 \in \mathrm{Z}^{d_2}_{M_2}(X)^0$. Then the biextension class $[E^{Z_2}_{Z_1}]$ belongs to the preimage $H^1_{M_1\text{-}f}(F, E^2) \subset H^1(F, E^2)$ of $H^1_f(F, M_1)$ under the natural map $H^1_f(F, E^1) \to H^1(F, M_1)$. This group sits in the (pushout) diagram of exact sequences[19]

$$\begin{tikzcd} 0 \arrow[r] & H^1(F, L(1)) \arrow[r] & H^1_{M_1\text{-}f}(F, E^2) \arrow[r] \arrow[l, bend right, "\sigma"'] & H^1_f(F, M_1) \arrow[r] & 0 \\ 0 \arrow[r] & H^1_f(F, L(1)) \arrow[r] \arrow[u] & H^1_f(F, E^2) \arrow[r] \arrow[u] \arrow[l, bend right, "\sigma_f"'] & H^1_f(F, M_1) \arrow[r] \arrow[u, equal] & 0 \end{tikzcd} \tag{10.2.2}$$

admitting canonical splittings σ, σ_f. These are obvious if $\ell \neq p$, as then $H^1(F, M_1) = 0$; for $\ell = p$, they are induced by (10.2.1) (see [Nek93, § 4]). Morevoer, the Kummer map identifies $H^1(F, L(1)) \cong F^\times \hat{\otimes} L(1)$.

Definition 10.2.1. Let M_1, M_2, Z_1, Z_2 be as above. We define

$$h_{X,\lambda}(Z_1, Z_2) := \lambda \circ \sigma([E^2_1]) \in \Gamma. \tag{10.2.3}$$

Remark 10.2.2. Since the conditions on the pair (M_1, M_2) are stable under subobejcts and extensions (see [DL24, Lemma A.14] for extensions when $\ell = p$), there is a maximal pair satisfying those; in particular we may omit (M_1, M_2) from the notation.

Remark 10.2.3. If $\ell = p$, it follows from the previous discussion that $\sigma([E^2_1]) \in \mathscr{O}^\times_F \hat{\otimes} L \subset F^\times \hat{\otimes} L$ if and only if $[E^2_1]$ is crystalline (that is, belongs to $H^1_f(F, E^2)$).

Lemma 10.2.4 (Base change)**.** *Consider the setup of Definition 10.2.1.*

(1) Let F'/F be a finite extension, and let $\lambda' := \lambda \circ \mathrm{N}_{F'/F}$. Then for any $Z_1 \in \mathrm{Z}^{d_1}_{M_1}(X)^0$, $Z'_2 \in \mathrm{Z}^{d_2}_{M_2}(X_{F'})^0$,

$$h_{X_F,\lambda}(Z_1, \mathrm{N}_{F'/F} Z'_2) = h_{X_{F'},\lambda'}(Z_{1,F'}, Z_2).$$

(2) Let $u\colon X' \to X$ be a finite étale morphism, and let $Z_1 \in \mathrm{Z}^{d_1}_{M_1}(X)^0$, $Z_2 \in \mathrm{Z}^{d_2}_{M_2}(X)^0$. Denote by Z'_i the pullback of Z_i to X'. Assume $\ell \neq p$. Then

$$h_{X,\lambda}(Z_1, Z_2) = \frac{1}{\deg u} h_{X',\lambda}(Z'_1, Z'_2).$$

Proof. Part (1) is [Nek95, (II.1.9.1)]. Part (2) follows from [LL21, Lemma B.3] and [DL24, Proposition A.7]. □

10.2.2. *Global height pairings for Selmer groups.* Let now F be number field and $\lambda\colon \Gamma_{F,L} \to \Gamma$ be an L-linear map.

Let M_1, M_2 be L-vector spaces endowed with continuous G_F-representations that are unramfied at all but finitely many places of F, and de Rham at all the p-adic places. Assume moreover that M_1, M_2 are endowed with a perfect G_F-equivariant pairing $\langle\, , \rangle\colon M_1 \otimes_L M_2 \to L(1)$, and that for each i and each finite place w of F, the representation M_i restricted to G_{F_w} satisfies the conditions (2), (3) of § 10.2.1.

[19] This diagram should also replace an incorrect one in [Dis17, (4.1.4)].

Under these conditions,[20] Nekovář [Nek93] defined a bilinear height pairing on the Bloch–Kato Selmer groups

$$h_{M_1,\lambda}\colon H^1_f(F, M_1) \otimes_L H^1_f(F, M_2) \longrightarrow \Gamma \tag{10.2.4}$$

as follows. For $i = 1, 2$ pick representatives E_i of the extension classes $[E_i] \in H^1_f(F, M_i)$, and let E_1^2 be a biextension fitting in a diagram (10.1.1) of G_F-representations. For each place w of F, one can then define $h_w^{E_1^2}([E_1], [E_2])$ by the right-hand side of (10.2.3) (where everything is viewed as a representation of G_w); the sum

$$h([E_1], [E_2]) \coloneqq \sum_w h_w^{E_1^2}([E_1], [E_2])$$

does not depend on the choice of E_1^2.

Lemma 10.2.5 (Projection formula). *Let (M_1, M_2) and (M_1', M_2') be as above. Let $\phi\colon M_1' \to M_1$ be a map of G_F-representations, and let $\phi^*(1)\colon M_2 \to M_2'$ be the dual map. Let $[E_1'] \in H^1_f(F, M_1')$, $[E_2] \in H^1_f(F, M_2)$. Denote by $E_2' \coloneqq \phi^*(1)_* E_2$, $E_1 \coloneqq \phi_* E_1'$ the pushouts. Then*

$$h_{M_1'}([E_1'], [E_2']) = h_{M_1}([E_1], [E_2]).$$

Proof. Let $E' \in H^1_f(F, E^{2\prime})$ be a biextension (as in (10.1.1)) of E_1' and $E^{2\prime} \coloneqq \phi^*(E_2^*(1)) = E_2'^*(1)$. The map $\phi\colon M_2'^*(1) \cong M_1' \to M_1 \cong M_2^*(1)$ induces by pullback a map $\phi\colon E^{2\prime} \to E^2$. Then a diagram chase shows that $\phi_* E' \in H^1_f(F, E^2)$ is a biextension of E_1 and E^2. □

10.2.3. *Decomposition in the case of algebraic cycles.* Let X be a proper smooth scheme over F of dimension $m-1$, and suppose that $M_i \subset H^{2d_i-1}(X_{\overline{F}}, L(d_i))$ are $L[G_F]$-submodules satisfying the above conditions with respect to a pairing $\langle\, ,\,\rangle$ that is the restriction of the Poincaré pairing. We then denote $h_{X,\lambda} \coloneqq h_{M_1,M_2,\lambda}$, for which we have

$$h_{X,\lambda}(\mathrm{cl}(Z_1), \mathrm{cl}(Z_2)) = \sum_{w\nmid\infty} h_{X_w,\lambda_w}(Z_1, Z_2), \tag{10.2.5}$$

where the sum runs over all the non-archimedean places of F, and $X_w \coloneqq X_{F_w}$, $\lambda_w \coloneqq \lambda_{|F_w^\times \hat{\otimes} L}$.

10.3. **Relation to arithmetic intersection theory.** We collect two results relating local heights away from p, and the crystalline property of biextensions at p, with arithmetic intersections. We start with some preliminaries. For more details on the background, see [LL21, Appendix B] and references therein.

10.3.1. *Extensions of algebraic cycles.* Let $\mathscr{X}$ be a regular scheme; for a closed subset $\mathscr{Y}$ (omitted from the notation if $\mathscr{Y} = \mathscr{X}$) we denote by $K_0^{\mathscr{Y}}(\mathscr{X})$ the K-group of complexes of coherent sheaves on $\mathscr{X}$ with cohomology supported in $\mathscr{Y}$. We denote by $\mathrm{F}^\bullet$ the filtration on $K_0^{\mathscr{Y}}(\mathscr{X})$ by the codimension of support. We have an L-linear map

$$\kappa\colon \mathrm{Z}^d(\mathscr{X})_L \longrightarrow \mathrm{F}^\bullet K_0(\mathscr{X})_L \tag{10.3.1}$$

such that if $\mathscr{Z} \subset \mathscr{X}$ is an integral subscheme, then $\kappa([\mathscr{Z}]) = [\mathscr{O}_{\mathscr{Z}}]$.

[20] These are not the most general possible; for instance, the crystalline condition at p-adic places is not necessary.

Let now F be a nonarchimedean local field and denote by k its residue field. Assume that the regular scheme $\mathscr{X}$ is endowed with a projective and flat map $\pi\colon \mathscr{X} \to \mathscr{O}_F$, and denote by X and $\mathscr{X}_k$ respectively the generic and special fibre of $\mathscr{X}$.

Definition 10.3.1. Let $Z \in \mathrm{Z}^d(X)_L$, and denote by $|Z| \subset X$ its support. We say that an element

$$\mathscr{Z} \in \mathrm{F}^d K_0^{\mathscr{X}_k \cup |Z|}(\mathscr{X})_L \subset \mathrm{F}^d K_0(\mathscr{X})_L$$

is an *extension* of Z if $\mathscr{Z}_{|X} \in \mathrm{F}^d K_0(X)_L$ coincides with $\kappa(Z)$.

10.3.2. *Intersection pairing.* Suppose that X has dimension $m-1 \geq 1$. For a pair of integers $d_1, d_2 \geq 0$ with $d_1 + d_2 = m$, and cycles $\mathscr{Z}_i \in \mathrm{F}^{d_i} K_0(\mathscr{X})$ with $|\mathscr{Z}_1| \cap |\mathscr{Z}_2| \subset |\mathscr{X}_k|$, we define their *intersection* by

$$(\mathscr{Z}_1 \cdot \mathscr{Z}_2) \coloneqq \chi(\pi_*(\mathscr{Z}_1 \cup \mathscr{Z}_2)),$$

where

$$\cup\colon \mathrm{F}^{d_1} K_0^{|\mathscr{Z}_1|}(\mathscr{X}) \otimes \mathrm{F}^{d_2} K_0^{|\mathscr{Z}_2|}(\mathscr{X}) \longrightarrow \mathrm{F}^m K_0^{\mathscr{X}_k}(\mathscr{X})$$

is the cup product, and $\chi\colon K_0(\operatorname{Spec} k) \to \mathbf{Z}$ is the Euler characteristic. The definition is extended linearly to cycles with coefficients in L.

10.3.3. *Arithmetic intersections and the crystalline property at p.* Consider the setup of § 10.2.1 with $\ell = p$.

Proposition 10.3.2. *Assume that $p > m$ or $m = 2$, and that X admits a proper smooth model $\mathscr{X}/\mathscr{O}_F$. If the supports of the Zariski closures $\mathscr{Z}_1$, $\mathscr{Z}_2$ of Z_1, Z_2 in $\mathscr{X}$ are disjoint, then the biextension $[E_1^2]$ is crystalline.*

Proof. If $p > m$, this is a special case of [DL24, Theorem A.8]. If $m = 2$, this is a special case of [Dis17, Proposition 4.3.1]. □

10.3.4. *Arithmetic intersections and local heights away from p.* Consider the setup of § 10.2.1 with $\ell \neq p$.

Proposition 10.3.3. *Assume that $m = 2n$ and $d_1 = d_2 = n$. Let $T_1, T_2 \in \mathrm{\acute{E}tCorr}(\mathscr{X})_L$, and assume that $Z_1.T_1$ and $Z_2.T_2$ have disjoint supports. Let $\mathscr{X}$ be a regular flat projective scheme over $\mathscr{O}_F$ with generic fibre X, and let $\mathscr{Z}_i \in \mathrm{F}^{d_i} K_0(\mathscr{X})$ be an extension of Z_i for $i = 1, 2$.*

Suppose that one of the following conditions holds:

(1) $\mathscr{X}$ is smooth over $\mathscr{O}_F$, $\mathscr{Z}_i$ is (the image under $\kappa =$ (10.3.1) of) the Zariski closure of Z_i, and $T_i = \mathrm{id}$;

(2) T_1, T_2 annihilate $H^{2n}(\mathscr{X}, L(n))$.

Then

$$h_\lambda(\mathscr{Z}_1.T_1, \mathscr{Z}_2.T_2) = -((\mathscr{Z}_1.T_1) \cdot (\mathscr{Z}_2.T_2))\, \lambda(\varpi)$$

where $\varpi \in F^\times$ is a uniformizer.

Proof. In case (1), this is a special case of [LL21, Proposition B.10] combined with [DL24, Proposition A.7, Remark A.6]. In case (2), this is [LL21, Proposition B.13] combined with [DL24, Proposition A.7, Remark A.6]. □

In favorable cases, correspondences satisfying condition (2) of the proposition can be found using Proposition 9.4.1 as in [LL21, LL22].

11. The arithmetic relative-trace formula

Let $V \in \mathscr{V}^{\circ,-}$ be an incoherent pair, and let $\mathrm{G} = \mathrm{G}^V$, $\mathrm{H} \coloneqq \mathrm{H}^V$. In this section, we define our cycles of interest, and a distribution $\mathscr{J} = \mathscr{J}_{K_p}$ on (part of) the Hecke algebra for $\mathrm{G}(\mathbf{A}^p)$ that encodes their p-adic heights. The main result of this section is the arithmetic RTF for $\mathscr{J}$ (Theorem 11.5.3).

We will denote $X_K \coloneqq \mathrm{Sh}_K(\mathrm{G})$, $Y_{K_\mathrm{H}} \coloneqq \mathrm{Sh}_{K_\mathrm{H}}(\mathrm{H})$. In § 11.1 we study the étale cohomology of X_K and define the Galois representation of interest. In § 11.2, we define and study the arithmetic diagonal cycles and Gan–Gross–Prasad cycles. In § 11.3 we define $\mathscr{J}$ by means of height pairings of those cycles, and give its spectral expansion. In § 11.4 we prove some vanishing results to decompose $\mathscr{J}$ as a sum indexed by the nonsplit places of F_0. Finally, in § 11.5 we state the geometric expansion of $\mathscr{J}$.

11.1. **Cohomology and automorphic Galois representations.**

Let L be an algebraic extension of $\mathbf{Q}_p$.

11.1.1. *Ordinary representations of* $\mathrm{G}(\mathbf{A})$. We say that $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$ is *ordinary* if for every place $v|p$ of F_0, the base-change $\mathrm{BC}(\pi_v)$ satisfies the ordinariness conditions of § 1.1.2. If $K_p \subset \mathrm{G}(F_{0,p})$ is a compact open subgroup, we say that π_p is K_p-*ordinary* if it is ordinary and moreover $\pi^{K_p} \neq 0$. These conditions define ind-subschemes

$$\mathscr{C}(\mathrm{G})^{\mathrm{ord}}_{K_p} \subset \mathscr{C}(\mathrm{G})^{\mathrm{ord}} \subset \mathscr{C}(\mathrm{G})_{\mathbf{Q}_p}.$$

We also denote by $\mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}$ and $\mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}$ their ind-subschemes of Galois orbits of distinguished representations. Finally, for the above decorations '?', we define $\widetilde{\mathscr{C}}(\mathrm{G})^?(L)$ as the corresponding sets of isomorphism classes of representations such that $\mathscr{C}(\mathrm{G})^?(L) = \widetilde{\mathscr{C}}(\mathrm{G})(\overline{L})/G_L$ (cf. § 2.5.3).

11.1.2. *Duals and Hecke actions.* If S is a finite set of places of F_0 and M is an admissible (left) $L[\mathrm{G}(\mathbf{A}^S)]$-module, we denote

$$M^* \coloneqq \varprojlim_{K^S \subset \mathrm{G}(\mathbf{A}^{S\infty})} M^{K^S,\vee}$$

the algebraic dual of M, whereas as usual we denote by $M^\vee = \varinjlim_{K^S} M^{K^S,\vee}$ the contragredient; for any compact subgroup $K' \subset \mathrm{G}(\mathbf{A}^{S\infty})$, we denote by $M^*_{K'}$ the K'-coinvariants (thus the natural map $M^{\vee,K'} \to M^*_{K'}$ is an isomorphism if K' is open). We have a map

$$\begin{aligned} M^\vee &\longrightarrow M^* \\ x &\longmapsto \lim_K x \circ e_K \end{aligned} \tag{11.1.1}$$

(where $e_K \colon M \to M^K$ is the natural K-projection). The left Hecke action on M induces a right action

$$T \colon \mathscr{H}(\mathrm{G}(\mathbf{A}^S), L) \longrightarrow \mathrm{Hom}(M^*, M^\vee).$$

11.1.3. *Hecke and Galois actions on the cohomology of unitary Shimura varieties.* For $i \in \mathbf{Z}$, we put

$$M^{i,K} := H^i(\mathrm{Sh}_K(\mathrm{G})_{\overline{F}}, \mathbf{Q}_p(n)), \qquad M^i := \varinjlim_K M^i_K. \tag{11.1.2}$$

where the limit is with respect to the pullback maps. For $? = \emptyset, K$, we also put $M^{\oplus,?} := \bigoplus M^{i,?}$; it has a natural (left) action by $\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty), \mathbf{Q}_p)_?$ and by the Galois group G_F.

Let $\diamond \in \mathbf{Z} \cup \{\oplus\}$. Then the $\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty), \mathbf{Q}_p)$-action on $M^\diamond$ makes it into an admissible $\mathrm{G}(\mathbf{A}^\infty)$-module, so that we may consider $M^{\diamond,*}$. It is helpful to think of $M^{\diamond,*}$ as the inverse limit of homology (and of $M^\diamond$, $M^{\diamond,\vee}$ as the direct limits of cohomology, respectively homology).

For $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$, let

$$\begin{aligned}\rho[\pi]^\diamond &:= \mathrm{Hom}_{\mathscr{H}(\mathrm{G}(\mathbf{A}^\infty))}(\pi^\vee, M_L^{\diamond,\vee}(1)),\\ M^{\diamond,\pi} &:= \pi^* \boxtimes \rho[\pi] \subset M_L^{\diamond,*}(1),\end{aligned}$$

so that we have a Hecke-equivariant map

$$\pi \longrightarrow \mathrm{Hom}_{G_F}(M^{\diamond,*}(1), \rho[\pi]) \tag{11.1.3}$$

factoring through $\mathrm{Hom}_{G_F}(M^{\diamond,\pi}, \rho[\pi])$. In fact, it is known (see [BW80, Theorem III.5.1]) that the temperedness implies

$$M^{\oplus,\pi} = M^{2n-1,\pi} \tag{11.1.4}$$

so that we will simply write $M^\pi := M^{2n-1,\pi}$, $\rho[\pi] := \rho[\pi]^{2n-1}$.

We put $M^K_{\pi^\vee} := (M^\pi_K)^\vee$ and $M_{\pi^\vee} := \varinjlim_K M_{\pi^\vee}$, so that $M^\pi = M^*_{\pi^\vee}(1)$. For $? \in \{\mathrm{temp}, \mathrm{t\text{-}ord}\}$, we put

$$M_{?,\overline{\mathbf{Q}}_p} := \bigoplus M_\pi \subset M^{\oplus}_{\overline{\mathbf{Q}}_p}, \qquad M^?_{\overline{\mathbf{Q}}_p} := \bigoplus M^\pi \subset M^{\oplus,*}_{\overline{\mathbf{Q}}_p}(1)$$

where the sums run over $\mathscr{C}(\mathrm{G})(\overline{\mathbf{Q}}_p)$ and $\mathscr{C}(\mathrm{G})_{\mathrm{ord}}(\overline{\mathbf{Q}}_p)$ respectively. These are base-changes of L-subspaces $M_? \subset M^{2n-1}$, $M^?_L \subset M^{2n-1,*}(1)$. Poincaré duality gives an isomorphism $M_K \cong M^*_K(1)$, which induces isomorphisms

$$M^K_? \cong M^?_K$$

for $? \in \{\mathrm{temp}, \mathrm{t\text{-}ord}\} \cup \widetilde{\mathscr{C}}(\mathrm{G})(\mathrm{L})$.

11.1.4. *Automorphic Galois representations and decomposition of the cohomology.* Assume from now on that the extension L of $\mathbf{Q}_p$ is finite, and denote by $\overline{\mathbf{Q}}_p$ an algebraic closure of L. Let $\pi = \pi_n \boxtimes \pi_{n+1} \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$.

Lemma 11.1.1. *For $\nu \in \{n, n+1\}$ there is a semisimple continuous representation*

$$\rho_{\pi_\nu, \overline{\mathbf{Q}}_p} \colon G_F \to \mathrm{GL}_\nu(\overline{\mathbf{Q}}_p)$$

characterized, up to isomorphism, by the property that for all but finitely many places w of F split over F_0, the restriction $\rho_{\pi_\nu,\overline{\mathbf{Q}}_p|\mathcal{G}_{F_w}}$ is unramified, and a geometric Frobenius at w acts with a characteristic polynomial equal to the Satake polynomial of π_w viewed as a representation of $\mathrm{GL}_\nu(E_w)$. If π_ν is stable, then

$$\rho_{\pi_\nu,\overline{\mathbf{Q}}_p} \cong \rho_{\mathrm{BC}(\pi_\nu),\overline{\mathbf{Q}}_p} \tag{11.1.5}$$

(where the latter is as in § 1.2).

Proof. The construction is as in [DL24, Lemma 4.10], using [LTX+22, Proposition 3.2.8] (due to Shin) instead of [Mok15]. Property (11.1.5) is immediate from the construction. □

Let

$$\rho_{\pi,\overline{\mathbf{Q}}_p}\colon G_F \to \mathrm{GL}_{n(n+1)}(\overline{\mathbf{Q}}_p)$$

be defined by $\rho_{\pi,\overline{\mathbf{Q}}_p}(-n) := \rho_{\pi_n,\overline{\mathbf{Q}}_p} \otimes \rho_{\pi_{n+1},\overline{\mathbf{Q}}_p}$. If $\rho\colon G_F \to \mathrm{GL}_n(L)$ is a continuous representation, denote by $\rho_{\overline{\mathbf{Q}}_p} := \rho \otimes_L \overline{\mathbf{Q}}_p$ the base-change and by $\rho^{\mathrm{ss}}_{\overline{\mathbf{Q}}_p}$ its semisimplification.

The following key hypothesis gives an explicit description of $\rho[\pi]$ (at least in the stable case).

Hypothesis 11.1.2. *Let $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$, and let $K \subset \mathrm{G}(\mathbf{A}^\infty)$ be an open compact subgroup. Then $\rho[\pi]^{\mathrm{ss}}_{\overline{\mathbf{Q}}_p}$ is is isomorphic to a direct summand of $\rho_{\pi,\overline{\mathbf{Q}}_p}$. Moreover, if π is stable then $\rho[\pi]^{\mathrm{ss}}_{\overline{\mathbf{Q}}_p} \cong \rho_{\pi,\overline{\mathbf{Q}}_p}$.*

Remark 11.1.3. Let $\Pi \in \mathscr{C}(\mathrm{G}')^{\mathrm{her},-}_{\mathbf{Q}_p}$, and let $L = \mathbf{Q}_p(\Pi)$. Let $\overline{\pi} \in \mathscr{C}(\mathrm{G}^V)(\overline{\mathbf{Q}}_p)$ be the preimage of Π under (2.5.1); a priori we know it is isomorphic to its G_L-conjugates but not that it arises from some $\pi \in \widetilde{\mathscr{C}}(\mathrm{G}^V)(L)$. Assume that Hypothesis 11.1.2 holds. By the definitions, the space

$$M^{\overline{\pi}}$$

is isomorphic to $\overline{\pi}^* \boxtimes \rho[\overline{\pi}]$ as a Hecke- and $\overline{\mathbf{Q}}_p[G_F]$-module, and it is a G_L-invariant subspace of $M^{2n-1,*}_{\overline{\mathbf{Q}}_p}(1)$. Let $M^\pi := (M^{\overline{\pi}})^{G_L} \subset M^{2n-1,*}_L(1)$, and define the $L[G_F]$-module

$$\rho_\Pi := (M^\pi)^{\mathrm{H}(\mathbf{A}^\infty)}$$

Then we have

$$\rho_\Pi \otimes_L \overline{\mathbf{Q}}_p \cong (\overline{\pi}^*)^{\mathrm{H}(\mathbf{A}^\infty)} \otimes_{\overline{\mathbf{Q}}_p} \rho[\overline{\pi}].$$

The first tensor factor is 1-dimensional, so that by Remark 2.5.7 and (11.1.5), the representation

$$\rho_\Pi\colon G_F \longrightarrow \mathrm{GL}_{n(n+1)}(L)$$

satisfies

$$(\rho_\Pi \otimes_L \overline{\mathbf{Q}}_p)^{\mathrm{ss}} \cong \rho_{\Pi_n,\overline{\mathbf{Q}}_p} \otimes \rho_{\Pi_{n+1},\overline{\mathbf{Q}}_p}(n).$$

(In fact, it is conjectured that $\rho_{\Pi_\nu,\overline{\mathbf{Q}}_p}$ is irreducible for $\nu = n, n+1$, so that the semisimplifcation should be superfluous.) This also implies that $\overline{\pi}$ has a model $\pi = \mathrm{Hom}_{L[G_F]}(M^\pi, \rho_\Pi)$ defined over L; in other words, for an incoherent $V \in \mathscr{V}^{\circ,-}$ we have $\widetilde{\mathscr{C}}(\mathrm{H}^V\backslash\mathrm{G}^V)^{\mathrm{st}}_{\mathbf{Q}_p} = \mathscr{C}(\mathrm{H}^V\backslash\mathrm{G}^V)^{\mathrm{st}}_{\mathbf{Q}_p}$.[21]

11.1.5. *Properties of automorphic Galois representations.*

Proposition 11.1.4. *Let $\pi \in \mathscr{C}(\mathrm{G})_{\mathbf{Q}_p}(L)$. The Galois representation $\rho := \rho_{\pi,\overline{\mathbf{Q}}_p}$ satisfies the following properties:*

(1) For every nonarchimedean place w of F, the representation $\rho_{|G_{F_w}}$ is pure of weight -1 in the sense of [DL24, Definition A.11].

(2) The representations ρ^{c} and $\rho^*(1)$ are isomorphic.

(3) For every place $v|p$ of F_0 and every place $w|v$ of F:

(a) if π_v is unramified, then $\rho_{|G_{F_w}}$ is crystalline;

[21]It is plausible that this kind of equality holds more generally, but we do not explore this here.

(b) if moreover π_v is ordinary, then $\rho_{|G_{F_w}}$ is Panchishkin-ordinary.

If Hypothesis 11.1.2 holds, then the conclusions (1)-(3) above also hold for $\rho = \rho[\pi]$ and $\rho = M^\pi_K$.

Proof. Part (1) is a fundamental result of Caraiani [Car12, Car14] (see also [TY07, Lemma 1.4 (3)]). Part (2) follows from the last statement in [DL24, Lemma 4.10] for $\rho_{\pi,\overline{\mathbf{Q}}_p}$, and from the Galois-equivariance of the Poincaré pairing for $\rho[\pi]$, M_π. The proof of part (3) is as in [DL24, Lemmas 4.9, 4.14]. (In fact the assumption on $\pi_{\nu,v}$ in (b) is stronger than the analogous assumption in *loc. cit.*; correspondingly each factor $\rho_{\pi_\nu|G_{F_w}}$ is also ordinary in the sense of [Nek93, Definition 1.29]; however, only Panchishkin-ordinariness is stable under tensor products.) ☐

For the rest of the paper, we will assume Hypothesis 11.1.2 for every[22] representation $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(\overline{\mathbf{Q}}_p)$.

11.2. **Gan–Gross–Prasad cycles.** We define our cycles and study an 'ordinary' modification.

11.2.1. *Arithmetic diagonal cycles.* We have a fundamental cycle

$$[Y]^\circ = ([Y_{K_\mathrm{H}}]^\circ) \in \varprojlim_{K_\mathrm{H}} \mathrm{Z}^0(Y_{K_\mathrm{H}})_\mathbf{Q},$$

where the transition maps on the right are pushforwards and $[Y_{K_\mathrm{H}}]^\circ = \mathrm{vol}(K_\mathrm{H}, dh)[Y_{K_\mathrm{H}}]$. Let $\jmath$ be (system of) arithmetic diagonal maps (8.2.2). The *arithmetic diagonal cycle*

$$Z := \jmath_*[Y]^\circ \in \varprojlim_{K} \mathrm{Z}^n(X_{\mathrm{G},K})_\mathbf{Q} \tag{11.2.1}$$

is well-defined. We denote by Z_K its image in $\mathrm{Z}^n(X_{\mathrm{G},K})_\mathbf{Q}$.

11.2.2. *Limits of Selmer groups.* Let L be a finite extension of $\mathbf{Q}_p$, and let $\pi \in \widetilde{\mathscr{C}}(\mathrm{G})(L)$. For $? \in \{\mathrm{temp}, \text{t-ord}, \pi\}$, define

$$H^1_f(F, M^?) := \varprojlim_K H^1_f(F, M^?_K), \qquad H^1_f(F, M_?) := \varinjlim_K H^1_f(F, M^K_?).$$

11.2.3. *GGP cycles and associated functionals.* Let

$$Z_{\pi,K} \in H^1_f(F, M_{\pi,K})$$

be the Hecke-eigencomponent of $\widetilde{\mathrm{cl}}(Z_K)$. Here, by the discussion in § 10.1, the fact that $Z_{\pi,K}$ belongs to the Bloch–Kato Selmer group is a consequence of the vanishing of $M_{\pi,K} \cap M_{2n}$ and Proposition 11.1.4.

Definition 11.2.1. The *Gan–Gross–Prasad cycle* of π is

$$Z_\pi := \varprojlim_K Z_{\pi,K} \quad \in H^1_f(F, M^\pi);$$

[22] In fact, it would be enough to assume it for the representation π in order to prove Theorem D for π, at the cost of some complication in the exposition.

The $\mathrm{H}(\mathbf{A})$-invariant functional associated to it via (11.1.3) will still be denoted by

$$\begin{aligned} Z_\pi \colon \pi &\longrightarrow H^1_f(E, \rho[\pi]) \\ \phi &\longmapsto Z_\pi\phi := \phi_* Z_\pi. \end{aligned}$$

From the $\mathrm{H}(\mathbf{A})$-invariance it follows that Z_π vanishes unless $\pi \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})$.

Remark 11.2.2. The linear functional Z_π valued in the Selmer group can be viewed as an arithmetic analog of the automorphic period functional P_π of (4.6.1).

11.2.4. *Ordinary cycles.* Suppose that every place $v|p$ of F_0 splits in F. For each v, we may fix a place $w|v$ of F and compatible bases of $V_{\nu,w}$, giving isomorphisms $\mathrm{G}_{F_{0,v}} \cong \mathrm{GL}_n \times \mathrm{GL}_{n+1}$, $\mathrm{H}_{F_{0,v}} \cong \mathrm{GL}_n$ as algebraic groups over $F_{0,v} = F_w$. Then we may and will use the notation, definitions and results of § 5; we generally also denote $\square_p := \prod_{v|p} \square_v$; for instance, $t_{0,p} = \prod_{v|v} t_{0,v}$, $N^\circ_{0,p} = \prod_{v|p} N^\circ_{0,v} \subset \mathrm{GL}_n(F_{0,p}) \times \mathrm{GL}_{n+1}(G_{F_{0,p}}) \cong \mathrm{G}(F_{0,p})$. We define an operator $e^{\mathrm{ord}} := \lim U_{t_0}^{N!}$; it acts on $M_{N^\circ_{0,p}}$ and on $\pi^{N^\circ_{0,p}}$ for any $\pi \in \mathscr{C}(\mathrm{G})_{\mathbf{Q}_p}$; the representation π is ordinary if and only if it is not annihilated by e^{ord}.

Let $K_p \subset \mathrm{G}(F_{0,p})$ be an open compact subgroup containing $N^\circ_{0,p}$, and let $c \geq 1$ be such that K_p contains $K_0^{\langle c+1\rangle}$. For positive integers r, N with $N! \geq r \geq c$, set $m_{0,r} = \prod_{v|p} m_{0,r,v}$ (cf. (5.1.4) for the definition of twisting matrices), and define

$$Z^{\dagger,N}_{K_p} := \prod_{v|p} q_v^{rd(n)} \cdot Z.T(m_{0,r} U_{t_{0,p}}^{N!-r} e_{K_p})_{\mathbf{Q}} \ \in \mathrm{Z}^n(X_{\mathrm{G},K_p}),$$

which is independent of r by Corollary 5.1.5.

We define the *ordinary arithmetic diagonal cycle* by

$$Z^{\mathrm{ord}}_{K_p} := \lim_{N\to\infty} \widetilde{\mathrm{cl}}(Z^{\dagger,N}_{K_p}) \quad \in \varprojlim_{K^p} (H^{2n}(X_{K^pK_p}, \mathbf{Q}_p(n))/\mathrm{Fil}^2).e^{\mathrm{ord}}.$$

For any $\pi \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}$, we define the ordinary GGP cycle

$$Z^{\mathrm{ord}}_{\pi,K_p} \in H^1_f(F, M^\pi_{K_p})$$

to be the eigencomponent of $Z^{\mathrm{ord}}_{K_p}$. By the definitions, for any sufficiently large r,

$$Z^{\mathrm{ord}}_{\pi,K_p} = \prod_{v|p} q_v^{rd(n)} \lim_{N\to\infty} Z_\pi.T(m_{0,r} U_{t_{0,p}}^{N!-r} e_{K_p}).$$

We have an induced $\mathrm{H}(\mathbf{A}^p)$-invariant functional still denoted by the same name

$$Z^{\mathrm{ord}}_{\pi,K_p} \colon \pi^{K_p} \longrightarrow H^1_f(E, \rho[\pi]).$$

It factors through $e^{\mathrm{ord}} e_{K_p}$.

11.2.5. *Norm relation.* Continue with the notation and assumptions of § 11.2.4. In order to study p-adic heights, it will be useful to know that $Z^{\dagger,N}_{\pi,K_p}$ is a norm from some ring class fields of F of conductors that are high powers of p.

Let T be the unitary group of the 1-dimensional hermitian space $(F, N_{F/F_0})$. We have a map

$$\mathrm{rec} \colon G_F \longrightarrow \overline{F^\times} \backslash \mathbf{A}_F^{\infty,\times} \longrightarrow \overline{\mathrm{T}(F_0)} \backslash \mathrm{T}(\mathbf{A}^\infty),$$

where the first map is the reciprocity law of class field theory and the second map is $x \mapsto x^{\mathrm{c}}/x$ (and the bars denote Zariski closures). For $v|p$ and $r \geq 0$, let $K_{T,v}^{(r)} \coloneqq \mathrm{T}(\mathscr{O}_{F_{0,v}}) \cap 1 + v^r \mathscr{O}_{F_{0,v}}$, let $\Gamma_r = \Gamma_r^{(v)} \coloneqq \mathrm{T}(F_0)\backslash \mathrm{T}(\mathbf{A}^\infty)/\mathrm{T}(\widehat{\mathscr{O}}_{F_0}^v)K_{T,v}^{(r)}$, and let

$$F_r = F_r^{(v)}/F$$

be the abelian extension such that $\mathrm{Gal}(F_r/F) \cong \Gamma_r$ under the reciprocity map. We have the norm map

$$\mathrm{N}_{F_r/F}\colon \mathrm{Z}^n(X_{\mathrm{G},K,F_r}) \longrightarrow \mathrm{Z}^n(X_{\mathrm{G},K}).$$

Lemma 11.2.3. *Fix a place $v|p$ of F_0. For any $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^{p\infty}), \mathscr{O}_L)_{K^p}$ and any integer r with $\max\{1, c(K_v) - 1\} \leq r \leq N!$, there exists a cycle $Z_r = Z_{K^p}^{\dagger,N}(f^p)_r^{(v)} \in \mathrm{Z}^n(X_{\mathrm{G},K,F_r})_{\mathscr{O}_L}$ such that*

$$Z_{K^p}^{\dagger,N}(f^p) = \mathrm{N}_{F_r/F}(Z_r).$$

Proof. We may assume $f^p = e_{K^p}$, and abbreviate $Z_K^{\dagger,N} = Z_{K^p}^{\dagger,N}(f^p)$. Let $K_H^p \coloneqq \mathrm{H}(\mathbf{A}^{p\infty}) \cap K^p$ and let $Y_r \coloneqq Y_{K_H^p K_{H,0,p}^{(r)}}$. Then by (5.1.8), the map $Y \xrightarrow{\jmath} X \xrightarrow{m_{0,r}} X \to X_K$ factors through Y_r, and we can write

$$Z_K^{\dagger,N} = \prod_{v|p} q_v^{rd(n)} \cdot (\jmath_*[Y_r]^\circ).T(m_{0,r}U_{t_0,p}^{N!-r}e_K) = \mathrm{vol}^\circ(K_{H,0,p}) \cdot (\jmath_*[Y_r]).T(m_{0,r}U_{t_0,p}^{N!-r}e_K)$$

(see (5.1.7) for $\mathrm{vol}^\circ(K_{H,0,p})$).

Let $\det\colon \mathrm{H} \to \mathrm{T}$ be the determinant map. For a compact open subgroup $K \subset \mathrm{H}(\mathbf{A}^\infty)$, by Shimura's reciprocity law we have an isomorphism of G_F-sets

$$\pi_0(Y_{K,\overline{F}}) \cong \mathrm{T}(F_0)\backslash\mathrm{T}(\mathbf{A}^\infty)/\det K.$$

Now we have $\det K_{H,0,v}^{(r)} = 1 + v^r\mathscr{O}_{F_{0,v}} = 1 + w^r\mathscr{O}_{F_{0,w}} \cong K_{S,v}^{(r)}$, where the last identification comes from the natural map $F_w^\times \subset F_v^\times \to S_v$. Thus we deduce a natural surjection $\mathrm{p}\colon \pi_0(Y_{r,\overline{F}}) \to \Gamma_r$. For each $\gamma \in \Gamma_r$, let $Y_{r,\gamma,\overline{F}} \subset Y_{r,\overline{F}}$ be the union of connected components in $\mathrm{p}^{-1}(\gamma)$; it arises as $Y_{r,\gamma} \times_{F_r} \overline{F}$ for an F_r-subvariety

$$Y_{r,\gamma} \subset Y_{r,F_r}.$$

Then for any $\gamma_0 \in \Gamma_r$, we have

$$Z_K^{\dagger,N} = \mathrm{vol}^\circ(K_{H,0,p}) \cdot \mathrm{N}_{F_r/F}(\jmath_*[Y_{r,\gamma_0}]).T(m_{0,r}U_{t_0,p}^{N!-r}e_K),$$

which belongs to $\mathrm{N}_{F_r/F}(\mathrm{Z}^n(X_{\mathrm{G},K,F_r})_{\mathbf{Z}_p})$ since $\mathrm{vol}^\circ(K_{H,0,p})$ is a p-unit. □

11.3. **The distribution and its spectral expansion.** From now until the end of the paper, we suppose that every place $v|p$ of F_0 splits in F and that $K_p \subset \mathrm{G}(F_{0,p})$ is the hyperspecial subgroup $\mathrm{G}(\mathscr{O}_{F_{0,p}})$.

11.3.1. *Height pairings.* Considering the setup of § 10.2.3, we denote by

$$h\colon H_f^1(F, M_{\text{t-ord}}^{K_p}) \times H_f^1(F, M_{K_p}^{\text{t-ord}}) \longrightarrow \Gamma_{F_0,L} \tag{11.3.1}$$

the pairing induced by the family $h_{X_K,\lambda}\colon H_f^1(F, M_{\text{t-ord}}^K)^{\otimes 2} \to \Gamma_{F_0,L}$ for $K = K^pK_p$, where

$$\lambda\colon \Gamma_{F,L} \longrightarrow \Gamma_{F_0,L}$$

is the natural surjection. It is well-defined by the projection formula (Lemma 10.2.5). Note that the conditions of § 10.2.1 for the definition of h (as well as for the definition of the pairing h_π from § 1.3.7) are satisfied by Proposition 11.1.4. We also have a pairing (abusively denoted by the same name)

$$h\colon H^1_f(F, M^{K_p}_{\text{t-ord}}) \times H^1_f(F, M^{K_p}_{\text{t-ord}}) \tag{11.3.2}$$

obtained from (11.3.1) by composing with the map induced by (11.1.1) on the second factor.

For a non-archimedean place w of F, we denote by h_w the corresponding local pairings (10.2.5) on pairs of (limits of) cycles with disjoint supports in $\mathrm{Z}^n_{\text{t-ord}}(X_{K,F_w})^0_L$ (if $w|p$) or $\mathrm{Z}^n_{\text{temp}}(X_{K,F_w})^0_L$ (if $w \nmid p$). For $w \nmid p$, this requires a projection formula for w-local heights, which is equivalent to Lemma 10.2.4 (2).

11.3.2. *Definition of the distribution.* For S a finite set of non-archimedean places of F_0, denote by

$$\mathscr{H}(\mathrm{G}(\mathbf{A}^S), L)^\circ_{K_S\text{-temp}} \subset \mathscr{H}(\mathrm{G}(\mathbf{A}^S), L)_{K_S}$$

the subalgebra of measures $f^S = f^{S\infty} f_\infty$ such that $f_\infty \in L f_\infty^\circ$ (where $f^\circ_\infty = (4.1.5)$) and $M^\oplus.T(f^S e_{K_S}) \subset M^{K_S}_{\text{temp}}$.

Define first

$$\begin{aligned} \mathscr{J}_{K_p}\colon (\mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-temp}})^2 &\longrightarrow \Gamma_{F_0,L} \\ (f_1^p, f_2^p) &\longmapsto h(Z^{\text{ord}}_{K_p}.T(f_1^p), Z^{\text{ord}}_{K_p}.T(f_2^p)), \end{aligned} \tag{11.3.3}$$

(where the right-hand side uses the pairing (11.3.2)).

Definition 11.3.1. For any $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p}$ that can be written as

$$f^p = f_1^p \star f_2^{p,\vee} \tag{11.3.4}$$

with $f_i^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-temp}}$, we define the *arithmetic relative-trace distribution* by[23]

$$\mathscr{J}_{K_p}(f^p) \coloneqq \mathscr{J}_{K_p}(f_1^p, f_2^p).$$

Remark 11.3.2. The definition is independent of the decomposition (11.3.4). Indeed, let $K^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$ be such that $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^{p\infty}))_{K^p}$, and let S be a finite set of finite places of F_0, not above p, such that $K^S \coloneqq K \cap \mathrm{G}(\mathbf{A}^{Sp\infty})$ is a maximal hyperspecial subgroup. Let $e_K^{\text{temp}} \in \mathscr{H}(\mathrm{G}(\mathbf{A}^{S\infty}))_{K^S}$ be an element acting as the idempotent projection $M^K \to M^K_{\text{temp}}$. Then by the projection formula (Lemma 10.2.5), for each decomposition (11.3.4) we have

$$\mathscr{J}_{K_p}(f_1^p, f_2^p) = h(Z^{\text{ord}}_{K_p}.T(f_1^p), Z^{\text{ord}}_{K_p}.T(f_2^p e_K^{\text{temp}})) = h(Z^{\text{ord}}_{K_p}.T(f^p), Z^{\text{ord}}_{K_p}.T(e_K^{\text{temp}})),$$

which shows that $\mathscr{J}_{K_p}(f^p)$ is well-defined.

Let now

$$f_{p,K_p,N} \coloneqq \prod_{v|p} q_v^{rd(n)} \cdot m_{0,r,p} U_{t_0,p}^{N!-r} e_{K_p}, \tag{11.3.5}$$

where $1 \le r \le N!$. By the definitions, we have

$$\mathscr{J}_{K_p}(f_1^p, f_2^p) = \lim_{N\to\infty} h(Z.T(f_1^p f_{p,K_p,N}), Z.T(f_2^p f_{p,K_p,N})).$$

[23]The abuse of notation with respect to (11.3.3) should cause no confusion.

It is independent of the choice of $r \leq N!$.

11.3.3. *Spectral expansion.* Let $\pi \in \widetilde{\mathscr{C}}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}(L)$. Denote by

$$h_{M_\pi}\colon H^1_f(F, M_\pi^{K_p}) \times H^1_f(F, M^{\pi^\vee}_{K_p}) \longrightarrow \Gamma_{F_0,L}$$

the restriction of h. For any $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ$, we define

$$\mathscr{I}_{\pi,K_p}(f^p) \coloneqq h_{M_\pi}(Z^{\mathrm{ord}}_{\pi,K_p}.T(f^p), Z^{\mathrm{ord}}_{\pi^\vee,K_p}) = \mathrm{Tr}^{h_\pi\circ\left(Z^{\mathrm{ord}}_{\pi,K_p}\boxtimes Z^{\mathrm{ord}}_{\pi^\vee,K_p}\right)}_{(,)_{\pi^{K_p}}}(T(f^p)),$$

where the pairing $(,)_{\pi^{K_p}}$ is the restriction of $(,)_\pi = (1.3.4)$ to $\pi^{K_p} \times \pi^{\vee,K_p}$. Then it is clear that if f^p is as in Definition 11.3.1, we have

$$\mathscr{I}_{K_p}(f^p) = \sum_{\pi\in\mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}} \mathscr{I}_{\pi,K_p}(f^p),$$

where for a Galois orbit $\pi = \{\pi^\sigma\} \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}$ of isomorphism classes of representations, we put $\mathscr{I}_{\pi,K_p} \coloneqq \sum \mathscr{I}_{\pi^\sigma,K_p}$.

11.4. **Decomposition over nonsplit places.** We will complete the arithmetic relative-trace formula by finding a geometric expansion for the distribution $\mathscr{I}_{K_p}$. Each term in the expansion will be a sum over all nonsplit finite places of F_0. The goal of this subsection is to show the preliminary result that $\mathscr{I}_{K_p}$ has a decomposition as a sum over nonsplit places, by proving some vanishing results for local height pairings at split (p-adic and non-p-adic) places.

11.4.1. *Decomposition over all places.* Let v be a non-archimedean place of F_0. We define

$$\mathscr{I}^{(v),N}_{K_p}(f_1^p, f_2^p) \coloneqq \sum_{w|v} h_w(Z^{\dagger,N}_{K_p}.T(f_1^p), Z^{\dagger,N,}_{K_p}.T(f_2^p))$$

for any f_1^p, $f_2^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-temp}}$ (respectively $\mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-t-ord}}$ if $v|p$) such that the two cycles involved have disjoint supports. Here, the sum ranges over the (one or two) places of F above v.

It is then clear from the definitions that for f_1^p, $f_2^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p\text{-t-ord}}$, we have a decomposition

$$\mathscr{I}_{K_p}(f_1^p, f_2^p) = \lim_{N\to\infty} \sum_{v\nmid\infty} \mathscr{I}^{(v),N}_{K_p}(f_1^p, f_2^p). \tag{11.4.1}$$

In the rest of this subsection, we show the vanishing of the contribution at split (p-adic and non-p-adic) places.

Remark 11.4.1. If $v \nmid p\infty$, we can more generally define

$$\mathscr{I}^{(v)}(f_1, f_2) \coloneqq \sum_{w|v} h_w(Z.T(f_1), Z.T(f_2))$$

for f_1, $f_2 \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^\circ_{\mathrm{temp}}$ such that the two cycles involved have disjoint supports; then

$$\mathscr{I}^{(v),N}_{K_p}(f_1^p, f_2^p) = \mathscr{I}^{(v)}(f_1^p f_{p,K_p,N}, f_2^p f_{p,K_p,N}). \tag{11.4.2}$$

11.4.2. *Auxiliary Shimura varieties.* Let v be a place of F_0 and w a place of F above v. Choose an "admissible CM type Φ (relative to v)" in the sense of [LL21, p.851] and a place u of the reflex field E above the place w of F such that u is unramified over w. (Note that Φ depends on v.) Recall from §9.1 that, by our assumption, the compact open subgroup $K_{\mathrm{Z}\mathbf{Q}}$ is maximal at v. We then have the auxiliary Shimura variety

$$X'_u := X'_{K/E_u} := \mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}(\widetilde{\mathrm{G}})_{E_u} \tag{11.4.3}$$

and its integral model $\mathscr{X}'_u := \mathcal{M}_{K_{\widetilde{\mathrm{G}}},\mathscr{O}_{E,u}}$ from §9.3. We observe that X'_u is of the form $X_{F_w} \times_{F_w} A$ for some finite étale F_w-algebra A.

We denote by

$$\mathscr{Z}' = \mathscr{Z}'_u := \widetilde{\jmath}_* \left(\mathrm{vol}(K_{\mathrm{H}})[\widetilde{\mathcal{M}}_{K_{\widetilde{\mathrm{H}}}}]\right)$$

the $\mathscr{O}_{E,u}$-integral model of the arithmetic diagonal cycle, where $\widetilde{\jmath}$ is as in § 9.3.3.

11.4.3. *Local heights at split places.* The following lemma will be useful for considerations both at places above p and away from p. We first need a definition.

Definition 11.4.2. We say that a pair $(f_{1,v}, f_{2,v}) \in \mathscr{H}(G_v)^2_{K_v}$ is K_v*-regular*, if $f_{1,v}$ has regular support and $f_{2,v} = e_{K_v}$.

If S is a finite set of finite places of F_0 and $v \notin S$ is another finite place of F_0, we say that a pair $(f_1^S, f_2^S) \in \mathscr{H}(\mathrm{G}(\mathbf{A}^S), L)^\circ_{K^S}$ is K*-regular* at v if we can write $K^S = K^{Sv}K_v$ and $f_i^S = f_{i,v} \otimes f_i^{Sv}$ with $(f_{1,v}, f_{2,v})$ K_v-regular.

Lemma 11.4.3. *Let v be a split place of F_0. Let $K = \prod_v K_v$ be an open subgroup of* $\mathrm{G}(\mathbf{A}^\infty)$. *Suppose that $f_1, f_2 \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^\circ_K$ satisfy:*

- *(f_1, f_2) is $K_{v'}$-regular support at some finite place $v' \neq v$;*
- *the subgroup $K_v = K_{n,v} \times K_{n+1,v}$ satisfies either of the following conditions:*
  - *(a) for some labelling $\{\nu, \nu'\} = \{n, n+1\}$, the subgroup $K_{\nu,v}$ is maximal hyperspecial and $K_{\nu',v}$ is the principal congruence subgroup of level $m \in \mathbf{Z}_{\geq 0}$ (cf.§9.2);*
  - *(b) for both $\nu = n, n+1$, the subgroup $K_{\nu,v}$ is Iwahori (that is, $G_{\nu,v}$-conjugate to the standard Iwahori $\mathrm{Iw}_{\nu,v,0}$).*

Then the following statements hold.

(i) The cycles $Z.T(f_1)$ and $Z.T(f_2)$ have disjoint support (on the generic fiber).

(ii) Abusing notation, we still let $T(f_i)$ denote the (flat) correspondence on the integral model $\mathscr{X}'_u$. Then the cycles $\mathscr{Z}'_u.T(f_1)$ and $\mathscr{Z}'_u.T(f_2)$ have disjoint supports in $\mathscr{X}'_u$.

Proof. Part (i) follows from [RSZ20, Theorem 8.5 (i)]. (The result in *loc. cit* only treats the auxiliary Shimura variety attached to $\widetilde{\mathrm{G}}$; but it implies the desired result for G.)

For (ii) case (a), the integral model with Drinfeld m-level structure at one factor and with hyperspecial level ($m = 0$) at the other factor is regular. The proof of [RSZ20, Theorem 8.5 (ii)] (only the case $f_2 = e_K$ was considered there) still applies to show that the cycles $\mathscr{Z}'_u.T(f_1)$ and $\mathscr{Z}'_u.T(f_2)$ have disjoint supports in $\mathscr{X}'_u$.

In case (b), the integral model is the resolution given in §9.3 of the moduli scheme and the Hecke correspondences are obtained by base change and hence remain finite flat. The cycles are obtained by strict transforms. Hence it suffices to show the disjointness before the resolution, which again follows from [RSZ20, Theorem 8.5 (ii)]. (Strictly speaking, the result of *loc. cit.* concerns the case of Drinfeld m-levels rather than Iwahori level. However, we may pull back the cycles to the moduli scheme with Drinfeld level for $m = 1$ and then apply that result.)

$\square$

Proposition 11.4.4. *Let $v \nmid p$ be a split place of F_0. Let $f_1,\ f_2 \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^{\circ}_{\mathrm{temp}}$ be as in Lemma 11.4.3. Assume furthermore that either K_v is hyperspecial or that $T(f_1^v)$, $T(f_2^v)$ annihilate $H^{2n}(\mathscr{X}'_u, L(n))$. Then*

$$\mathscr{J}^{(v)}(f_1, f_2) = 0.$$

Proof. We show that $h_w(Z.T(f_1), Z.T(f_2)) = 0$ for each of the two places $w|v$. By Lemma 10.2.4 (2), it suffices to show the vanishing of the local height after pull-back to the auxiliary Shimura variety $\mathrm{Sh}_{K_{\widetilde{\mathrm{G}}}}(\widetilde{\mathrm{G}})$ over E_u. Finally, under our assumption, Proposition 10.3.3 further reduces the question to the vanishing of the arithmetic intersection pairing on the integral model $\mathscr{X}'_u$ over $\mathscr{O}_{E,u}$. This last vanishing follows from Lemma 11.4.3 (*ii*). $\square$

11.4.4. *Vanishing at p-adic places.*

Proposition 11.4.5. *Let v be a place of F_0 above p (hence split in F). If $n > 1$, assume $p > 2n$. Let $f^p = f_1^p \star f_2^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p, L)^{\circ}_{K_p\text{-t-ord}}$, and assume that the pair (f_1^p, f_2^p) has regular support. Then*

$$\lim_{N\to\infty} \mathscr{J}^{(v),N}_{K_p}(f^p) = 0.$$

Proof. Write $Z_1 \coloneqq Z^{\dagger,N}_{K_p}.T(f_1^p)$, $Z_2 \coloneqq Z^{\dagger,N}_{K_p}.T(f_2^p)$, and let K^p be such that f_1, f_2 are right-K^p-invariant. For any finite extension E of F_w, denote by $\lambda'_E\colon E^\times\hat{\otimes}L \to E^\times\hat{\otimes}L$ the identity map, and by $h_{X_{K,E}} \coloneqq h_{X_{K,E},\lambda'_E}$ the corresponding height pairing. We will show that

$$h_{X_{K,F_w}}(Z_1, Z_2) \quad \in p^{N!-C}\mathscr{O}^\times_{F_w}\hat{\otimes}\mathscr{O}_L$$

for some constant C; after taking limits, this implies the desired vanishing. Up to multiplying by a nonzero scalar, we may assume that $f_i^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), \mathscr{O}_L)^\circ$.

By Lemma 11.2.3, for some constant C' cancelling the denominators of f_i, and for any sufficiently large $r \leq N!$, we have $Z_i = \mathrm{N}_{F_r/F}(Z_{i,r})$ for some $Z_{i,r} \in p^{-C'}\mathrm{Z}^n(X_{\mathrm{G},K,F_r})_{\mathscr{O}_L}$. Denote by $F_{w,r}$ the localization of F_r at its unique place above w. First, we show that

$$h_{X_{K,F_{w,r}}}(Z_1, Z_{2,r}) \quad \in \mathscr{O}^\times_{F_{w,r}}\hat{\otimes}L. \tag{11.4.4}$$

By Lemma 10.2.4 (1) (which applies thanks to the observation made after (11.4.3)), it is enough to show the same result for the corresponding height pairing of arithmetic diagonal cycles on the auxiliary Shimura variety (11.4.3). This follows from Lemma 11.4.3 (ii), Proposition 10.3.2, and Remark 10.2.3.

By the integrality results of [Nek95, Proposition II.1.11], we have in fact

$$h_{X_{K,F_{w,r}}}(Z_1, Z_{2,r}) \quad \in p^{-C''_r-C'}\mathscr{O}^\times_{F_{w,r}}\hat{\otimes}\mathscr{O}_L, \tag{11.4.5}$$

for a constant C_r'' that, similarly to [DL24, Proof of Proposition 4.35], can be bounded as follows. Let $\mathrm{T} := M_{\text{t-ord},K,L} \cap H^{2n-1}(X_{K,\overline{F}}, \mathscr{O}_L(n))/(\text{tors})$, and denote by

$$\mathrm{N}_\infty H^1_f(F_{w,r}, \mathrm{T}) := \bigcap_{s \geq r} \mathrm{Im}\left[\mathrm{Tr}_{F_{w,s}/F_{w,r}} \colon H^1_f(F_{w,s}, \mathrm{T}) \longrightarrow H^1_f(F_{w,s}, \mathrm{T})\right].$$

Then $p^{C_r''} \leq c_r'' := |H^1_f(F_{w,r}, \mathrm{T})/\mathrm{N}_\infty H^1_f(F_{w,r}, \mathrm{T})|$. However c_r'' is bounded independently of r: this follows by the same argument as for [DL24, Lemma 4.37] from the fact that $M_{\text{t-ord},K,L}$, as a representation of G_{F_w}, is crystalline, Panchishkin-ordinary, and pure of weight -1 (Proposition 11.1.4). Thus in (11.4.5) we may replace $C_r'' + C'$ by a constant C''.

Finally, by Lemma 10.2.4 (1), we have

$$p^{C''} \cdot h_{X_{K,F_w}}(Z_1, Z_2) = p^{C''} \mathrm{N}_{F_{w,r}/F_w}(h_{X_K,F_{w,r}}(Z_{1,F_{w,r}}, Z_{2,r})) \quad \in \mathrm{N}_{F_{w,r}/F_w}(\mathscr{O}^\times_{F_{w,r}} \hat{\otimes} \mathscr{O}_L).$$

By the definition of $F_{w,r}$ and local class field theory, $\mathrm{N}_{F_{w,r}/F_w}(\mathscr{O}^\times_{F_{w,r}} \hat{\otimes} \mathscr{O}_L) \subset p^{r-C'''}(\mathscr{O}^\times_{F_w} \hat{\otimes} \mathscr{O}_L)$ for some constant C'''. This completes the proof. $\square$

11.5. **The arithmetic relative-trace formula.** The previous subsection shows that, for suitable f_1^p, f_2^p, we have a decomposition

$$\mathscr{J}_{K_p}(f_1^p, f_2^p) = \lim_{N \to \infty} \sum_{v \nmid \infty \text{ nonsplit}} \mathscr{J}^{(v),N}_{K_p}(f_1^p, f_2^p).$$

We state a geometric expansion of $\mathscr{J}^{(v),N}_{K_p}$ (in fact, $\mathscr{J}^{(v)}$) for inert places v. When F/F_0 is unramified, we then deduce a geometric expansion of $\mathscr{J}_{K_p}$, thus completing the corresponding RTF.

11.5.1. *Local arithmetic intersection numbers and geometric expansions at inert places.* Let $v \nmid 2p$ be an inert finite place of F_0 and let w be the unique place of F above v. We define for $\delta \in \mathrm{G}^{V(v)}_{\mathrm{rs}}(F_{0,v})$,

$$\mathscr{J}_{\delta,v}(e_{K_v}) := -(\delta \cdot \mathcal{N}_{n,v}, \mathcal{N}_{n,v})\, \lambda(\varpi_w), \tag{11.5.1}$$

where, in the right hand side, $(-,-)$ denotes the arithmetic intersection number on the unitary Rapoport–Zink space $\mathcal{N}_{n,v} \times_{\mathrm{Spf}\, \mathscr{O}_{\breve{F}_v}} \mathcal{N}_{n+1,v}$ (resp. the small resolution in [ZZh]) if K_v is hyperspecial (resp. K_v is vertex parahoric), relative to the quadratic field extension $F_w/F_{0,v}$. Since $\mathscr{J}_{\delta,v}(e_{K_v})$ only plays an intermediate role, we refer to [MZ] (resp. [ZZh]) for the unexplained notation in the hyperspecial (resp. parahoric) case.

Recall the matching of global orbits $\underline{\delta}$ of (3.5.4), and the characteristic function $\mathbf{1}_{V'}$ of those orbits matching one from a given $V' \in \mathscr{V}^\circ$ from § 7.3.1.

Proposition 11.5.1. *Let $v \nmid 2p$ be an inert finite place of F_0. Let f_1, $f_2 \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^\circ_{\mathrm{temp}}$ and let $f = f_1 \star f_2^\vee$. Suppose that:*

(1) (f_1, f_2) is K-regular at a place different from v;

(2) $f_{1,v} = f_{2,v} = e_{K_v}$ where K_v is a vertex parahoric subgroup of type (t,t) (cf. §9.3);

(3) K_v is hyperspecial or $T(f_1)$, $T(f_2)$ annihilate $H^{2n}(\mathscr{X}'_u, L(n))$.

Then

$$\begin{aligned}\mathscr{J}^{(v)}(f_1,f_2) &= \sum_{\delta\in \mathrm{B}_{\mathrm{rs}}^{V(v)}(F_0)} J_\delta^v(f^v)\mathscr{J}_{\delta,v}(f_v)\\ &= \sum_{\gamma\in \mathrm{B}'_{\mathrm{rs}}(F_0)} \mathbf{1}_{V(v)}(\gamma)J^v_{\underline{\delta}(\gamma)}(f^v)\mathscr{J}_{\underline{\delta}(\gamma),v}(f_v).\end{aligned}$$

Proof. It suffices to show the first equality. Similar to the proof of Proposition 11.4.4, by the base change property of Lemma 10.2.4(2) we have

$$\mathscr{J}^{(v)}(f_1,f_2) = \frac{1}{\deg(X'_u/X_w)} h_u(Z'.T(f_1), Z'.T(f_2)),$$

where h_u denotes the local height on X'_u over $\mathscr{O}_{E,u}$. Under our assumption, by Proposition 10.3.3 we have

$$h_u(Z'.T(f_1), Z'.T(f_2)) = (\mathscr{Z}'.T(f_1), \mathscr{Z}'.T(f_2))\, \lambda(\mathrm{Nm}_{E_u/F_w}\varpi_u).$$

Since $\lambda_{|F_w^\times}$ is necessarily unramified and E_u/F_w is an unramified extension, we have

$$\lambda(\mathrm{Nm}_{E_u/F_w}\varpi_u) = \deg(E_u/F_w)\lambda(\varpi_w).$$

In the hyperspecial case, by [RSZ20, Theorem 8.15] (the statement there is for the sum over all places of E above w, but the proof contains the formula for each place u), we obtain

$$(\mathscr{Z}'_u.T(f_1), \mathscr{Z}'_u.T(f_1)) = \deg(X'_u/X_w)\sum_{\delta\in \mathrm{B}_{\mathrm{rs}}^{V(v)}(F_0)} J_\delta^v(f^v)\mathscr{J}_{\delta,v}(f_v).$$

The vertex parahoric case is similar and we omit the details. □

Remark 11.5.2. We could relax condition (2) in the proposition to allow vertex parahoric subgroup K_v of type $(t,t+\epsilon)$ with $\epsilon\in\{0,1\}$. But this implicitly violates the convention in §2.1.3, so that we would need to renormalize the matching of orbits that appears in the statement of the proposition.

11.5.2. *The arithmetic relative-trace formula.* We are ready to deduce the following relative-trace formula for $\mathscr{J}_{K_p}$.

Theorem 11.5.3 (Arithmetic relative-trace formula)**.** *Suppose that:*

- *F/F_0 is unramified,*
- *$p>2n$ if $n>1$,*
- *all places $v|2p$ of F_0 are split in F.*

Suppose also that there is a finite set S of places of F_0, not above p or ∞, and a compact open subgroup $K^p=\prod_{v\nmid p}K_v$ satisfying:

- *K_v is (self-dual) hyperspecial for $v\notin S$,*
- *for every split place $v\in S$, $K_v = K_{n,v}\times K_{n+1,v}$ where either at least one of the factors is maximal hyperspecial, or both are Iwahori,*
- *for every inert $v\in S$, K_v is a vertex parahoric subgroup of type (t,t) (cf. §9.3),*

For $i=1,2$, let $f_i^p = f_i^p = f_i^{Sp}\otimes\otimes_{v|S} f_{i,v}\in\mathscr{H}(\mathrm{G}(\mathbf{A}^p),L)^\circ_{K^p\text{-t-ord}}$ satisfy the following properties:

- *for every inert v, $f_{1,v}=f_{2,v}=e_{K_v}$,*

– *for two (necessarily split) places* $v \in S$*, the pair* $(f_{1,v}, f_{2,v})$ *is* K_v*-regular (in the sense of Definition 11.4.2);*

– *for every finite place* $v \in S$*,* $T(f_i^{Sp})$ *annihilates* $H^{2n}(\mathscr{X}'_u, L(n))$ *for some place* u *of* E *that is unramified over* v*.*

Let $f^p := f_1^p \star f_2^{p,\vee} \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K^p}$*. Then we have a spectral and a geometric expansion*

$$\begin{aligned}\mathscr{J}_{K_p}(f^p) &= \sum_{\pi \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}} \mathscr{J}_{\pi, K_p}(f^p)\\ &= \int_{\mathrm{B}'_{\mathrm{rs}}(F_0)^\circ} \sum_{\substack{v \nmid p\infty \\ \text{nonsplit}}} \mathbf{1}_{V(v)}(\gamma) J^{vp}_{\underline{\delta}(\gamma)}(f^{vp}) \mathscr{J}_{\underline{\delta}(\gamma), v}(f_v)\, dI^{\mathrm{ord}}_{\gamma, p, K'_p},\end{aligned}$$

where $dI^{\mathrm{ord}}_{\gamma,p,K'_p} = dI^{\mathrm{ord}}_{\gamma,p,K'_p}(\mathbf{1}_p)$ *is as in* (7.1.3) *for* $K'_p = \mathrm{G}'(\mathscr{O}_{F_{0,p}})$*.*

Proof. The spectral expansion was noted in § 11.3.3. We establish the geometric expansion. By (11.4.1), we have

$$\mathscr{J}(f^p) = \lim_{N \to \infty} \sum_{v \nmid \infty} \mathscr{J}^{(v),N}_{K_p}(f_1^p, f_2^p).$$

By Propositions 11.4.4, 11.4.5, only the terms corresponding to nonsplit places $v \nmid p$ contribute. (We use the 'second' place of regular support to apply Proposition 11.4.4 to the 'first' one.) By (11.4.2) and Proposition 11.5.1, we then have

$$\mathscr{J}(f^p) = \lim_{N\to\infty} \sum_{\substack{v \nmid p\infty \\ \text{nonsplit}}} \sum_{\gamma \in \mathrm{B}'_{\mathrm{rs}}(F_0)} \mathbf{1}_{V(v)}(\gamma) J^{vp}_{\underline{\delta}(\gamma)}(f^{vp}) \mathscr{J}_{\underline{\delta}(\gamma), v}(f_v) \cdot J_{\underline{\delta}(\gamma)}(f_{p, K_p, N} \star f^\vee_{p, K_p, N}).$$

The asserted form of the geometric expansion then follows, via Lemma 3.5.6 and Lemma 5.3.5, from the definition of $dI^{\mathrm{ord}}_{\gamma,p,K'_p}$. □

Epilogue

12. Comparison of RTFs and proof of the main theorem

In this concluding section, we compare the arithmetic distribution $\mathscr{J}_{K_p}$ with the derivative $\partial \mathscr{I}_{K'_p}$ of the analytic distribution, and deduce our main theorem. Throughout this section we assume:

– F/F_0 is unramified,

– $p > 2n$ if $n > 1$,

– all places $v | 2p$ of F_0 are split in F.

12.1. **Comparison of relative-trace formulas.** The comparison will be based on the following local result.

Theorem 12.1.1 ([Zha21, MZ, ZZh]). *Let v be an inert place of F_0 and assume that either of the following conditions on $K_v \subset G_v$, $K'_v \subset G'_v$ hold:*

(1) K_v is hyperspecial, and $K'_v = \mathrm{G}(\mathscr{O}_{F_{0,v}})$;

(2) $K_v = K_{n,v} \times K_{n+1,v}$ is a vertex parahoric subgroup of type (t,t) (cf. §9.3), and $K'_v = K'_{n,v} \times K'_{n+1,v}$ where $K'_{\nu,v}$ is the stabilizer in $\mathrm{G}'_\nu(F_{0,v})$ of both the vertex lattice defining $K_{\nu,v}$ and its dual lattice.

Suppose that $\gamma \in B'_{\mathrm{rs},v}$ matches an orbit $\delta = \underline{\delta}(\gamma) \in B_{\mathrm{rs},v,V_v}$ for the hermitian pair V_v with $\epsilon(V_v) = -1$ (cf. (1.3.1)). Then

$$\mathscr{J}_{\delta,v}(e_{K_v}) = \partial \mathscr{I}_{\gamma,v}(e_{K'_v}).$$

Proof. By the definitions, the identity is equivalent to

$$-(\delta \cdot \mathcal{N}_{n,v}, \mathcal{N}_{n,v}) = \partial \mathscr{I}_\gamma(e_{K'_v})/\lambda(\varpi_w) = \frac{d}{ds}\bigg|_{s=0} I^{\mathbf{C}}_{\gamma,v}(e_{K'_v}, |\cdot|^s_{F_v})/(-\log q_{0,v}^2) \tag{12.1.1}$$

(where w is the place of F above v, and the 'division' in the second term has the obvious meaning).

In the hyperspecial case, the identity (12.1.1) is the Arithmetic Fundamental Lemma conjecture proved in [Zha21, MZ]. In the vertex parahoric case, (12.1.1) (an instance of Arithmetic Transfer conjecture) is recently proved by Z. Zhang [ZZh].

There are two points where the formulation in those works appears different. First, they consider a version with derivatives of 'inhomogenous' orbital integrals; this is verified to be equivalent to the above homogeonous version as in [RSZ18, Proposition 14.1 (ii)]. Second, their identity apparently differs from ours by a sign -1: the reason is that their orbital integral contains a transfer factor defined as in § 2.4 *ibid.*; under our assumptions on γ and v, that transfer factor (in its inhomogeneous version), evaluated at a preimage $\gamma' \in G'_{\mathrm{rs},v}$ of γ, differs from our $\kappa_v(\gamma')$ by -1. $\square$

We can now make the global comparison.

Theorem 12.1.2 (Comparison of RTFs). *Let S, $K^p = \prod_{v\nmid p} K_v$, and*

$$f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p}$$

be as in Theorem 11.5.3. Write $S = S^{\mathrm{spl}} \sqcup S^{\mathrm{in}}$ as a union of sets of split and inert places.

Let $K'_p \coloneqq \mathrm{G}'(\mathscr{O}_{F_{0,p}})$ and let $K'^p = \prod_{v\nmid p} K'_v \subset \mathrm{G}'(\mathbf{A}^{p\infty})$ be a compact open subgroup satisfying:

- *for every $v \notin S$, $K'_v = \mathrm{G}'_\nu(\mathscr{O}_{F_{0,v}})$ is hyperspecial;*
- *for every inert $v \in S$, $K'_v = K'_{n,v} \times K'_{n+1,v}$ and $K'_{\nu,v}$ is the stabilizer in $\mathrm{G}'_\nu(F_{0,v})$ of both the vertex lattice defining $K_{\nu,v}$ and its dual lattice.*

Let

$$f'^p = f'^{Sp} \otimes f'_{S^{\mathrm{spl}}\infty} \otimes f'_{S^{\mathrm{in}}} \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ_{K'^p\text{-rs, qc}}$$

be a quasicuspidal Gaussian with weakly regular semisimple support whose factors satisfy the following properties:

- *$f'^{S^{\mathrm{spl}}p} = \otimes_v f'_v$ with $f'_v = e_{K'_v}$;*
- *$f'_{S^{\mathrm{spl}}\infty}$ matches $f_{S^{\mathrm{spl}}\infty}$;*

Then $\mathscr{I}_{K'_p}(f'^p, \mathbf{1}) = 0$ *and*

$$\mathscr{J}_{K_p}(f_p) = \partial\mathscr{I}_{K'_p}(f'_p).$$

Proof. We first show that f^p and f'^p match under the assumption. By our conditions and the Jacquet–Rallis Fundamental lemma (Proposition 3.5.4), $f^{S^{\rm in}p}$ and $f'^{S^{\rm in}p}$ match. The theorem of [ZZh] on transfer at vertex parahoric levels shows that f_v and f'_v match at the places in $S_{\rm in}$ as well.

It follows that the function f'^p is incoherent, hence $\mathscr{I}_{K'_p}(f'^p, \mathbf{1}) = 0$ by Proposition 7.3.1.

Next we compare the geometric expansions of both sides of the desired equality, given by Theorem 11.5.3 and Proposition 7.3.1 (3) respectively. By the identity of Theorem 12.1.1, these are equal term by term. The proof is complete. □

12.2. **Test Hecke measures.** We find some $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ$, $f'^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ$ to which the comparison may be applied, and that isolate a given pair of representations over L.

We will from now admit the following local hypothesis:

Hypothesis 12.2.1. *Let v be an inert place of F_0 and let $\pi_v = \pi_{n,v} \boxtimes \pi_{n+1,v}$ be a representation of G_v such that $\pi_{n,v}$ is either unramified or almost unramified, and $\pi_{n+1,v}$ is almost unramified. Let $f_v = e_{K_v}$ where $K_v \subset G_v$ is a vertex parahoric subgroup of type (t,t) for $t = n$ if π_n is unramified and $t = 1$ if π_n is almost unramified. Then*

$$J_{\pi_v}(f_v) \neq 0.$$

The special case of type $(t,t) = (n,n)$ is proved in [Dan] (note that [Dan] considered the equivalent problem for the parahoric subgroup of type $(0,1)$).

Lemma 12.2.2. *Let $\pi \in \mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\rm ord, st}_{K_p}(L)$ and let $\Pi = \mathrm{BC}(\pi)$. Assume that:*

- *for every place v of F_0 that is split in F/F_0, at least one of $\pi_{n,v}$ and $\pi_{n+1,v}$ is unramified;*
- *for every place v of F_0 that is inert in F/F_0, $\pi_{n,v}$ and $\pi_{n+1,v}$ are either unramified or almost unramified, and if $\pi_{n,v}$ is almost unramified then $\pi_{n+1,v}$ is also almost unramified.*

Then there exist:

- *a finite set S of places of F_0, not above p or ∞,*
- *open compact subgroups $K^p = \prod_{v\nmid p} K_v \subset \mathrm{G}(\mathbf{A}^{p\infty})$ and $K'^p = \prod_{v\nmid p} K'_v \subset \mathrm{G}'(\mathbf{A}^{p\infty})$,*
- *Hecke measures $f_1^p, f_2^p, f^p := f_1^p \star f_2^{p,\vee} \in \mathscr{H}(\mathrm{G}(\mathbf{A}^p), L)^\circ_{K_p}$ and $f'^p \in \mathscr{H}(\mathrm{G}'(\mathbf{A}^p), L)^\circ_{K'_p\text{-rs, qc}}$,*

such that:

- *$(S, K^p, f_1^p, f_2^p, K'^p, f'^p)$ satisfy the conditions of Theorem 11.5.3 and of Theorem 12.1.2;*
- *$M^{\oplus,*}.T(f_i^p e_{K_p}) \subset \bigoplus_{\pi'\in \mathrm{BC}^{-1}(\Pi)=\Pi} M_{\pi'}^{K_p}$;*
- *$\Pi'(f'^p e_{K_p}) = 0$ for every $\Pi \neq \Pi' \in \mathscr{C}$;*
- *$\otimes_{v\nmid p} J_{\pi_v}(f^p) = \otimes_{v\nmid p} I_{\Pi,v}(f'^p) \neq 0$.*

Proof. We construct f_1, f_2^p, f'^p as products whose various factors take care of the required conditions.

Regularity of the supports. Let $S^{\rm rs} = \{v_+, v_-\}$ be a set consisting of two split places of F_0 at which Π is an unramified regular principal series (cf. Lemma 4.3.3). We use '$\pm$' instead of '$v_\pm$' as a subscript for the sake of legibility in this paragraph. Let

$$f'_\pm \in \mathscr{H}(G'_\pm, L)$$

be an element with $\pm$-regular support such that $I_{\Pi_\pm}(f'_\pm, \mathbf{1}) \neq 0$ as provided by Lemma 4.3.1 (3); we take any sufficiently small $K'_\pm$ such that $f'_\pm \in \mathscr{H}(G'_\pm, L)_{K'_\pm}$. For each $v \in S^{\rm rs}$, upon a choice of a basis of V_v, we have the matching $f_{v,1} \in \mathscr{H}(G_v, L)$; by that Lemma, we may arrange that $f_{v,1}$ is bi-invariant under an Iwahori subgroup $K_v \subset G_v$. We put $f_{v,2} \coloneqq e_{K_v}$. Thus $f_v = f_{v,1} \star f_{v,2}^\vee = f_{v,1}$ still matches f'_v and has K_v-regular support. For $i = 1, 2$, we put

$$f_{S^{\rm rs},i} = \otimes_{v \in S^{\rm rs}} f_{v,i}, \qquad f'_{S^{\rm rs}} = \otimes_{v \in S^{\rm rs}} f'_v.$$

Any global Hecke measure with component $f'_{S^{\rm rs}}$ has weakly regular semisimple support (Definition 3.3.5) since $\mathrm{G}'_{\rm rs} = \mathrm{G}'_{\rm reg^+} \cap \mathrm{G}'_{\rm reg^-}$.

Choices at places of ramification. Let S^R be the finite set of places $v \notin S^{\rm rs}p\infty$ of F_0 where at least one of $\pi_{n,v}, \pi_{n+1,v}$ is ramified. Then for every *split* $v \in S^R$, we let $K_v = K_{n,v} \times K_{n+1,v}$ such that $\pi^{K_v} \neq 0$ and $K_{\nu,v}$ is hyperspecial if $\pi_{\nu,v}$ is unramified. Then we pick any $f_{v,1}, f_{v,2}, f_v \coloneqq f_{v,1} \star f_{v,2}^\vee \in \mathscr{H}(G_v, L)_{K_v}$ such that $J_{\pi_v}(f_v) \neq 0$. For every *inert* $v \in S^R$, we let K_v be the vertex parahoric subgroup such that $\pi_v^{K_v} \neq 0$ and we let $f_v = e_{K_v}$. More precisely, there are two cases:

- if $\pi_{n,v}$ is unramified and $\pi_{n+1,v}$ is almost unramified, we let K_v be a vertex parahoric subgroup of type (n, n);
- if both $\pi_{n,v}$ and $\pi_{n+1,v}$ are almost unramified, we let K_v be a vertex parahoric subgroup of type $(1, 1)$.

We put $f_{S^R,i} = \otimes_{v \in S^R} f_{v,i}$ and we let $f'_{S^R} \in \mathscr{H}(G'_{S^R})$ match $f_{S^R} \coloneqq f_{S^R,1} \star f_{S^R,2}^\vee$.

Isolation of π and Π. Now take $S = S^R \cup S^{\rm rs}$. For $v \notin Sp$, we let K_v, K'_v be hyperspecial, and form $K = \prod_v K_v$, $K' = \prod_v K'_v$. Consider the split Hecke algebras

$$\begin{aligned} \mathbb{T} = \mathbb{T}^{\rm spl} &\coloneqq \bigotimes_{\substack{v \nmid Sp \\ \text{split}}} \mathscr{H}(G_v, L)_{K_v} && \subset \mathscr{H}(\mathrm{G}(\mathbf{A}^{Sp}, L)^\circ_{K^S} \\ \mathbb{T}' = \mathbb{T}'^{\rm spl} &\coloneqq \bigotimes_{\substack{v \nmid Sp \\ \text{split}}} \mathscr{H}(G'_v, L)_{K'_v} \otimes_L \mathscr{H}(G'_\infty, L)^\circ && \subset \mathscr{H}(\mathrm{G}'(\mathbf{A}^{Sp}, L)^\circ_{K'^{Sp}}. \end{aligned}$$

Let $f_{\pi,1} = f_{\pi,2} \in \mathbb{T}$ be an element acting as the idempotent projection from $M_K^{\oplus,*}$ onto $\bigoplus_{\pi' \in \mathrm{BC}^{-1}(\Pi)} M_{\pi'}^K$, which exists by Lemma 4.6.2 (for Σ the finite set of representations occurring in $M_K^{\oplus,*}$). Let $f'_\pi \in \mathbb{T}'$ be an element supported at the finite places and matching $f_\pi \coloneqq f_{\pi,1} \star f_{\pi,2}^\vee$.

Let $f'_\Pi \in \mathbb{T}'$ be an element such that $\Pi(f'_\Pi) = \mathrm{id}$ and that for each $\iota\colon L \hookrightarrow \mathbf{C}$, $R(f'^\iota_\Pi)$ sends $\mathscr{A}(\mathrm{G}')^K$ into $\Pi^{\iota,K}$, which exists by Proposition 4.3.2; let $f_{\Pi,1} \in \mathbb{T}$ be a matching element and let $f_{\Pi,2}$ be the unit of $\mathbb{T}$.

Annihilation of absolute cohomology. For every place $v \in S$, by the vanishing theorem of Proposition 9.4.2 (1) (applied to the maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ corresponding to the eigensystem attached to π), there exists $f_{\{v\},1} = f_{\{v\},2} \in \mathbb{T}$ which annihilates $H^{2n}(\mathscr{X}'_u, L(n))$ (for $\mathscr{X}'_u$ as in § 11.4.2), and acts by a non-zero scalar on the line $\bigotimes_{\substack{v\nmid Sp\\ \text{split}}} \pi_v^{K_v}$. Let $f'_{\{v\}} \in \mathbb{T}'$ be an element supported at the finite places and matching $f_{\{v\}} \coloneqq f_{\{v\},1} \star f_{\{v\},2}^{\vee}$.

Assembly. For $i = 1, 2, \emptyset$, we define

$$f_i^{Sp} = f_{\pi,i} f_{\Pi,i} \otimes \otimes_{v\in S} f_{\{v\},i} \in \mathbb{T}, \qquad f'^{Sp} = f'_\pi f'_\Pi \otimes \otimes_{v\in S} f_{\{v\}} \in \mathbb{T}',$$

viewed naturally as elements in $\mathscr{H}(\mathrm{G}(\mathbf{A}^{Sp}), L)^\circ_{K^{Sp}}$, $\mathscr{H}(\mathrm{G}'(\mathbf{A}^{Sp}), L)^\circ_{K'^{Sp}}$. Then we define

$$f_i^p = f_{S,i} f_i^{Sp}, \qquad f'^p = f'_S f'^{Sp}.$$

Then it is easy to see that, by construction, f_i^p satisfies the required conditions. To check the condition on spherical characters, we use

$$\otimes_{v\nmid p} J_{\pi_v}(f^p) = \otimes_{v\notin Sp} J_{\pi_v}(f^{Sp}) \prod_{v\in S} J_{\pi_v}(f_v).$$

The product over $v \in S$ does not vanish by construction; the first factor is the product of $\otimes_{v\notin Sp} J_{\pi_v}(e_{K^{Sp}}) \neq 0$ and of the eigenvalue of f^{Sp} acting on the line $\pi^{K^{Sp}}$, which is a non-zero scalar. $\square$

12.3. **Proof of the main theorem.** We first reduce the identity

$$h_\pi(Z_\pi(\phi), Z_{\pi^\vee}(\phi')) = e_p(\mathrm{M}_\Pi)^{-1} \cdot \frac{1}{4} \partial \mathscr{L}_p(\mathrm{M}_\Pi) \cdot \alpha(\phi, \phi') \tag{12.3.1}$$

of Theorem D to the factorization

$$\mathscr{J}_{\pi,K_p}(f^p) = \frac{1}{4} \partial \mathscr{L}_p(\mathrm{M}_\Pi) \cdot \otimes_{v\nmid p} J_{\pi_v}(f^p). \tag{12.3.2}$$

Lemma 12.3.1. *Let $\pi \in \mathscr{C}(\mathrm{G})^{\mathrm{ord}}_{K_p}$, and let $\Pi = \mathrm{BC}(\pi)$, $L = \mathbf{Q}_p(\pi)$. The following are equivalent:*

(1) For every $\phi \in \pi$, $\phi' \in \pi^\vee$, the identity (12.3.1) *holds.*

(2) For some $\phi \in \pi$, $\phi' \in \pi^\vee$ such that $\alpha(\phi, \phi') \neq 0$, the identity (12.3.1) *holds.*

(3) For every $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^\circ$, the factorization (12.3.2) *holds.*

(4) For some $f^p \in \mathscr{H}(\mathrm{G}(\mathbf{A}), L)^\circ$ such that $\otimes_{v\nmid p} J_{\pi_v}(f^p) \neq 0$, the factorization (12.3.2) *holds.*

Proof. It is trivial that (1) implies (2), and (3) implies (4). The two converse implications follow from multiplicity one and the nonvanishing of α.

We prove that (3) is equivalent to (1). It is clear that (1) is equivalent to

$$\mathrm{Tr}^{h\circ Z_\pi \boxtimes Z_{\pi^\vee}}_{(,)_\pi}(\tau) = e_p(\mathrm{M}_\Pi)^{-1} \cdot \frac{1}{4} \partial \mathscr{L}_p(\mathrm{M}_\Pi) \cdot \mathrm{Tr}^\alpha_{(,)_\pi}(\tau) \tag{12.3.3}$$

for all $\tau \in \mathrm{End}(\pi)$, and equivalently for some τ such that $\mathrm{Tr}^\alpha_{(,)_\pi}(\tau) \neq 0$. Thus it is enough to show that (12.3.2) is equivalent to (12.3.3) for some such τ.

Choose a factorization $(,)_\pi = (,)_{\pi^p}(,)_{\pi_p}$. For any $N \geq 1$, let $f_{p,K_p,N} \coloneqq (11.3.5) \in \mathscr{H}(G_p, L)$, let $f^\star_{p,K_p,N} \coloneqq f_{p,K_p,N} \star f^\vee_{p,K_p,N}$, and for $? \in \{\emptyset, \vee\}$, let

$$\pi_p^?(f_{p,K_p}) \coloneqq \lim_{N\to\infty} \pi_p^?(f_{p,K_p,N}) \in \mathrm{End}(\pi_p).$$

(This does not depend on the integer $1 \leq r \leq N!$ implicit in (11.3.5).) Let

$$\pi_p(f^\star_{p,K_p}) \coloneqq \pi_p(f_{p,K_p}) \circ (\pi_p^\vee(f_{0,p,K_p}))^\vee,$$

where $(-)^\vee$ denotes the transpose with respect to $(,)_{\pi_p}$. Then by the definition in § 11.3.3, we have

$$\mathrm{Tr}^{h\circ Z_\pi\boxtimes Z_{\pi^\vee}}_{(,)_\pi}(\pi^p(f^p)\pi_p(f^\star_{p,K_p}))=\mathscr{J}_{\pi,K_p}(f^p) \tag{12.3.4}$$

On the other hand, it is clear from the definitions that

$$\mathrm{Tr}^{\alpha}_{(,)_\pi}(\pi^p(f^p)\pi_p(f^\star_{p,K_p}))=\otimes_{v\nmid p}J_{\pi_v}(f^p)\cdot\lim_{N\to\infty}J_{\pi_p}(f^\star_{p,K_p,N}) \tag{12.3.5}$$

Now by Lemma 3.5.6, $f^\star_{p,K_p,N}$ matches the function $f'_{p,K'_p,N}$ attached to $U^{N!}_{t_p}$ as in Lemma 5.3.5. By the definitions and Corollary 5.3.4, we then have

$$\lim_{N\to\infty}J_{\pi_p}(f^\star_{p,K_p,N})=\lim_{N\to\infty}I_{\Pi_p}(f'_{p,K'_p,N})=e_p(\mathrm{M}_\Pi). \tag{12.3.6}$$

(Recall that $e_p(\mathrm{M}_\Pi)$ is the product of the factors $e(\Pi_v,\mathbf{1}_v)$ of (5.3.5).) Thus by (12.3.4), (12.3.5), (12.3.6), the identity (12.3.2) for f^p is equivalent to (12.3.3) for $\tau=\pi^p(f^p)\pi_p(f^\star_{p,K_p})$. This completes the proof. □

We may now prove Theorem D based on the comparison of relative-trace formulas in Theorem 12.1.2.

Proof of Theorem D. By Lemma 12.3.1, it suffices to prove

$$\mathscr{J}_{\pi,K_p}(f^p)=\frac{1}{4}\partial\mathscr{L}_p(\mathrm{M}_\Pi)\cdot\otimes_{v\nmid p}J_{\pi_v}(f^p) \tag{12.3.7}$$

for any f^p such that $\otimes_{v\nmid p}J_{\pi_v}(f^p)\neq 0$.

Let S, K^p, f^p, f'^p be as in Lemma 12.2.2. By construction, $\otimes_{v\nmid p}J_{\pi_v}(f^p)\neq 0$, the elements f^p and f'^p match (geometrically), and Theorem 12.1.2 is applicable and it gives

$$\mathscr{J}_{K_p}(f_p)=\partial\mathscr{I}_{K'_p}(f'_p).$$

By Theorem 11.5.3 and Proposition 7.3.1 (2), we have an equality of spectral expansions

$$\sum_{\pi\in\mathscr{C}(\mathrm{H}\backslash\mathrm{G})^{\mathrm{ord}}_{K_p}}\mathscr{J}_{\pi,K_p}(f^p)=\sum_{\Pi\in\mathscr{C}(\mathrm{G}')^{\mathrm{her,ord},V}_{K_p}}\partial\mathscr{I}_{\Pi,K_p}(f'^p),$$

but by construction only the terms corresponding to π and Π may be nonzero. We deduce that

$$\mathscr{J}_{\pi,K_p}(f^p)=\frac{1}{4}\partial\mathscr{L}_p(\mathrm{M}_\Pi)\cdot\otimes_{v\nmid p}I_{\Pi_v}(f'^p),$$

which is equivalent to the desired factorization (12.3.7) by the (spectral) matching of f^p and f'^p. □

Proof of Theorem C. The main implication follows immediately from Theorem D, upon choosing the unique distinguished π such that $\Pi=\mathrm{BC}(\pi)$. The strengthened implication then follows from [LTX$^+$22] (or [LaSk] under a different condition), as observed in Remark 1.3.2. □

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