THETA CYCLES AND THE BEILINSON–BLOCH–KATO CONJECTURES

by

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Abstract. — We introduce 'canonical' classes in the Selmer groups of certain Galois representations with a conjugate-symplectic symmetry. They are images of special cycles in unitary Shimura varieties, and defined uniquely up to a scalar. The construction is a slight refinement of one of Y. Liu, based on the conjectural modularity of Kudla's theta series of special cycles. For 2-dimensional representations, Theta cycles are (the Selmer images of) Heegner points. In general, they conjecturally exhibit an analogous strong relation with the Beilinson–Bloch–Kato conjectures in rank 1, for which we gather the available evidence.

Contents

1. Introduction

The purpose of this largely expository note is to introduce certain Selmer classes of algebraic cycles, and discuss their relation to the Beilinson–Bloch–Kato (BBK) conjectures. These classes, called *Theta cycles*, should play an analogous role to Heegner points on elliptic curves, in that the Bloch–Kato Selmer group $H^1_f(E,\rho)$ of a relevant Galois representation ρ should be 1-dimensional precisely when its Theta cycle is nonzero (cf. [BST, Kim] and references therein for the case elliptic curves). Moreover, the BBK conjectures, reviewed in § 2, predict that the 1-dimensionality of the Selmer group is equivalent to the (complex or, for suitable primes, p-adic) L-function of ρ vanishing toọrder 1 at the center, and Theta cycles allow to approach this conjecture.

The following theorem summarises the state of our knowledge on the topic. Unexplained notions or loose formulations will be defined and made precise in the main body of the paper.

We fix a rational prime p and denote by $\mathbf{Q}^\circ \subset \overline{\mathbf{Q}}_p$ the extension of $\mathbf{Q}$ generated by all roots of unity, and we fix an embedding $\iota^\circ\colon \mathbf{Q}^\circ \hookrightarrow \mathbf{C}$. We set $\Sigma := \{\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C} \mid \iota_{|\mathbf{Q}^\circ} = \iota^\circ\}$.

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Theorem A. — *Let E be a CM field with Galois group G_E, and let*

$$\rho\colon G_E \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

be an irreducible, geometric Galois representation of weight -1 and even dimension n. Suppose that ρ is conjugate-symplectic, automorphic, and has minimal regular Hodge–Tate weights.

If $n \geqslant 4$, assume that the maximal totally real subfield F of E is not $\mathbf{Q}$, and that Hypothesis 4.1 on the cohomology of unitary Shimura varieties holds.

1. *Assume Hypothesis 4.3 on the modularity of generating series of special cycles. The construction of § 4.3 attaches to ρ a pair $(\Lambda_\rho, \Theta_\rho)$, well-defined up to isomorphism, consisting of a $\overline{\mathbf{Q}}_p$-line Λ_ρ together with a $\overline{\mathbf{Q}}_p$-linear map*

$$\Theta_\rho\colon \Lambda_\rho \to H^1_f(E,\rho),$$

whose image is spanned by classes of algebraic cycles.

2. *Suppose that E and ρ are 'mildly ramified' and that ρ is crystalline at p-adic places.*
   (a) *Assume Hypothesis 4.3, as well as Conjecture 5.3 on the injectivity of certain Abel–Jacobi maps, and that p is unramified in E. For any $\iota \in \Sigma$, denote by $L_\iota(\rho,s)$ the complex L-function of ρ with respect to ι. Then*[1]

$$\mathrm{ord}_{s=0} L_\iota(\rho,s) = 1 \implies \Theta_\rho \neq 0.$$

   (b) *Suppose that E/F is totally split above p, that $p > n$, and that for every place $w|p$ of E, the representation ρ_w is Panchishkin–ordinary. Denote by $\mathscr{X}_F$ the $\overline{\mathbf{Q}}_p$-scheme of continuous p-adic characters of G_F that are unramified outside p, by $\mathfrak{m} \subset \mathscr{O}(\mathscr{X}_F)$ the ideal of functions vanishing at $\mathbf{1}$, and by $L_p(\rho) \in \mathscr{O}(\mathscr{X}_F)$ the p-adic L-function of ρ. Then*

$$\mathrm{ord}_{\mathfrak{m}} L_p(\rho) = 1 \implies \textit{Hypothesis 4.3 holds and } \Theta_\rho \neq 0.$$

3. *Assume Hypothesis 4.3 and that ρ has 'sufficiently large' image. Then*

$$\Theta_\rho \neq 0 \implies \dim_{\overline{\mathbf{Q}}_p} H^1_f(E,\rho) = 1.$$

Examples of representations ρ satisfying the general assumptions of the theorem arise from symmetric powers of elliptic curves: namely, if A is a modular elliptic curve over F with rational Tate module V_pA, then by [NT] one may consider the natural representation $\rho_{A,n}$ of G_E on $\mathrm{Sym}^{n-1} V_pA_E(1-n/2)$ (see [DL, § 1.4] for more details); in particular, for $n = 2$ we obtain the representation V_pA_E already studied (when $F = \mathbf{Q}$) by Gross–Zagier, Perrin-Riou and Kolyvagin in the 1980s.

Part 1 of the theorem, which builds on constructions of Kudla and Y. Liu, is the main focus of this note; it is explained in § 4, after reviewing the representation-theoretic preliminaries in § 3. The construction is canonical up to a representation-theoretic choice described in Remark 3.5. (However, there is a 'standard' choice, and part 3 of the theorem indicates that this ambiguity is quite innocuous.)

In § 5, we state a pair of formulas for the Bloch–Beĭlinson and the Nekovář heights of Theta cycles, which are essentially reformulations of a breakthrough result of Li and Liu [LL21, LL22], and of its

[1] The order of vanishing of $L_\iota(\rho,s)$ at $s=0$ is conjecturally independent of ι, cf. Conjecture 2.2.

p-adic analogue by Liu and the author [DL]. They imply the assertions of Part 2, and take the shape

$$\langle \Theta_\rho(\lambda), \Theta_{\rho^*(1)}(\lambda')\rangle_\star = c_\star \cdot L'_\star(\rho, 0)\cdot \zeta_\star(\lambda,\lambda'),$$

where '$\star$' stands for the relevant decorations, $c_\star$ are constants, and $\zeta_\star$ are canonical trivialisations of $\Lambda_\rho \otimes \Lambda_{\rho^*(1)}$.

Part 3 is the subject of the forthcoming [Dis/b] (itself relying on forthcoming work of Jetchev–Nekovář–Skinner), on which we only give some brief remarks in § 5.4; in particular, we sketch the relevance of the perspective proposed here for the results obtained there.

All the constructions and results should have analogues in the odd-dimensional case, in the symplectic case, and for more general Hodge–Tate types. We hope to return to some of these topics in future work.

Acknowledgements. — It will be clear to the reader that this note is little more than an attempt to look from the Galois side, and the multiplicity-one side, at ideas of Kudla and Liu. I would like to thank Yifeng Liu for all I learned from him during and after our collaboration, and Elad Zelingher for a remark that sparked it. I am also grateful to Yannan Qiu and Eitan Sayag for helpful conversations or correspondence, and to Chao Li and Yifeng Liu for many useful comments on a first draft.

This text is based on a talk given at the Second JNT Biennial Conference in Cetraro, Italy, and I would like to thank the organisers for the opportunity to speak there. One of the participants reminded me of Tate's similarly named 'θ-cycles' in the theory of mod-p modular forms [Joc82, § 7]: besides the context, the capitalisation should also dispel any risk of confusion. Homonymous objects also occur in neuroscience, in connection with a pattern of brain activity typical of "a drowsy state transitional from wake to sleep" [McN19, pp. 60-61]; I am grateful to the Cetraro audience for not indulging in this confusion either.

2. The conjecture of Beĭlinson–Bloch–Kato–Perrin-Riou

Let E be a number field with Galois group G_E, and let

$$\rho\colon G_E \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

be an irreducible Galois representation that is geometric in the sense of [FM95], and pure of weight -1 at all finite places (in the sense of [DL, Definition A.11] – where at non-p-adic places, we take the functor of [Tat79, (4.2.1)] in place of the functor $\mathrm{WD}(\cdot)$ of *loc. cit.*).

2.1. Chow and Selmer groups. — A typical source of representations as above is the cohomology of algebraic varieties. In fact, define a *motivation* of ρ to be an element of[(2)]

$$\mathrm{Mot}_\rho \coloneqq \varinjlim_{(X,k)} \mathrm{Mot}_\rho(X,k), \qquad \text{where} \qquad \mathrm{Mot}_\rho(X,k) \coloneqq \mathrm{Hom}_{\overline{\mathbf{Q}}_p[G_E]}(H^{2k-1}_{\text{ét}}(X_{\overline{E}}, \overline{\mathbf{Q}}_p(k)), \rho),$$

and the limit runs over all pairs consisting of a smooth proper variety $X_{/E}$ and an integer $k \geqslant 1$ (this is a directed system by Künneth's fromula). We refer to (X,k) as a source of $f \in \mathrm{Mot}_\rho$ if f is in the

(2)Throughout this paper, if $R \to R'$ is a ring map that can be understood from the context, and X is an R-scheme or an R-module, we write $X_{R'} \coloneqq X \otimes_R R'$.

image of $\mathrm{Mot}_\rho(X,k)$. We say that ρ is *motivic* if Mot_ρ is nonzero. According to the conjecture of Fontaine–Mazur, every geometric irreducible Galois representation is motivic.

To a representation ρ as above is attached its Bloch–Kato [BK90] Selmer group $H^1_f(E,\rho)$.[(3)] To a variety $X_{/E}$ as above is attached its Chow group $\mathrm{Ch}^k(X)$ of codimension-k algebraic cycles on X up to rational equivalence (with coefficients in $\mathbf{Q}$). A central object of arithmetic interest is its subgroup $\mathrm{Ch}^k(X)^0_{\overline{\mathbf{Q}}_p} := \mathrm{Ker}[\mathrm{Ch}^k(X) \to H^{2k}_{\text{ét}}(X_{\overline{E}},\overline{\mathbf{Q}}_p(k))]$ (where the map is the cycle class). It is endowed with an Abel–Jacobi map

$$\mathrm{AJ}\colon \mathrm{Ch}^k(X)^0_{\overline{\mathbf{Q}}_p} \to H^1(E, H^{2k-1}_{\text{ét}}(X_{\overline{E}},\overline{\mathbf{Q}}_p(k)))$$

(see [Nek93, § 5.1]) whose image is conjectured to land in $H^1_f(E,H^{2k-1}_{\text{ét}}(X_{\overline{E}},\overline{\mathbf{Q}}_p(k)))$. We can define an analogue of the image of AJ for the representation ρ by

$$H^1_f(E,\rho)^{\mathrm{mot}} := \sum_{f'\in \mathrm{Mot}_\rho} f'_*\mathrm{AJ}(\mathrm{Ch}^k(X)^0_{\overline{\mathbf{Q}}_p}) \cap H^1_f(E,\rho) \subset H^1_f(E,\rho),$$

where we have denoted by (X,k) any source of the motivation f'. By an evocative abuse of nomenclature, we refer to elements of $H^1_f(E,\rho)^{\mathrm{mot}}$ as cycles.

Remark 2.1. — If $\rho = H^{2k_0-1}_{\text{ét}}(X_{0,\overline{E}},\overline{\mathbf{Q}}_p(k_0))$ for a variety X_0 and an integer k_0, then we expect that $H^1_f(E,\rho)^{\mathrm{mot}} = \mathrm{AJ}(\mathrm{Ch}^{k_0}(X_0)^0_{\overline{\mathbf{Q}}_p})$. This equality is implied by the Tate conjecture [Tat65, Conjecture 1] for $X\times X_0$.

2.2. The conjecture. — We say that ρ is (Panchishkin-) *ordinary* (see [Nek93, § 6.7], [PR92, § 2.3.1] for more details) if for each place $w|p$, there is a (necessarily unique) exact sequence of De Rham G_{E_w}-representations $0\to \rho_w^+ \to \rho_{|G_{E_w}} \to \rho_w^- \to 0$, such that $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(\rho_w^+) = \mathbf{D}_{\mathrm{dR}}(\rho_w^-)/\mathrm{Fil}^0 = 0$. For any subfield $F\subset E$, let

$$\mathscr{X}_F := \mathrm{Spec}\,\mathbf{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]\otimes_{\mathbf{Z}_p}\overline{\mathbf{Q}}_p,$$

where F_∞/F is the abelian extension with $\mathrm{Gal}(F_\infty/F)$ isomorphic (via class field theory) to the maximal $\mathbf{Z}_p$-free quotient of $F^\times\backslash \mathbf{A}_F^\times/\widehat{\mathscr{O}}_F^{p,\times}$.

One can conjecturally attach to ρ entire L-functions

$$L_\iota(\rho,s)$$

for $\iota\colon L\hookrightarrow \mathbf{C}$ and, (at least) if ρ is ordinary, a p-adic L-function

$$L_p(\rho)\in \mathscr{O}(\mathscr{X}_F)$$

interpolating suitable modifications of the L-values $L_\iota(\rho\otimes\chi_{|G_E},0)$ for finite-order characters $\chi\in\mathscr{X}_F$ (see [PR95], at least when taking $F=\mathbf{Q}$).

Denote by $\mathfrak{m}=\mathfrak{m}_F\subset \mathscr{O}(\mathscr{X}_F)$ the maximal ideal of functions vanishing at the character $\mathbf{1}$ of $\mathrm{Gal}(F_\infty/F)$, and by $\mathrm{ord}_{\mathfrak{m}}$ the corresponding valuation. The integer $\mathrm{ord}_{\mathfrak{m}}L_p(\rho)$ is conjecturally independent of the choice of F.

(3) N.B.: the subscript f has nothing to do with names of objects elsewhere in this text. Galois cohomology and Selmer groups are usually defined for representations with coefficients in finite extensions of $\mathbf{Q}_p$. However, it is well-known that we can write $\rho=\rho_0\otimes_L\overline{\mathbf{Q}}_p$ for some finite extension $L\subset\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and some representation $\rho_0\colon G_E\to \mathrm{GL}_n(L)$ (and similarly for the other representations considered in this paper). Then we define $H^1_f(E,\rho):=H^1_f(E,\rho_0)\otimes_L\overline{\mathbf{Q}}_p$.

Conjecture 2.2 **(Beĭlinson, Bloch–Kato, Perrin-Riou** [Beĭ84, BK90, PR95])

Let $\rho\colon G_E \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ *be an irreducible geometric representation of weight* -1. *Let* $r \geqslant 0$ *be an integer. The following conditions are equivalent:*

(a)$_\infty$ *for any* $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$, *we have*

$$\mathrm{ord}_{s=0} L_\iota(\rho, s) = r;$$

(b) $\dim_{\overline{\mathbf{Q}}_p} H^1_f(E, \rho)^{\mathrm{mot}} = \dim_{\overline{\mathbf{Q}}_p} H^1_f(E, \rho) = r.$

If moreover ρ *is ordinary and* $\rho_w^{+,*}(1)^{G_{E_w}} = 0$ *for every* $w|p$, *then the above conditions are equivalent to*

(a)$_p$ $\mathrm{ord}_{\mathrm{m}} L_p(\rho) = r$;

Remark 2.3. — The first equality in (b) generalises the conjectural finiteness of the p^∞-torsion in the Tate–Shafarevich group of an elliptic curve. The extra condition in (a)$_p$ serves to avoid the phenomenon of exceptional zeros, cf. [Ben14].

In the following pages, under some restrictions on ρ we will define elements in $H^1_f(E, \rho)^{\mathrm{mot}}$ whose nonvanishing is conjecturally equivalent to the conditions of Conjecture 2.2 with $r = 1$. The construction will be automorphic; in the next section, we give the representation-theoretic background.

3. Descent and theta correspondence

Suppose for the rest of this paper that E is a CM field with totally real subfield F. We denote by $\mathrm{c} \in \mathrm{Gal}(E/F)$ the complex conjugation, and by $\eta\colon F^\times\backslash\mathbf{A}^\times \to \{\pm 1\}$ the quadratic character attached to E/F.

3.1. p**-adic automorphic representations. —** We denote by $\mathbf{A}$ the adèles of F; if S is a finite set of places of F, we denote by $\mathbf{A}^S$ the adèles of F away from S. If G is a group over F and v is a place of F, we write $G_v := \mathrm{G}(F_v)$; if S a finite set of places of F, we write $G_S := \prod_{v\in S} G(F_S)$. (For notational purposes, we will identify a place of $\mathbf{Q}$ with the set of places of F above it.) We denote by $\psi\colon F\backslash\mathbf{A} \to \mathbf{C}^\times$ the standard additive character with $\psi_\infty(x) = e^{2\pi i \mathrm{Tr}_{F_\infty/\mathbf{R}} x}$, and we set $\psi_E := \psi \circ \mathrm{Tr}_{E/F}$. We view $\psi_{|\mathbf{A}^\infty}$ as valued in $\mathbf{Q}^\circ$ via the embedding ι°.

Unitary groups. — Fix a positive integer n. For a place v of F, we denote by $\mathscr{V}_v$ be the set of isomorphism classes of (nondegenerate) E_v/F_v-hermitian spaces of dimension n; this consists of one element if v splits in E, of two elements if v is finite nonsplit, and of $n+1$ elements if v is real. We denote by $\mathscr{V}^+$ the set of isomorphism classes of E/F-hermitian spaces of dimension n that are positive definite at all archimedean place, and by $\mathscr{V}^-$ the set of isomorphism classes of E/F-hermitian spaces of dimension n that are positive definite at all archimedean place but one, at which the signature is $(n-1, 1)$. We denote by $\mathscr{V}^\circ$ the set of isomorphism classes of $\mathbf{A}_E/\mathbf{A}$-hermitian spaces of dimension n such that for all but finitely many places v, the Hasse–Witt invariant $\epsilon(V_v) := \eta_v((-1)^{\binom{n}{2}} \det V_v) = +1$, and that V_v is positive definite at all archimedean places. We put $\epsilon(V) := \prod_v \epsilon(V_v)$, and write $\mathscr{V}^{\circ,\epsilon} \subset \mathscr{V}^\circ$ for the set of spaces with $\epsilon(V) = \epsilon \in \{\pm\}$.

We have a natural identification $\mathscr{V}^{\circ,+} = \mathscr{V}^+$. We will mostly be interested in $\mathscr{V}^{\circ,-}$, which we refer to as the set of *incoherent* E/F-hermitian spaces, cf. [Gro]. If $V \in \mathscr{V}^{\circ,-}$, then for every archimedean place v of F, there exists a unique $V(v) \in \mathscr{V}^-$ over F such that $V(v)_w \cong V_w$ if $w \neq v$.

For $V \in \mathscr{V}$, let $\mathrm{H}_V = \mathrm{U}(V)$; if $V \in \mathscr{V}^\circ$ with $\epsilon(V) = -1$, we still use the notation $\mathrm{H}_V(\mathbf{A}^S) \coloneqq \prod_{v \notin S} H_{V_v}$, $H_{V_v} \coloneqq \mathrm{U}(V_v)(F_v)$, and we refer to (the symbol)

$$\mathrm{H}_V$$

as an *incoherent* unitary group.

Suppose from now on that $n = 2r$ is even. We define the quasisplit unitary group over F

$$\mathrm{G} = \mathrm{U}(W),$$

where $W = E^n$ equipped with the skew-hermitian form $\begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}$ (here 1_r is the identity matrix of size r).

Definition 3.1. — 1. A *relevant* complex automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_E)$ is an irreducible cuspidal automorphic representation satisfying:
   (i) $\Pi \circ \mathrm{c} \cong \Pi^\vee$;
   (ii) for every archimedean place w of E, the representation Π_w is induced from the character $\arg^{n-1} \otimes \arg^{n-3} \otimes \ldots \otimes \arg^{1-n}$ of the torus $(\mathbf{C}^\times)^n = (E_w^\times)^n \subset \mathrm{GL}_n(E_w)$; here $\arg(z) \coloneqq z/|z|$.
2. A *possibly relevant* complex automorphic representation π of $\mathrm{G}(\mathbf{A})$ is an irreducible cuspidal automorphic representation such that for every archimedean place v of F, the representation π_v is the holomorphic discrete series representation of Harish-Chandra parameter $\{\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{3-n}{2}, \frac{1-n}{2}\}$. We say that π is *relevant* if it is possibly relevant and *stable* as defined at the beginning of § 3.2 below.
3. Let $V \in \mathscr{V}^{\circ,-}$ and let v be an archimedean place of F. A *possibly relevant* complex cuspidal automorphic representation σ of $\mathrm{H}_{V(v)}(\mathbf{A})$ is an irreducible cuspidal automorphic representation such that σ_v is one of the n discrete series representation of $H_{V(v)_v} = U(n-1,1)$ of Harish-Chandra parameter $\{\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{3-n}{2}, \frac{1-n}{2}\}$, and for every other archimedean place $v' \neq v$ of F, we have $\sigma_{v'} = \mathbf{1}$ (as a representation of $H_{V(v)_{v'}} = U(n)$). We say that σ is *relevant* if it is possibly relevant and stable.

Definition 3.2. — 1. A *relevant* p-adic automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_E)$ is a representation of $\mathrm{GL}_n(\mathbf{A}_E^\infty)$ on a $\overline{\mathbf{Q}}_p$-vector space, such that for every $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$, the representation $\iota\Pi$ is the finite component of a (unique up to isomorphism) relevant complex automorphic representation Π^ι.
2. A *possibly relevant*, respectively *relevant* p-adic automorphic representation π of $\mathrm{G}(\mathbf{A})$ is representation of $\mathrm{G}(\mathbf{A}^\infty)$ on a $\overline{\mathbf{Q}}_p$-vector space, such that for every $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$, the representation $\iota\pi$ is the finite component of a (unique up to isomorphism) possibly relevant, respectively relevant, complex automorphic representation π^ι of $\mathrm{G}(\mathbf{A})$.
3. Let $V \in \mathscr{V}^{\circ,-}$. A *possibly relevant*, respectively *relevant*, p-adic automorphic representation σ of $\mathrm{H}_V(\mathbf{A})$ is representation of $\mathrm{H}_V(\mathbf{A}^\infty)$ on a $\overline{\mathbf{Q}}_p$-vector space, such that for every $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ and every archimedean place v of F, the representation $\iota\sigma$ is the finite component of a (unique up to isomorphism) possibly relevant, respectively relevant, complex automorphic representation $\sigma^{\iota,(v)}$ of $\mathrm{H}_{V(v)}(\mathbf{A})$.

3.2. Automorphic descent. — For a place v of F, we denote by BC_v the base-change map from L-packets of tempered representations of G_v to tempered representations of $\mathrm{GL}_n(E_v)$, which is injective by [Mok15, Lemma 2.2.1]. We denote by BC_{G} and $\mathrm{BC}_{\mathrm{H}_V}$ the base-change maps from automorphic representations of the unitary groups $\mathrm{G}(\mathbf{A})$ or $\mathrm{H}_V(\mathbf{A})$ to automorphic representations of $\mathrm{GL}_n(\mathbf{A}_E)$, respectively; we simply write BC when there is no risk of confusion. We say that a cuspidal automorphic representation of a unitary group is *stable* if its base-change is still cuspidal.

Remark 3.3. — We have the following properties of the base-change maps.

(a) By [LTX+22, Proposition C.3.1], if Π is a relevant representation of $\mathrm{GL}_n(\mathbf{A}_E)$, then: the preimage of Π under $\mathrm{BC}_{\mathrm{H}_V}$ consists of relevant representations of $\mathrm{H}_V(\mathbf{A})$; the preimage of Π under BC_{G} contains a relevant representation of $\mathrm{G}(\mathbf{A})$.
(b) If v is a finite place, the base-change maps may be defined for representations with coefficients over $\overline{\mathbf{Q}}_p$, compatibly with any extensions of scalars $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$.
(c) As a consequence of (a) and (b), BC extends to a map from relevant *p-adic* automorphic representations of $\mathrm{G}(\mathbf{A})$ and $\mathrm{H}_V(\mathbf{A})$ to relevant p-adic automorphic representations of $\mathrm{GL}_n(\mathbf{A}_E)$.

Descent to a quasisplit unitary group. — We fix the auxiliary choice of a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ with torus T and unipotent radical N, and (the T-orbit of) a generic linear homomorphism $\Psi\colon \mathrm{N}(F)\backslash \mathrm{N}(\mathbf{A}) \to \mathbf{C}^\times$; we call this choice (N, Ψ) a *Whittaker datum*. A relevant complex or p-adic automorphic representation π of $\mathrm{G}(\mathbf{A})$ is called *Ψ-generic* if it for every finite place, π_v is Ψ_v-generic in the sense that it has a non-vanishing $(N_v, \Psi_{|N_v})$-Whittaker functional .

Proposition 3.4. — *Let Π be a relevant p-adic automorphic representation of* $\mathrm{GL}_n(\mathbf{A}_E)$*. Then there exists a relevant p-adic automorphic representation π of* $\mathrm{G}(\mathbf{A})$*, unique up to isomorphism, which is Ψ-generic and satisfies* $\mathrm{BC}(\pi) = \Pi$.

Proof. — By [GRS11] and [Mor18], for each ι there exists a relevant cuspidal automorphic representation π^ι of $\mathrm{G}(\mathbf{A})$ that is Ψ-generic and satisfies $\mathrm{BC}(\pi^\iota) = \Pi^\iota$. By [Var17, Ato17], for each finite place v, each local L-packet of G_v contains a unique Ψ-generic representation, which (together with the injectivity of BC_v) implies that π^ι is unique up to isomorphism. Then by Remark 3.3 (b), the collection (π^ι) arises from a well-defined relevant p-adic automorphic representation π of $\mathrm{G}(\mathbf{A})$. $\square$

Remark 3.5. — Our construction of Theta cycles will be based on the choice of a relevant representation π with $\mathrm{BC}(\pi) = \Pi$, which is not unique. For definiteness, we may pick a Whittaker datum Ψ (for which, as explained in [KMSW, § 0.2.2, § 1.6.1], there is a standard choice), and take π to be the Ψ-generic representation given by Proposition 3.4.

3.3. Theta correspondence. — Let π be a relevant p-adic representation of G with $\mathrm{BC}(\pi) = \Pi$. We will need to further transfer π to a representation of unitary groups H_V for $V \in \mathscr{V}^{\circ,-}$.

Local correspondence and duality. — We first review the local theory. Let v be a finite place of F, and let C be either $\overline{\mathbf{Q}}_p$ or $\mathbf{C}$. For $V_v \in \mathscr{V}_v$, let $\omega_{V_v} = \omega_{V_v, \psi_v}$ be the Weil representation of $H_{V_v} \times G_v$ (with respect to the character ψ_v) over C, a model of which is recalled in § 4.2 below. Whenever $\square$ is some smooth admissible representation of a group $G^?$, we denote by $\square^\vee$ the contragredient, and by $(\ ,\ )_\square$ the natural pairing on $\square \times \square^\vee$.

The first part of the following result (for nonsplit finite places) is known as *theta dichotomy*.

Proposition 3.6. — *Let π_v be an tempered irreducible admissible representation of G_v over $C = \overline{\mathbf{Q}}_p$ or $C = \mathbf{C}$.*

1. *There exists a unique $V_v \in \mathscr{V}_v$ such that*
$$\sigma_v^\vee := (\pi_v^\vee \otimes \omega_{V_v})_{G_v} \neq 0.$$
2. *The representation $\sigma_v^\vee$ is tempered and irreducible. Its contragredient σ_v satisfies $\mathrm{BC}(\sigma_v) = \mathrm{BC}(\pi_v)$, and the space*
$$\mathrm{Hom}_{H_{V_v} \times G_v}(\sigma_v \otimes \pi_v^\vee \otimes \omega_{V_v}, C)$$
is 1-dimensional over C.
3. *The representation $(\pi_v \otimes \omega_{V_v}^\vee)_{G_v}$ is canonically identified with σ_v.*
4. *Denote by ϑ each of the projection maps $\pi_v^\vee \otimes \omega_{V_v} \to \sigma_v^\vee$, $\pi_v \otimes \omega_{V_v}^\vee \to \sigma_v$. Then the map*
$$\zeta_v(f,\varphi,\phi;f',\varphi',\phi') := (f,\vartheta(\varphi,\phi))_{\sigma_v} \cdot (f',\vartheta(\varphi',\phi'))_{\sigma_v^\vee}$$
defines a canonical generator
$$\zeta_v \in \mathrm{Hom}_{H_{V_v} \times G_v}(\sigma_v \otimes \pi_v^\vee \otimes \omega_{V_v}, C) \otimes_C \mathrm{Hom}_{H_{V_v} \times G_v}(\sigma_v^\vee \otimes \pi_v \otimes \omega_{V_v}^\vee, C),$$
with the property that if π_v and σ_v are unramified and $f,\varphi,\phi,f',\varphi',\phi'$ are spherical vectors, then
$$\zeta_v(f,\varphi,\phi;f',\phi',\varphi') = (f,f')_{\sigma_v}(\varphi,\varphi')_{\pi_v^\vee} \cdot (\phi,\phi')_{\omega_v^\vee}.$$

Proof. — We drop all subscripts v. We start by recalling the first two statements. Consider first the case that v is finite and E is a field. Then $\sigma_V^\vee = (\pi^\vee \otimes \omega_V)_G$ is the (a priori, 'big') theta lift of $\pi^\vee$ as defined in [Har07, (2.1.5.1)]. By the local theta dichotomy proved in Theorem 2.1.7 (iv) *ibid.* and [GG11, Theorem 3.10], there is exactly one $V \in \mathscr{V}$ such that $\sigma_V^\vee$ is nonzero; we fix this V and drop it from then notation. Then the other properties of $\sigma := (\sigma^\vee)^\vee$ are consequences of [GI16, Theorem 4.1] (which collects results from [Wal90, GT16, GS12, GI14]). For the case $E = F \oplus F$, see [Mín08].

We now turn to the other two statements. For a character $\chi : F^\times \to C^\times$, let

$$b_n(\chi) := \prod_{i=1}^{n} L(i, \chi\eta^{i-1}). \tag{3.1}$$

If $C = \mathbf{C}$, then we have a canonical element

$$\check{\zeta} \in \mathrm{Hom}_G(\pi^\vee \otimes \omega_V, \mathbf{C}) \otimes_{\mathbf{C}} \mathrm{Hom}_G(\pi \otimes \omega_V^\vee, \mathbf{C})$$

given by

$$\check{\zeta}(\varphi,\phi;\varphi',\phi') := \frac{b_n(1)}{L(1/2,\Pi)} \int_G (g\varphi,\varphi')_{\pi^\vee} \cdot (\omega(g)\phi,\phi')_\omega \, dg, \tag{3.2}$$

where dg is the measure of [DL, § 2.1 (G7)], $\Pi := \mathrm{BC}(\pi)$. It is a generator by [HKS96, § 6], where the regularisation of the integral is also taken care of. (For the well-known comparison between the definition in *loc. cit.* and the one given here, see [Sak17, Lemma 3.1.2].) When π (hence σ) are unramified and all the vectors are spherical, by [Yam14, Propositions 7.1, 7.2] we have

$$\check{\zeta}(\varphi,\phi;\varphi',\phi') = (\varphi,\varphi')_{\pi^\vee} \cdot (\phi,\phi')_{\omega^\vee}. \tag{3.3}$$

If $C = \overline{\mathbf{Q}}_p$, then for any $\iota \in \Sigma$ we have a tetralinear form $\check{\zeta}^\iota$ as above, and by [DL, Lemma 3.30], there is a $\check{\zeta} \in \mathrm{Hom}_G(\pi^\vee \otimes \omega_V, \overline{\mathbf{Q}}_p) \otimes_{\overline{\mathbf{Q}}_p} \mathrm{Hom}_G(\pi \otimes \omega_V^\vee, \overline{\mathbf{Q}}_p)$ such that $\zeta \otimes_{\overline{\mathbf{Q}}_p, \iota} \mathbf{C} = \zeta^\iota$ for every $\iota \in \Sigma$.

Now, we may view $\check{\zeta}$ as a map

$$\check{\zeta} : (\pi^\vee \otimes \omega_V)_G \otimes (\pi \otimes \omega_V^\vee)_G \to C \tag{3.4}$$

that is, by inspection, invariant under the diagonal action of H on both factors. It follows that $\check{\zeta}$ gives the duality of our third statement. The fourth statement then follows from the definitions and (3.3). □

Remark 3.7. — A more symmetrically defined exalinear form would be

$$(f, \varphi, \phi; f', \varphi', \phi') \mapsto \int_{H_V} \int_G (hf, f')_\sigma \cdot (g\varphi, \varphi')_{\pi^\vee} \cdot (\omega(h,g)\phi, \phi')_\omega \, dg\, dh,$$

where the integral in dg is regularised as remarked after (3.2). If σ is a discrete series, the integral in dh converges and its value equals that of ζ_v, times the formal degree of σ – for which [BP21] gives a formula in terms of adjoint gamma factors. In general, regularising the integral in dh amounts to regularising the inner product of two matrix coefficients of σ. A regularisation has been proposed by Qiu [Qiu12a, Qiu12b]; however the definition of the resulting generalised formal degree is partly conjectural, and no precise (even conjectural) formula for it appears in the literature.

Global correspondence. — We have the following global variant of Proposition 3.6.

Proposition 3.8. — *Let Π be a relevant p-adic automorphic representation of $\mathrm{GL}_n(\mathbf{A}_E)$, and set $\epsilon = \epsilon(1/2, \Pi)$. Let $\mathscr{R}_{\Pi,\mathrm{G}}$ be the set of isomorphism classes of relevant automorphic representations π of $\mathrm{G}(\mathbf{A})$ with $\mathrm{BC}(\pi) = \Pi$, and let $\mathscr{R}_{\Pi,\mathrm{H}}$ be the set of pairs (V, σ), with $V \in \mathscr{V}^{\circ,\epsilon}$ and σ an isomorphism class of relevant p-adic automorphic representations of $\mathrm{H}_V(\mathbf{A})$.*

The relation

$$\mathrm{Hom}_{\mathrm{G}(\mathbf{A}^\infty) \times \mathrm{H}(\mathbf{A}^\infty)}(\sigma^\infty \otimes \pi^{\infty,\vee} \otimes \omega_V^\infty, \overline{\mathbf{Q}}_p) \neq 0 \tag{3.5}$$

defines a bijection between $\mathscr{R}_{\Pi,\mathrm{G}}$ and $\mathscr{R}_{\Pi,\mathrm{H}}$.

Proof. — Take any $\iota \colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$. After base-change to $\mathbf{C}$ via ι, given π, the existence of V with $\varepsilon(V) = \epsilon(1/2, \Pi)$ and of a representation $\sigma^{\infty,\iota}$ of $\mathrm{H}_V(\mathbf{A}^\infty)$ satisfying (3.5) follows from the explicit form of theta dichotomy in terms of the doubling epsilon factors of [Har07], whose product over all places coincides with the standard central epsilon factor of Π by [LR05]. Again by [LTX+22, Proposition C.3.1], we have that $\sigma^{\infty,\iota}$ is the finite component of relevant automorphic representation σ^ι; and as in Remark 3.3 (c), the collection σ^ι arises from a relevant p-adic automorphic representation σ.

The bijective property of the resulting map $\mathscr{R}_\mathrm{G} \to \mathscr{R}_\mathrm{H}$ follows from [GI16, Theorem 4.1 (iv)] and the following archimedean fact (see [NZ01] or [PT02, Theorem 4.1 (4)]): if $v | \infty$ and π_v is the holomorphic discrete series of $U(\frac{n}{2}, \frac{n}{2})$ with Harish–Chandra parameter $\{\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{3-n}{2}, \frac{1-n}{2}\}$, then π_v has a nonzero theta lift to H_{V_v}, with $V_v \in \mathscr{V}_v$, exactly for V_v positive-definite, in which case the theta lift σ_v is the trivial representation of H_{V_v}. □

4. Theta cycles

4.1. Assumptions on the Galois representation. — Let again $\rho\colon G_E \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be irreducible, geometric, and of weight -1. We denote by $\rho^{\mathrm{c}}\colon G_E \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ the representation defined by $\rho^{\mathrm{c}}(g) = \rho(cgc^{-1})$, where $c \in G_E$ is any fixed lift of c. (A different choice of lift would yield an isomorphic representation.)

We suppose from now on that the following conditions are satisfied:

1. ρ is *conjugate-symplectic* in the sense that there exists a perfect pairing
$$\rho \otimes_{\overline{\mathbf{Q}}_p} \rho^{\mathrm{c}} \to \overline{\mathbf{Q}}_p(1)$$
such that for the induced map $u\colon \rho^c \to \rho^*(1)$ (where * denotes the linear dual) and its conjugate-dual $u^*(1)^{\mathrm{c}}\colon \rho^{\mathrm{c}} \to \rho^{\mathrm{c},*}(1)^{\mathrm{c}} = \rho^*(1)$, we have $u = -u^*(1)^{\mathrm{c}}$;
2. $n = 2r$ is even;
3. for every $\jmath\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}_p$, the $\jmath$-Hodge–Tate weights of ρ are the n integers $\{-r, -r+1, \ldots, r-1\}$ (where the convention is that the cyclotomic character has weight -1);
4. ρ is *automorphic* in the sense that for each $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$, there is a cuspidal automorphic representation Π^ι of $\mathrm{GL}_n(\mathbf{A}_E)$ such that $L_\iota(\rho, s) = L(\Pi^\iota, s + 1/2)$;

Associated automorphic representations. — A collection $(\Pi^\iota)_{\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}}$ as in Condition 4 is uniquely determined up to isomorphism if it exists, by the multiplicity-one theorem for automorphic forms on GL_n; it is conjectured to always exist. Moreover, every Π^ι is relevant in the sense of Definition 3.1.1, where Condition 1 implies property (i) in the definition, and Condition 3 implies property (ii). It is then clear that $(\Pi^\iota)_\iota$ arises from a unique (up to isomorphism) relevant p-adic automorphic representation
$$\Pi = \Pi_\rho$$
of $\mathrm{GL}_n(\mathbf{A}_E)$ (Definition 3.2.1). We denote by $\pi = \pi_\rho$ the relevant p-adic representation of $\mathrm{G}(\mathbf{A})$ associated with Π as in Proposition 3.4,[(4)] and by
$$(V, \sigma) = (V_\rho, \sigma_\rho)$$
the pair associated with π as in Proposition 3.8. We also put $\mathrm{H} = \mathrm{H}_V$.

4.2. Models of the representations. — We now fix some concrete models of the representations ω, π, and σ.

Weil representations. — We fix the well-known model of the representation $\omega = \otimes'_{v\nmid\infty} \omega_{V,v}$ on $\mathscr{S}(V^r_{\mathbf{A}^\infty}, \overline{\mathbf{Q}}_p)$ associated with ψ, on which $\mathrm{H}(\mathbf{A}^\infty)$ acts by right translations, whereas the action of $\mathrm{G}(\mathbf{A}^\infty)$ is recalled in [DL, §4.1 (H7)].

Denote by $\dagger$ the involution on G given by conjugation by the element $\begin{pmatrix} 1_r & \\ & -1_r \end{pmatrix}$ inside $\mathrm{GL}_n(E)$; it acts on any $\mathrm{G}(R)$-module for any E-algebra R. The representation $\omega^\dagger$ is a model of the Weil representation attached to ψ^{-1}.

Siegel-hermitian modular forms and their q-expansion. — The representation π may be realised in spaces of hermitian modular forms, which we briefly review.

In [DL, § 2.2], we have defined the following objects.[(5)]

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(4) As noted in Remark 3.5, any other relevant π with $\mathrm{BC}(\pi) = \Pi$ would be equally good.

(5) In this discussion, most new notation will be introduced by equalities whose right-hand sides reproduce the corresponding notation in [DL].

- A $\mathbf{C}$-vector space $\mathscr{H}_{\mathbf{C}} = \mathscr{A}^{[r]}_{r,\mathrm{hol}}$ of holomorphic forms for the group G.
- For any $\overline{\mathbf{Q}}_p$-algebra R, an R-module $\mathscr{H}_R = \mathscr{H}_r^{[r]} \otimes_{\mathbf{Q}_p} R$ of (classical) p-adic automorphic forms for G, such that for each $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$, we have an isomorphism
$$\mathscr{H}_{\overline{\mathbf{Q}}_p} \otimes_{\iota} \mathbf{C} \to \mathscr{H}_{\mathbf{C}}, \qquad \Phi \otimes 1 \mapsto \Phi^{\iota}.$$
  In fact, only the case where E/F is totally split above p was considered in [DL], where $\mathscr{H}_r^{[r]}$ is the direct limit, over open compact subgroups $K \subset \mathrm{G}(\mathbf{A}^{\infty})$, of subspaces of sections of a certain line bundle on a Siegel hermitian variety $\Sigma(K)_{/\mathbf{Q}_p}$; let us explain why the splitting condition is not necessary for our purposes. Define a *p-adic CM type* of E to be a set Φ of $[F:\mathbf{Q}]$ embeddings $i\colon E \hookrightarrow \overline{\mathbf{Q}}_p$ such that $i \in \Phi$ if and only if $i \circ \mathrm{c} \notin \Phi$; in the totally split case, the choice of a p-adic CM type is equivalent to the choice of a set $\mathtt{P}_{\mathrm{CM}}$ as in [DL, §2.1 (F2)], which intervenes in the construction of $\Sigma(K)$ as a moduli scheme by fixing a *signature type* for test objects in the sense of [LTX$^+$22, Definition 3.4.3]. However, this construction, and the comparison complex Siegel hermitian varieties of [DL, Lemma 2.1], go through with any p-adic CM type Φ (with the innocuous difference that, in general, $\Sigma(K)$ and $\mathscr{H}_r^{[r]}$ will only be defined over a finite extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$).
- A space $\mathrm{SF}_R = \mathrm{SF}_r(R)$ of formal q-expansions with coefficients in the (arbitrary) ring R, and a Siegel–Fourier expansion map $\mathbf{q}_{\infty} = \mathbf{q}_r^{\mathrm{an}}\colon \mathscr{H}_{\mathbf{C}} \to \mathrm{SF}_{\mathbf{C}}$. By the argument at the end of the proof of [DL, Proposition 4.19] (based on Lemma 2.11 *ibid.*), we deduce a $\overline{\mathbf{Q}}_p$-linear q-expansion map
$$\mathbf{q}_p\colon \mathscr{H}_{\overline{\mathbf{Q}}_p} \to \mathrm{SF}_{\overline{\mathbf{Q}}_p}$$
  satisfying $\iota\mathbf{q}_p(\Phi) = \mathbf{q}_{\mathbf{C}}(\Phi^{\iota})$ for every $\Phi \in \mathscr{H}_{\overline{\mathbf{Q}}_p^{\circ}}$ and every embedding $\iota \in \Sigma$.

By [DL, Lemma 3.14] (based on [Mok15]), for a relevant p-adic automorphic representation π, the space $\mathrm{Hom}_{\mathrm{G}(\mathbf{A}^{\infty})}(\pi, \mathscr{H}_{\overline{\mathbf{Q}}_p})$ is 1-dimensional, and $\pi^{\vee,\dagger}$ is also relevant. We identify $\pi = \pi_{\rho}$ with the corresponding subspace of $\mathscr{H}_{\overline{\mathbf{Q}}_p}$. Then $\pi_{\rho^*(1)}$ is isomorphic to $\pi^{\vee,\dagger}$.

Shimura varieties and their cohomology. — We assume from now on that $\varepsilon(\rho) = -1$. (The opposite case will be trivial for our purposes in Definition 4.5 below.) Then $V \in \mathscr{V}^{\circ,-}$, and we have an inverse system
$$(X_K)_{K \subset \mathrm{H}(\mathbf{A}^{\infty})}$$
of $(n-1)$-dimensional smooth varieties over E, with the property that for every archimedean place w of F, with underlying place v of F, the variety $X_{V,K} \times_{E,w} \mathbf{C}$ is isomorphic to the complex Shimura variety $X_{V(v),wK}$ associated with the unitary group $\mathrm{H}_{V(v)}$ and the Shimura datum attached to w that is the complex conjugate to the one defined in [Liu21, § C.1] (and thus coincides with the one specified in [LTX$^+$22, § 3.2] and used in [LL21, DL]); see also [Gro, ST].

From now on we assume that each X_K is projective, which is the case if and only if either $F \neq \mathbf{Q}$, or $n = 2$, $F = \mathbf{Q}$ and $\varepsilon(V_v) = -1$ for some finite place v. In fact, in the remaining non-compact case for $n = 2$, the curve X_K (closely related to a classical modular curve) can be canonically compactified by adding finitely many cusps; in this case the constructions make sense, and the theorems hold true, after replacing X_K by its compactification.

Let
$$H^{2r-1}_{\text{ét}}(X_{\overline{E}},\overline{\mathbf{Q}}_p(r)) := \varprojlim_{K\subset \mathrm{H}(\mathbf{A}^\infty)} H^{2r-1}_{\text{ét}}(X_{K,\overline{E}},\overline{\mathbf{Q}}_p(r)),$$
where the transition maps are pushforwards. For each K, we have a spherical Hecke algebra for H acting on X_K; let $\mathfrak{m}_{\rho,K}$ be the Hecke ideal denoted by $\mathfrak{m}^{\mathrm{R}}_\pi$ in [LL21, Definition 6.8]. We denote by
$$M_{\rho,K} := H^{2r-1}_{\text{ét}}(X_{K,\overline{E}},\overline{\mathbf{Q}}_p(r))_{\mathfrak{m}_{\rho,K}}$$
the localisation, and we set
$$M_\rho := \varprojlim_K M_{\rho,K} \subset H^{2r-1}_{\text{ét}}(X_{\overline{E}},\overline{\mathbf{Q}}_p(r)).$$

We will assume the following hypothesis, which is a special case of [LL21, Hypothesis 6.6] (it is known for $n=2$, and it is expected to be confirmed in general in a sequel to [KSZ]).

Hypothesis 4.1. — *For each open compact $K\subset \mathrm{H}(\mathbf{A}^\infty)$, we have a Hecke- and Galois-equivariant decomposition*

$$M_{\rho,K} \cong \bigoplus_{\sigma'} \rho\otimes\sigma'^{\vee,K}, \tag{4.1}$$

where the direct sum runs over the isomorphism classes of relevant p-adic automorphic representation σ' of $\mathrm{H}_V(\mathbf{A})$ with $\mathrm{BC}(\sigma')=\Pi$.

We thus have an $\mathrm{H}(\mathbf{A}^\infty)$-equivariant map

$$\sigma \longrightarrow \mathrm{Hom}_{\overline{\mathbf{Q}}_p[G_E]}(H^{2r-1}_{\text{ét}}(X_{\overline{E}},\overline{\mathbf{Q}}_p(r)),\rho), \tag{4.2}$$

and we identify σ with the image of this map. We also put $M_{\sigma,K} := \rho\otimes\sigma^{\vee,K}\subset H^{2r-1}_{\text{ét}}(X_{K,\overline{E}},\overline{\mathbf{Q}}_p(r))$, and

$$M_\sigma := \varprojlim_K M_{\sigma,K} \subset M_\rho \subset H^{2r-1}_{\text{ét}}(X_{\overline{E}},\overline{\mathbf{Q}}_p(r)). \tag{4.3}$$

Then $\sigma = \mathrm{Hom}_{\overline{\mathbf{Q}}_p[G_E]}(M_\sigma,\rho) := \varinjlim_K \mathrm{Hom}_{\overline{\mathbf{Q}}_p[G_E]}(M_{\sigma,K},\rho)$.

Denote by $\mathrm{Fil}^\bullet \subset H^{2r}_{\text{ét}}(X_K,\mathbf{Q}_p(r))$ the filtration induced by the Hochschild–Serre spectral sequence $H^i(E,H^{2r-i}_{\text{ét}}(X_K,\mathbf{Q}_p(r)))\Rightarrow H^{2r}_{\text{ét}}(X_K,\mathbf{Q}_p(r))$. By the argument for [DL, Lemma 4.7], we have a canonical Hecke-equivariant projection
$$H^{2r}_{\text{ét}}(X_K,\overline{\mathbf{Q}}_p(r))/\mathrm{Fil}^2) \to H^1(E,M_{\rho,K}).$$

Lemma 4.2. — *The image of the composition*
$$[-]_\rho\colon \mathrm{Ch}^r(X_K)^0_{\overline{\mathbf{Q}}_p} \xrightarrow{\mathrm{AJ}} H^{2r}_{\text{ét}}(X_K,\overline{\mathbf{Q}}_p(r))/\mathrm{Fil}^2 \to H^1(E,M_{\rho,K})$$
is contained in $H^1_f(E,M_{\rho,K})$

Proof. — As in [DL, Lemma 4.24], using [NN16, Theorem B] in place of [Nek00] for p-adic places. □

4.3. Construction. — We proceed in four steps. The first three steps follow works of Kudla and collaborators [Kud97, Kud03, KRY06], and of Liu and collaborators [Liu11a, DL].

0. Special cycles in X. — For each $x \in V^r_{\mathbf{A}^\infty}$ and each open compact $K \subset \mathrm{H}(\mathbf{A}^\infty)$, we have a codimension-$r$ special cycle

$$Z(x)_K \in \mathrm{Ch}^r(X_K)$$

defined in [Liu11a, § 3A]. Putting

$$T(x) := ((x_i, x_j)_V)_{ij},$$

where $(\,,\,)_v$ is the hermitian form on V, we recall the definition in two basic cases. Denote by $\mathrm{Herm}_r(F)^+$ the set of $r \times r$ matrices over E that satisfy $T^c = T^t$ and that are totally positive semidefinite. First, $Z(x)_K = 0$ if $T(x) \notin \mathrm{Herm}_r(F)^+$. Second, assume that $T(x) \in \mathrm{Herm}_r(F)^+$ is positive definite. Let $V_x \subset V$ be the incoherent hermitian space that is (place by place) the orthogonal complement of the span of $(x_1, \ldots, x_r)$. The corresponding embedding $\mathrm{U}(V_x) \hookrightarrow \mathrm{U}(V)$ of incoherent unitary groups. induces a map of towers of Shimura varieties $\alpha_x : X_{V_x} \to X_V$; then we define $Z(x)_K \in \mathrm{Ch}^r(X_{V,K})$ to be the class of the image cycle.

1. Theta kernel. — The special cycles just defined may be assembled into a generating series. Let $\phi \in \omega$. For every $K \subset \mathrm{H}_V(\mathbf{A}^\infty)$ fixing ϕ, we define

$${}^{\mathrm{q}}\Theta(\phi)_{\rho,K} := \mathrm{vol}(K) \sum_{x \in K \backslash V^r_{\mathbf{A}^\infty}} \phi(x)[Z(x)_K]_\rho\, q^{T(x)},$$

where $\mathrm{vol}(K)$ is as in [LL21, Definition 3.8]. Then ${}^{\mathrm{q}}\Theta(\phi)_{\rho,K}$ is an element of $H^1_f(E, M_{\rho,K}) \otimes_{\overline{\mathbf{Q}}_p} \mathrm{SF}_{\overline{\mathbf{Q}}_p}$, and the construction is compatible under pushforward in the tower X_K. (The reason why we prefer our $\Theta(\phi)_{-,\rho}$ to be compatible with pushforwards rather than pullbacks is that this allows to pair them, in Step 3, with elements of the automorphic representation σ under the identification (4.2).)

The following conjecture, which is a variant of [DL, Conjecture 4.17], asserts the modularity of the generating series, and from now on we will assume it holds.

Hypothesis 4.3. — *For every $\phi \in \omega$ and any $K \subset \mathrm{H}_V(\mathbf{A}^\infty)$ fixing ϕ, there exists a unique*

$$\Theta(\phi)_{\rho,K} \in H^1_f(E, M_{\rho,K}) \otimes_{\overline{\mathbf{Q}}_p} \mathscr{H}_{\overline{\mathbf{Q}}_p}$$

such that for every $g \in \mathrm{G}(\mathbf{A}^\infty)$, we have

$$\mathbf{q}_p(\Theta(\omega(g)\phi)_{K,\rho}) = {}^{\mathrm{q}}\Theta(\omega(g)\phi)_{\rho,K}.$$

Remark 4.4. — A recent piece of evidence for this modularity conjecture is provided in [DL, Theorem 4.21], which is recalled as part of Theorem 5.5.2; moreover,(6) an analogous conjecture for orthogonal Shimura varieties can be deduced from [Kud21]. Hypothesis 4.3 is implied by the variant for Chow groups of [LL21, Hypothesis 4.5]. See Remark 4.6 *ibid.* for comments on the supporting evidence for that conjecture until then, to which we should add the recent [Xia22]. For the history, which traces back to the work of Gross–Kohnen–Zagier on generating series of Heegner points [GKZ87], see [Li23, Remark 3.5.5], cf. also *ibid.* § 6.4.

2. Arithmetic theta lifts. — Denote by $\Phi \mapsto \Phi_\pi$ the Hecke-eigenprojection $\mathscr{H}_{\overline{\mathbf{Q}}_p} \to \pi$, and by $\langle\,,\,\rangle_{\pi^\vee} : \pi^\vee \otimes \pi \to \overline{\mathbf{Q}}_p$ the canonical duality. (We also use the same names for any base-change.)

(6) I am grateful to Yifeng Liu for bringing this to my attention.

Then for every $\varphi \in \pi^\vee$, we may define

$$\Theta(\varphi,\phi)_K := \langle \varphi, \Theta(\phi)_{K,\rho,\pi}\rangle_{\pi^\vee} \quad \in H^1_f(E, M_{\rho,K}). \tag{4.4}$$

Since the map $(\varphi,\phi)\mapsto \Theta(\varphi,\phi)_K$ is equivariant under the action of $\overline{\mathbf{Q}}_p[K\backslash \mathrm{H}_V(\mathbf{A}^\infty)/K]$, Proposition 3.8 implies that $\Theta(\varphi,\phi)_K$ belongs to the subspace $H^1_f(E, M_{\sigma,K})\subset H^1_f(E,M_{\rho,K})$.

3. Theta cycles. — For every $f\in\sigma, \varphi\in\pi^\vee$, and any $K\subset \mathrm{H}_V(\mathbf{A}^\infty)$ fixing f and ϕ, we define

$$\Theta_\rho(f,\varphi,\phi) := f_*\Theta(\varphi,\phi)_K \in H^1_f(E,\rho).$$

The following definition then satisfies the first property asserted in Theorem A.

Definition 4.5. — Let ρ be a Galois representation satisfying the assumptions of § 4.1.

If $\varepsilon(\rho)=+1$, we may put $\Lambda_\rho = \overline{\mathbf{Q}}_p$ and $\Theta_\rho := 0$.

If $\varepsilon(\rho)=-1$, assume that $F\neq \mathbf{Q}$ and that Hypotheses 4.1 and 4.3 hold, and let π, V, σ be as above. Then we define

$$\Lambda_\rho := (\sigma\otimes\pi^\vee\otimes\omega)_{\mathrm{H}(\mathbf{A}^\infty)\times \mathrm{G}(\mathbf{A}^\infty)},$$

and

$$\begin{aligned}\Theta_\rho\colon \Lambda_\rho &\to H^1_f(E,\rho),\\ [(f,\varphi,\phi)] &\mapsto \Theta_\rho(f,\varphi,\phi).\end{aligned}$$

Remark 4.6. — Suppose that $n=2$ and that $\rho = V_pA_E$ for a modular abelian variety A of GL_2-type over F. Then the image of Θ_ρ consists of classes of Heegner points. This follows by comparing the height formulas for the two objects in [YZZ12] and [Liu11b], against the backdrop of [Nek07]. A direct comparison is also possible: for $n=2$, all the $Z(x)$ are CM points on unitary Shimura curves, which can be related along the lines of [Car86, § 4] to the modular curves and the quaternionic Shimura curves used to construct Heegner points in [GZ86, YZZ12].

5. Relation to L-functions and Selmer groups

We continue to denote by ρ a Galois representation satisfying the assumptions of § 4.1.

5.1. Complex and p-adic L-functions.

— For every $\iota\colon \overline{\mathbf{Q}}_p\hookrightarrow \mathbf{C}$, and every finite-order character $\chi'\colon G_E\to \overline{\mathbf{Q}}_p^\times$, we have the L-function

$$L_\iota(\rho\otimes\chi', s) = L(s+1/2, \Pi^\iota\otimes\iota\chi'),$$

which is holomorphic and has a functional equation with center at $s=0$ and sign $\varepsilon(\rho)$.

At least under the following assumption, we also have a p-adic L-function.

Assumption 5.1. — *The extension E/F is totally split above p, and for every place $w|p$ of E, the representation ρ_w is crystalline and Panchishkin-ordinary.*

We need to make the auxiliary choice of an isomorphism $\alpha\colon \pi^{\vee,\dagger}\to\pi_{\rho^*(1)}$ (where $\pi=\pi_\rho, \pi_{\rho^*(1)}\subset \mathscr{H}_{\overline{\mathbf{Q}}_p}$), which yields for each $\iota\colon \overline{\mathbf{Q}}_p\hookrightarrow\mathbf{C}$, an element $\mathrm{P}_{\rho,\iota} = \mathrm{P}_{\rho,\alpha,\iota}(\rho)\in\mathbf{C}^\times$ such that

$$\iota(\varphi_1^\dagger,\varphi_2)_{\pi^\vee} = \frac{((\alpha\varphi_1)^{\iota,\dagger},\varphi_2^\iota)_{\mathrm{Pet}}}{\mathrm{P}_{\rho,\iota}}$$

for every $\varphi_1 \in \pi^{\vee,\dagger}$, $\varphi_2 \in \pi$; here

$$(\varphi, \varphi')_{\rm Pet} := \int_{\mathrm{G}(F)\backslash \mathrm{G}(\mathbf{A})} \varphi(g)\varphi'(g)\, dg$$

where dg is the measure of [DL, § 2.1 (G7)].

For a character χ of G_F, we put $\chi_E := \chi_{|G_E}$, and $b_n(\chi) := \prod_{v\nmid\infty} b_n(\chi_v)$, where the factors are as in (3.1); we also define a constant

$$c_\infty = \left((-1)^r 2^{-r^2-r}\pi^{r^2}\frac{\Gamma(1)\cdots\Gamma(r)}{\Gamma(r+1)\cdots\Gamma(2r)}\right)^{[F:\mathbf{Q}]}.$$

Finally, we denote by $\mathscr{K}(\mathscr{X}_F)$ the fraction field of $\mathscr{O}(\mathscr{X}_F)$.

Proposition 5.2. — *Suppose that ρ satisfies Assumption 5.1. There is a meromorphic function*

$$L_p(\rho) = L_{p,\alpha}(\rho) \quad \in \mathscr{K}(\mathscr{X}_F)$$

characterised by the following property: for every finite-order character $\chi \in \mathscr{X}_F(\overline{\mathbf{Q}}_p)$ and every embedding $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$, we have

$$\iota L_p(\rho)(\chi) = \iota e_p(\rho,\chi)\cdot \frac{c_\infty L_\iota(\rho\otimes\chi_E, 0)}{b_n(\chi)\mathrm{P}_{\rho,\iota}}.$$

Here, $\iota e_p(\rho,\chi) = \prod_{w|v|p}\iota e_{w,\iota}(\rho,\chi) \in \iota\overline{\mathbf{Q}}_p$, in which the product ranges over the p-adic places of E and of F, and

$$\iota e_w(\rho,\chi) := \gamma(\iota\mathrm{WD}(\rho_w^+\otimes\chi_{E,w}), \psi_{E,w})^{-1}\frac{b_{n,v}(\chi)}{L_\iota(\rho_w\otimes\chi_{E,w})}.$$

where the Deligne–Langlands γ-factor and Fontaine's functor $\iota\mathrm{WD}$ are as recalled in [Dis/a, (1.1.4)].

Proof. — This follows by multiplying the incomplete p-adic L-function of [DL, Theorem 1.4] by local L-factors at ramified and p-adic places, as in the proof of Proposition 3.39 *ibid.*[(7)] □

5.2. Pairings. — Let ρ be a representation satisfying the assumptions of Definition 4.5, and let π_ρ, V, σ_ρ, Λ_ρ, and Θ_ρ be the associated objects. We denote by π_v and σ_v the local components of π_ρ and σ_ρ at the place v (which are well-defined up to isomorphism).

Dual Theta cycles. — The representation $\rho^*(1)$ also satisfies those assumptions, and we have the corresponding map

$$\Theta_{\rho^*(1)}\colon \Lambda_{\rho^*(1)} \to H^1_f(E, \rho^*(1)).$$

Pairings. — Let $\langle\, ,\rangle\colon M_\rho\otimes M_{\rho^*(1)} \to \overline{\mathbf{Q}}_p(1)$ be the pairing induced by Poincaré duality. Then we define a pairing

$$(\, ,\,)_\sigma \colon \sigma_\rho\otimes\sigma_{\rho^*(1)} \to \overline{\mathbf{Q}}_p \tag{5.1}$$

by $(f,f')_\sigma := f\circ u(f'^*(1))$, where $f'^*(1)\colon \rho^*_{\sigma^*(1)}(1)\to M^*_{\rho^*(1)}(1)$ is the transpose, and $u\colon M^*_{\rho^*(1)}(1)\to M_\rho$ is the isomorphism induced by $\langle\, ,\rangle$. Thus $\sigma_{\rho^*(1)}$ is identified with $\sigma_\rho^\vee = \sigma^\vee$.

(7) Before [DL], a p-adic L-function that extends $L_p(\rho)$ to a larger space was constructed in [EHLS20]; the rationality property proved there is weaker than stated here.

We also have a canonical pairing on $\omega \otimes \omega^\dagger$ defined by

$$(\phi,\phi')_\sigma = \int_{V^r_{\mathbf{A}^\infty}} \phi(x)\phi'(x)\,dx \tag{5.2}$$

for the product of ψ-selfdual measures. Thus $\omega^\dagger$ is identified with $\omega^\vee$. Similarly, if we denote $\iota\Box := \Box\otimes_{\overline{\mathbf{Q}}_p,\iota}\mathbf{C}$, and complex conjugation in $\mathbf{C}$ by a bar, we have $\overline{\iota\omega}=\omega^\vee$. Let $\mathrm{vol}(H_\infty)$ be the volume of $\mathrm{H}(F_\infty)$ for the measure denoted $\frac{1}{b_{2r}(0)}\,dh_v^\natural$ in [LL21, Definition 3.8], which is a rational number by [DZ, Lemma 2.2.1].

Then:

- for every isomorphism $\alpha\colon \pi_\rho^{\vee,\dagger}\to\pi_{\rho^*(1)}$, we have a pairing

$$\zeta_\alpha := \mathrm{vol}(H_\infty)\cdot\otimes_{v\nmid\infty}\zeta_v\circ(\,)^\dagger\circ j_\alpha\colon \Lambda_\rho\otimes\Lambda_{\rho^*(1)}\to\overline{\mathbf{Q}}_p, \tag{5.3}$$

  where j_α identifies the factor $\pi^\vee_{\rho^*(1)}$ of $\Lambda_\rho\otimes\Lambda_{\rho^*(1)}$ with $\pi_\rho^\dagger$ via the dual of α, and $(\,)^\dagger$ maps $\pi_\rho^\dagger\otimes\omega$ to $\pi_\rho\otimes\omega^\dagger=\pi_\rho\otimes\omega^\vee$;
- for every $\iota\in\Sigma$ we have an identification $j_\iota\colon \iota\pi^\vee_{\rho^*(1)}\xrightarrow{\cong}\overline{\iota\pi_\rho}$ via the restriction of $(\,,\,)_{\mathrm{Pet}}$ to $\overline{\pi^\iota_\rho}\otimes\pi_{\rho^*(1)}$. Then we obtain a pairing

$$\zeta_\iota := \mathrm{vol}(H_\infty)\cdot\otimes_{v\nmid\infty}\zeta_v\circ\overline{(\,)}\circ j_\iota\colon \iota\Lambda_\rho\otimes\iota\Lambda_{\rho^*(1)}\to\mathbf{C}$$

  where $\overline{(\,)}$ maps $\overline{\iota\pi_\rho}\otimes\iota\omega$ to $\iota\pi_\rho\otimes\overline{\iota\omega}=\iota\pi_\rho\otimes\iota\omega^\vee$.

p-adic height pairing. — Assume that ρ is Panchishkin-ordinary. Then the construction of Nekovář [Nek93] (see [DL, § 4.2] for a verification of the assumptions) yields a p-adic height pairing

$$\langle\,,\,\rangle\colon H^1_f(E,\rho)\otimes H^1_f(E,\rho^*(1))\to\Gamma_F\hat{\otimes}\overline{\mathbf{Q}}_p.$$

Complex height pairings. — On the other hand, assume that p is unramified in E, and let $K_p^\circ=\prod_{v|p}H_v\subset H_p$ be a product ot maximal hyperspecial subgroups. Then for open compact $K^p\subset\mathrm{H}(\mathbf{A}^{p\infty})$, setting $K:=K^pK_p^\circ\subset\mathrm{H}(\mathbf{A}^\infty)$, the variety X_K has good reduction at all p-adic places. Define

$$\mathrm{Ch}^r(X_K)^{\langle p\rangle}\subset\mathrm{Ch}^r(X_K)^0$$

to be the $\mathbf{Q}$-subspace of algebraic cycles whose class in $H^{2r}(X_{K,E_w},\mathbf{Q}_p(r))$ is trivial for every finite place $w\nmid p$ of E. Li and Liu [LL21] observed that the construction of Beĭlinson [Beĭ87] unconditionally defines a height pairing

$$\langle\,,\,\rangle^{\mathrm{BB}}\colon \mathrm{Ch}^r(X_K)^{\langle p\rangle}_{\mathbf{C}}\otimes_{\mathbf{C}}\mathrm{Ch}^r(X_K)^{\langle p\rangle}_{\mathbf{C}}\to\mathbf{C}\otimes_{\mathbf{Q}}\mathbf{Q}_p \tag{5.4}$$

that is $\mathbf{C}$-linear in the first factor and $\mathbf{C}$-antilinear in the second factor. (It is conjectured that the pairing takes values in $\mathbf{C}\subset\mathbf{C}\otimes_{\mathbf{Q}}\mathbf{Q}_p$; this turns out to be the case in the application to Theta cycles.)

In order to descend this pairing to Selmer groups, we need to assume a case of a standard conjecture on the injectivity of Abel–Jacobi maps. Whenever $K\subset\mathrm{H}(\mathbf{A}^\infty)$ is an open compact subgroup that is understood from the context, denote $\mathfrak{m}_\rho=\mathfrak{m}_{\rho,K}$, $\mathfrak{m}_{\rho^*(1)}=\mathfrak{m}_{\rho^*(1),K}$

Conjecture 5.3. — *For $\rho^? \in \{\rho, \rho^*(1)\}$ and for each open compact subgroup $K^p \subset \mathrm{H}(\mathbf{A}^{p\infty})$, the Abel–Jacobi map*

$$\mathrm{AJ}_{p,K^pK_p^\circ}\colon \left(\mathrm{Ch}^r(X_{K^pK_p^\circ})^{\langle p\rangle}_{\overline{\mathbf{Q}}_p}\right)_{\mathfrak{m}_{\rho^?}} \to H^1_f(E, M_{\rho^?,K^pK_p^\circ}) \tag{5.5}$$

is injective.

Assume that ρ is crystalline at all p-adic places. Fix a maximal hyperspecial subgroup $K_p^\circ \subset \mathrm{H}(\mathbf{A}^{p\infty})$, and assume that Conjecture 5.3 holds. Denote by $H^1_f(E, M_{\rho^?,K^pK_p^\circ})^X$ the image of (5.5), and let

$$H^1_f(E,\rho^?)^{X_{K_p^\circ}} := \sum_{\sigma',K^p}\ \sum_{f'\in(\sigma')^{K^pK_p}} f'_* H^1_f(E, M_{\rho^?,K^pK_p^\circ})^X$$

where the first sum is as in (4.1) for $\rho^?$. Then for every $\iota\colon \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ and every $K = K^pK_p^\circ$, we have a pairing

$$\langle\ ,\ \rangle^\iota_K\colon H^1_f(E, M_{\rho,K})^X \otimes_{\overline{\mathbf{Q}}_p} H^1_f(E, M_{\rho^*(1),K})^X \otimes_{\overline{\mathbf{Q}}_p,\iota} \mathbf{C} \to \mathbf{C}\otimes_{\mathbf{Q}} \mathbf{Q}_p \tag{5.6}$$

transported from (5.4) via the maps $\mathrm{AJ}_{p,K^pK_p^\circ}\otimes_\iota 1$. We may deduce from it a pairing

$$\langle\ ,\ \rangle^\iota\colon H^1_f(E,\rho)^{X_{K_p^\circ}} \otimes_{\overline{\mathbf{Q}}_p} H^1_f(E,\rho^*(1))^{X_{K_p^\circ}} \otimes_{\overline{\mathbf{Q}}_p,\iota} \mathbf{C} \to \mathbf{C}\otimes_{\mathbf{Q}} \mathbf{Q}_p \tag{5.7}$$

defined as follows.

For $i = 1,2$ let

$$c_i = f_{i,*}\mathrm{AJ}_{p,K} c'_i$$

for some $K = K^pK_p^\circ$, some $f_i \in \sigma_i^K$, and some

$$c'_1 \in \left(\mathrm{Ch}^r(X_{K^pK_p^\circ})^{\langle p\rangle}_{\overline{\mathbf{Q}}_p}\right)_{\mathfrak{m}_\rho}, \qquad c'_2 \in \left(\mathrm{Ch}^r(X_{K^pK_p^\circ})^{\langle p\rangle}_{\overline{\mathbf{Q}}_p}\right)_{\mathfrak{m}_{\rho^*(1)}}.$$

If $\sigma_1 \ncong \sigma_2^\vee$, we put

$$\langle c_1, c_2\rangle^\iota := 0. \tag{5.8}$$

If $\sigma_1 \cong \sigma_2^\vee$, we have the pairing $(\ ,\ )_{\sigma_1}$ of (5.1) on $\sigma_1\otimes\sigma_2$, through which we identify $\sigma_2 = \sigma_1^\vee$. Let

$$\mathrm{t}_K(f_1\otimes f_2) \in \mathrm{Hom}(\sigma_1^{\vee,K}, \sigma_2^K) = \mathrm{End}(\sigma_1^{\vee,K}) = \mathrm{End}_{\overline{\mathbf{Q}}_p[G_E]}(M_{\sigma_1,K})$$

be given by

$$\mathrm{t}_K(f_1\otimes f_2)(v_1) = \mathrm{vol}(K)\cdot(v_1,f_1)_{\sigma_1}\cdot f_2,$$

and let

$$\mathrm{t}(f_1\otimes f_2)(v_1) = \mathrm{vol}(K)\cdot \mathrm{t}_K(f_1\otimes f_2); \tag{5.9}$$

the normalising volume factor makes t into a well-defined map $\sigma_1\otimes\sigma_1^\vee \to \mathrm{End}_{\overline{\mathbf{Q}}_p[G_E]}(M_{\sigma_1})$. The existence of a Hecke correspondence acting as $\mathrm{t}(f_1\otimes f_2)$ implies that the action of $\mathrm{t}(f_1\otimes f_2)$ on Selmer groups preserves the subspace $H^1_f(E, M_{\sigma_1,K})^{X_{K_p^\circ}}$. Then we define

$$\langle c_1, c_2\rangle^\iota := \langle \mathrm{t}(f_1\otimes f_2)c'_1, c'_2\rangle^\iota_K. \tag{5.10}$$

The definition of (5.7) in the general case follows from (5.8), (5.10) by bilinearity.

Remark 5.4. — In the p-adic case, we also have $\Gamma_F\hat{\otimes}\overline{\mathbf{Q}}_p$-valued Nekovář pairings $\langle\ ,\ \rangle_K$ analogous to (5.6) (whose construction takes as input the pairing on $M_{\rho,K}\otimes M_{\rho^*(1),K}$ deduced from Poincaré duality). The analogous formula to (5.10) holds true as a consequence of the definitions and the projection formula [DZ, Lemma A.2.5].

5.3. The height formulas. — We may now state the main known results on Theta cycles. They parallel those of [GZ86, PR87, Kol88] on Heegner points.

We will say that E and ρ are *mildly ramified* if E and π_ρ satisfy the hypotheses of [DL, Assumption 1.6], except possibly for the ones about p-adic places.

Theorem 5.5. — *Suppose that $F\neq\mathbf{Q}$ or $n=2$, that E and ρ are mildly ramified, and that ρ is crystalline at all p-adic places. Assume Hypotheses 4.1.*

1. *Assume the Modularity Hypothesis 4.3, Conjecture 5.3, and that p is unramified in E. Then for every $\lambda\in\Lambda_\rho$, $\lambda'\in\Lambda_{\rho^*(1)}$ and for every $\iota\in\Sigma$, we have*
$$\langle\Theta_\rho(\lambda),\Theta_{\rho^*(1)}(\lambda')\rangle^\iota=\frac{c_\infty L'_\iota(\rho,0)}{b_n(\mathbf{1})}\cdot\zeta_\iota(\lambda,\lambda')$$
in $\mathbf{C}$.
2. *Suppose that Assumption 5.1 holds and that $p>n$. Let $\alpha\colon\pi_\rho^{\vee,\dagger}\cong\pi_{\rho^*(1)}$. Then:*
   - *if the order of vanishing of $L_{p,\alpha}(\rho)$ at $\mathbf{1}$ is one, then the Modularity Hypothesis 4.3 holds, and for every $\lambda\in\Lambda_\rho$, $\lambda'\in\Lambda_{\rho^*(1)}$, we have*
$$\langle\Theta_\rho(\lambda),\Theta_{\rho^*(1)}(\lambda')\rangle=e_p(\rho,\mathbf{1})^{-1}\cdot\mathrm{d}L_{p,\alpha}(\rho)(\mathbf{1})\cdot\zeta_\alpha(\lambda,\lambda').$$
in $\Gamma_F\hat{\otimes}\overline{\mathbf{Q}}_p=T_\mathbf{1}^*\mathscr{X}_F$.
   - *if the order of vanishing of $L_{p,\alpha}(\rho)$ at $\mathbf{1}$ is not one and the Modularity Hypothesis 4.3 holds, then for every $\lambda\in\Lambda_\rho$, $\lambda'\in\Lambda_{\rho^*(1)}$, we have*
$$\langle\Theta_\rho(\lambda),\Theta_{\rho^*(1)}(\lambda')\rangle=0.$$

Proof. — Write $\lambda=[(f,\varphi,\phi)]$, $\lambda'=[(f',\varphi',\phi')]$. Consider the p-adic case. The Modularity result is [DL, Theorem 4.21], after projection $H^1_f(E,M_\rho)\to H^1_f(E,M_{\sigma'})$ for any relevant σ' with $\mathrm{BC}(\sigma')=\Pi$; but this is equivalent to the modularity in $H^1_f(E,\rho)$ by Hypothesis 4.3.

For the first height formula, by the definitions and Remark 5.4, it is equivalent to prove

$$\langle\mathrm{t}(f\otimes f'^{\vee})\Theta(\varphi,\phi),\Theta(\varphi',\phi')\rangle=e_p(\rho,\mathbf{1})^{-1}\cdot\mathrm{d}L_{p,\alpha}(\rho)(\mathbf{1})\cdot\check{\zeta}_\alpha(\mathrm{t}(f\otimes f'^{\vee})\vartheta(\varphi,\phi');\vartheta(\phi,\phi'))\tag{5.11}$$

where the Θ's are the arithmetic theta liftings for ρ and $\rho^*(1)$ as in (4.4), and
$$\check{\zeta}_\alpha=\mathrm{vol}(H_\infty)\cdot\otimes_{v\nmid\infty}\check{\zeta}_v\circ(\ )^\dagger\circ j_\alpha$$
is defined analogously to (5.3) based on the pairings (3.4). Pick a $K\subset\mathrm{H}_V(\mathbf{A}^\infty)$ fixing f,f',ϕ,ϕ', and let $T\in\mathscr{H}(\mathrm{H}(\mathbf{A}^\infty))$ be a Hecke operator acting as $\mathrm{vol}(K)^{-1}\mathrm{t}(f\otimes f'^{\vee})$ on $\sigma^\vee$. Then (5.11) is equivalent to [DL, Theorem 1.8 (1)] in level K for
$$(\varphi,T\phi;\varphi',\phi').$$
(Note that our definitions of the arithmetic theta lifts $\Theta(-,-)$ differ from those of [DL] by a factor $\mathrm{vol}(K)$; in the height formula, one factor is accounted for by (5.9), and another by the normalisation

of height pairings in *loc. cit.*. The term $\mathrm{vol}^{\natural}(K)$ in [DL] equals our $\mathrm{vol}(H_\infty)\mathrm{vol}(K)$: this difference is accounted for by the factor $\mathrm{vol}(H_\infty)$ in the pairing ζ_α.)

The p-adic height vanishing formula is likewise equivalent to [DL, Theorem 1.8 (2)].

The complex case is similarly reduced to [LL22, Theorem 1.8]. As ρ is crystalline at all $w|p$, the representations Π_w, σ_w, π_w are unramified, so that we can take representatives $(f, \varphi, \phi; f', \varphi', \phi')$ of $\lambda, \lambda' \neq 0$ that are fixed by a maximal hyperspecial K_p°. Then the fact that $\langle\ ,\ \rangle^\iota$ is well-defined on Theta cycles follows from the definitions and [LL21, Proposition 6.10 (3)]. □

Part 2 of Theorem A is then an immediate consequence of Theorem 5.5. For a beautiful exposition of some key aspects of the proofs of the formulas in [LL21, LL22, DL], see [Li23].

The proof of Theorem 5.5 suggests that from the point of view of height formulas, Theta cycles offer no material advantage over previous constructions. This is not so from the point of view of Euler systems, as we explain next.

5.4. An Euler system. — The main technique for bounding Selmer groups is that of Euler systems, originally introduced by Kolyvagin to study Heegner points [Kol88, Kol90]. Roughly speaking, an Euler system for a representation ρ of G_E is a collection of integral Selmer classes defined over certain abelian extensions of ρ and satisfying certain compatibility relations; the (one) class defined over E itself is called the base class of the Euler system.

In a forthcoming work, Jetchev–Nekovář–Skinner theorise a variant of this notion, that we shall call a *JNS* Euler system. It is adapted to conjugate-symplectic representations over CM fields, where the abelian extensions are ring class fields ramified at the primes of E split over the totally real subfield F (see [Ski]). Their main result is that if ρ has 'sufficiently large' image, then the existence of a JNS Euler system with nontrivial base class z implies that z generates the Selmer group of ρ: for a precise statement (when $F = \mathbf{Q}$), see [ACR23, Theorem 8.3 and Remark 8.4], where JNS Euler systems are called 'split anticyclotomic Euler systems' (*ibid.*, Definition 8.1).

The following is the main result of the forthcoming [Dis/b]. Granted the results of Jetchev–Nekovář–Skinner, it implies part 3 of Theorem A.

Theorem 5.6. — *Let $\rho\colon G_E \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a representation satisfying the assumptions of § 4.1. Then for any $\lambda \in \Lambda_\rho$, there exists a JNS Euler system based on $\Theta_\rho(\lambda)$.*

Multiplicity-one principles are remarkably useful to prove relations between special cycles and, in particular, compatibility relations in Selmer groups – as first observed in [YZZ12] and [LSZ22]. The proof of Theorem 5.6 is no exception: this is the main technical advantage of having constructed a cycle depending on one parameter only.

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